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Multiscale Algorithm for the Numerical Inversion of Elliptic Operators

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Abstract

This work develops an efficient and fast algorithm to invert elliptic operators associated to Lipschitz coefficients. The method is based on multiresolution analysis and uses localization properties of wavelets. The numerical inverse of the operator is defined through the coupling between a Galerkin approximation and a preconditioner based on wavelets. The inverse uses a standard residual correction method that may be applied to solve some partial differential equations. According to the algorithm the time derivation operators is approximated by a finite difference scheme. The work presents numerical simulations and provides comparisons with a standard Galerkin method. The results are quite reasonable when compared to the exact solution. Numerical examples show that the proposed method is efficient.

AMS Subject Classifications: 65T60, 65N22.
Keywords: Elliptic operator, Galerkin method, multiresolution analysis, wavelets, preconditioner.

1 Introduction

Beginning from 1980s, wavelets have been used for the resolution of partial differential equations, they have been applied to obtain representations of integral and differential operators in many physical problems (see [1, 10]). The wavelet algorithms for solving partial differential equations (PDEs) usually are based on Galerkin technics or on the collocation method and most of them can handle easily periodic boundary conditions. Several studies testify the use of wavelets to solve PDEs. For example, in [5, 8], Haar wavelet solutions for reaction-diffusion and nonlinear evolution equations were

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proposed. In [2], a wavelet numerical method for the analysis of a convection-diffusion equation was presented. For more accounts of related topics, we refer the reader to [3].

For this work, the purpose is the definition and the implementation of a multiscale algorithm based on wavelets for the inversion of elliptic operators associated to variable coefficients. We focus on operators like $L = I - \mathrm{div}(A\nabla)$, where A is chosen such that the operator is connected to a coercive and hermitian form. The main results are the derivation of a preconditioner of L and the construction of an explicit approximation for the solution of the problem $Lu = f$. After introducing the problem in Section 2, we present in Section 3 some background on multiresolution analysis and wavelets. Section 4 describes a wavelet preconditioner of L and we derive an explicit expression of the inverse using a standard iterative correction algorithm. This construction is obtained through the coupling between, on one hand, the inverse of the Galerkin approximation in a low dimension space and, on the other hand, a preconditioner of the operator; this step corresponds to the refinement procedure offered by wavelets. Section 5 describes the iterative algorithm for solving the problem $Lu = f$. Section 6 is devoted to the numerical implementation of the inversion algorithm in a periodic framework, a short discussion on the complexity and numerical results are given. A final discussion on the obtained results and possible extensions of this work is presented in Section 7.

2 Presentation of the Problem

This section gives the assumptions we make on the operator to be inverted. Our interest stands in the approximation of the solution of the problem

$$Lu = f, \tag{2.1}$$

where L is defined from $H^2(\mathbb{R})$ to $L^2(\mathbb{R})$ as $L = I - \mathrm{div}(A\nabla)$. We choose A such that L is connected to a V-elliptic and hermitian form. For the sake of simplicity, we take $L = I - \dfrac{\partial}{\partial x}\left(\nu(x)\dfrac{\partial}{\partial x}\right)$, where I is the identity operator. After using a finite difference approximation of $\dfrac{\partial}{\partial t}$ in the diffusion equation $\dfrac{\partial \theta}{\partial t} = \dfrac{\partial}{\partial x}\left(\nu(x)\dfrac{\partial \theta}{\partial x}\right) + s$, the problem (2.1) can be found where ν plays the role of a diffusion coefficient. We assume that ν is a bounded and Lipschitzian function of $L^2(\mathbb{R})$ and that it exists a constant $\nu_0 > 0$, such that $\nu(x) \geq \nu_0, \forall x \in \mathbb{R}$. The operator L is then associated to a sesquilinear form $B(\cdot,\cdot)$ defined on $H^1(\mathbb{R})$ by

$$B(f,h) = \int_{I\!R} \nu(x)\frac{\partial f}{\partial x}\frac{\partial \bar{h}}{\partial x}(x)dx + \int_{I\!R} f(x)\bar{h}(x)dx, \quad f,h \in H^1(\mathbb{R}).$$

The form B is continuous and coercive, so the Lax–Milgram lemma ensures the existence of L^{-1} and of any of its Galerkin approximation derived on finite dimension subspaces.

3 Multiresolution Analysis and Wavelets

Classical approaches to wavelet construction deal with multiresolution analysis (MRA). In [7], we define a multiresolution analysis as a sequence of embedded spaces of approximation of $L^2(\mathbb{R})$, V_j, $j \in \mathbb{Z}$ satisfying the following properties: the spaces V_j are generated by the orthonormal bases $\{\phi_{jk}, k \in \mathbb{Z}\}$ where $\phi_{jk}(x) = 2^{j/2}\phi(2^j x - k)$ and ϕ is r-regular [1]. A multiresolution analysis is related to particular functions called wavelets obtained by dilation and translation of a given function ψ, the mother wavelet. Wavelets are essentially fast decaying and localized in time and frequency. Let now W_j be the orthogonal complement of V_j in V_{j+1} : $V_{j+1} = V_j \oplus W_j$, it exists a wavelet ψ such that the family $\{\psi_{jk} = 2^j\psi(2^j x - k), k \in \mathbb{Z}\}$ is an orthonormal basis of W_j. The function ψ has the same regularity properties as ϕ and, moreover, the function ψ has zero moments, i.e., $\forall \alpha \leq r$, $\int x^\alpha \psi(x)dx = 0$ and $\forall f \in L^2(\mathbb{R})$, $f(x) = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle f, \psi_{jk}\rangle \psi_{jk}(x)$.

Orthogonal or biorthogonal multiresolution analysis lead naturally to hierarchical algorithms for the expression of the scalars $\langle u, \psi_{jk}\rangle$ in term of the scalars $\langle u, \phi_{pk}\rangle$, $p > j$, $k \in \mathbb{Z}$, (for details, see [4,9]). For practical implementations, it implies the existence of filters H_j and G_j that are used in the following way:

$$\begin{aligned} \langle u, \psi_{jk}\rangle &= \sum_{j,l} \langle u, \phi_{j+1,l}\rangle G_j(2k - l) \\ \langle u, \phi_{jk}\rangle &= \sum_{j,l} \langle u, \phi_{j+1,l}\rangle H_j(2k - l). \end{aligned}$$

In numerical applications, these relations are implemented using convolution and decimation algorithms of complexity $O(N)$ or $O(N \log(N))$, where N is the number of variables. The operator $\{\langle u, \phi_{pk}\rangle, k \in \mathbb{Z}\} \mapsto \{\langle u, \psi_{jk}\rangle, j < p, k \in \mathbb{Z}\}$ is called the wavelet decomposition, while we call wavelet reconstruction to the inverse.

4 Preconditioning and Inversion Algorithm

The operator Π_j, $j \in \mathbb{Z}$ (resp. Π_j^*, $j \in \mathbb{Z}$), stands for the orthogonal projection from $L^2(\mathbb{R})$ on V_j (resp. the extension operator from V_j on L^2). Given an integer p, we consider the Galerkin approximation of L in the space V_p : $L_p = \Pi_p L \Pi_p^* : V_p \to V_p$. Choosing any integer $q < p$, we have

$$V_p = V_q \bigoplus_{p>j\geq q} W_j.$$

A classical Galerkin method is used to derive the approximation of L^{-1} on the subspace V_q, and a refinement procedure is applied on the subspaces W_j. Let $A_{p,q} : V_q \to V_q$ be

[1] $\forall m \in \mathbb{N}$, $n \leq r$, it exists a constant C_m such that $\left|\frac{\partial^n \phi(x)}{\partial x^n}\right| \leq C_m(1 + |x|)^{-m}$.

the inverse of the Galerkin approximation of L_p defined as

$$\pi_q L_p \pi_q^* A_{p,q} = I_{V_q},$$

where I_{V_q} is the identity operator on V_q, π_q is the orthogonal projection from V_p on V_q and π_q^* is the extension operator from V_q to V_p. The canonical extension of $A_{p,q}$ to V_p is still denoted by $A_{p,q}$. Let $D = -i\dfrac{\partial}{\partial x}$ and $D^2 = -\dfrac{\partial^2}{\partial x^2}$ so that $L = I + D\nu D$. We define an operator $P_{p,q}$ by

$$\begin{cases} P_{p,q}(\psi_{jk}) = \tau_{jk}, & q \leq j < p, \\ P_{p,q}(\psi_{jk}) = 0, & \text{otherwise}, \end{cases}$$

where the functions τ_{jk} are defined by

$$\nu(k2^{-j} + 2^{-j-1})(\Pi_p D^2 \Pi_p^*)\tau_{jk} = \psi_{jk}.$$

Following [6], we prove that it exists an operator $U_{p,q}$ such that

$$L_p(A_{p,q} + P_{p,q}) = I_p - U_{p,q},$$

where I_p is the identity operator in V_p and $A_{p,q} + P_{p,q}$ is a preconditioner of L_p. The operator $U_{p,q}$ satisfies the following property.

Theorem 4.1. *It exists a constant $C > 0$, independent of q and p, such that*

$$||U_{p,q}^2|| \leq C2^{-q}.$$

4.1 Construction of the Inverse Operator

The inversion scheme of L_p is divided into two steps:

- The first step is constructed from a coupling between the inverse of the Galerkin approximation of L in V_q, $A_{p,q}$, and the operator $P_{p,q}$ obtained using the wavelet basis of $W_j, p > j \geq q$.
- The second step is an iterative refinement of the first step approximation obtained by a classical method of residual correction.

Thanks to Theorem 4.1, and using Neumann's theorem, we deduce that

$$L_p^{-1} = (A_{p,q} + P_{p,q}) \sum_{k \geq 0} U_{p,q}^k.$$

5 Iterative Algorithm

A standard relaxed gradient method applied to the problem

$$L_p(A_{p,q} + P_{p,q})v = g, \quad g = (A_{p,q} + P_{p,q})f,$$

gives

$$\begin{cases} v^{(n+1)} = v^{(n)} + \alpha \left(g - L_p(A_{p,q} + P_{p,q})v^{(n)}\right), \\ v^{(0)}, \end{cases}$$

with α a relaxation parameter. The error $e^{(n)} = v^{(n)} - v$ satisfies

$$e^{(n+1)} = (I - \alpha L_p(A_{p,q} + P_{p,q}))^n e^{(0)}$$

and, therefore, the method converges as soon as $\rho(I - \alpha L_p(A_{p,q} + P_{p,q})) < 1$ where $\rho(L)$ stands for the spectral radius of the operator L. We get with $u^{(n)} = (A_{p,q} + P_{p,q})v^{(n)}$ the recursive equation for the correction $c^{(n)} = u^{(n)} - u^{(n-1)}$:

$$\begin{cases} c^{(n+1)} = (I - \alpha(A_{p,q} + P_{p,q})L_p)c^{(n)}, \\ c^{(o)} = (A_{p,q} + P_{p,q})f. \end{cases}$$

For $\alpha = 1$, the method converges for large values of q. Indeed, we get

$$\rho(I - L_p(A_{p,q} + P_{p,q})) < C2^{-q/2}$$

and $\rho(I - L_p(A_{p,q} + P_{p,q})) < 1$ for large enough values of q. Moreover, for $\alpha = 1$, the numerical solution of the problem $L_p u = f$ obtained after n iterations corresponds to

$$u^{(n)} = \sum_{k=0}^{n} f_k, \quad \begin{cases} f_k = (I - (A_{p,q} + P_{p,q})L_p)f_{k-1}, \\ f_0 = (A_{p,q} + P_{p,q})f. \end{cases}$$

6 Numerical Implementation

The implementation of the algorithm has been performed for the resolution of elliptic problems with periodic conditions; a spline multiresolution analysis of order m is used.

1. *Application of* $A_{p,q}$. The application of $A_{p,q}$ is equivalent to the resolution of a linear system of size 2^q. For any basis of V_q, $\{\phi_{q,l}, 0 \leq l < 2^q\}$, the general term of the linear system is $[M_q]_{l,l'} = \langle L\phi_{q,l}, \phi_{q,l'} \rangle$. Since $L = I - \dfrac{\partial}{\partial x}\left(\nu(x)\dfrac{\partial \cdot}{\partial x}\right)$, this matrix is ill conditioned $(\mathrm{cond}(M_q) = O(4^q))$. Therefore, the choice of the wavelet basis for the calculation is theoretically optimal since a diagonal preconditioner of M_q is available. For practical implementation, the basis that provides the maximum of zero coefficients in M_q is the B-spline basis. Indeed, when using the B-splines of order m, the matrix M_q is a $(2m-1)$ band matrix. Due to

the general form of ν, the calculation of each term of M_q should be made using a quadrature formula. Periodicity allows to use a simple formula such as trapezoidal rule. The resolution of the linear system is performed using a Cholesky factorization (M_q is semi definite positive), eventually adapted to the band structure of the matrix.

2. *Application of* $P_{p,q}$. The operator $P_{p,q}$ is defined by

$$P_{p,q}g = \sum_{\substack{0\leq k\leq 2^j-1 \\ q\leq j\leq p-1}} \langle g, \psi_{jk}\rangle \tau_{jk}$$

for all $g \in L^2$. The application of $P_{p,q}$ on a function g reduces to replace ψ_{jk} by τ_{jk} in the wavelet decomposition of g. Specific filters are used in a biorthogonal MRA [6].

6.1 Complexity

1. The complexity associated to the application of $A_{p,q}$ is

$$M2^q\frac{(2^q+1)}{2} + \frac{1}{3}2^{3q} + 2^{2q}.$$

The first contribution deals with the calculation of the terms of the symmetric matrix M_q using a quadrature formula on M points; the second contribution deals with the application of a Cholesky method for the resolution of the linear system.

2. The complexity of the application of $P_{p,q}$ is $O(p2^p)$ and is mainly related to the use of orthogonal and biorthogonal wavelet transformations.

3. The complexity associated to the application of L_p is again $O(p2^p)$. Finally, one iteration $u_{p,q}^{(n)} \mapsto u_{p,q}^{(n+1)}$ requires $O(p2^p)$ operations as soon as $q \leq \frac{p}{3}$. If we assume that the iteration number is bounded, then the global complexity of the algorithm is $O(p2^p)$. Estimations have shown that the CPU time ratio between one iteration and the global resolution using a Galerkin matrix inversion method is of order fifteen (see [7]).

6.2 Tests

We provide some numerical results for various values of $\nu(x)$ and f. We call $u_{p,q}^{(n)}$ the output of our algorithm after n iterations with $\alpha = 1$ and we will compare it to u, the solution of the initial problem $Lu = f$, to u_p the solution of the weak formulation $L_p u_p = \Pi_{V_p}(f)$ and to u_G the Galerkin approximation of u in V_p. We use the following definitions: $E_p = \dfrac{||u - \Pi_p u||_2}{||u||_2}$ is the relative error between the exact solution u and the

orthogonal projection in V_p, $\Pi_p u$; $E_G = \dfrac{\|u - u_G\|_2}{\|u\|_2}$ is the relative error between u and the Galerkin solution in V_p; $E^n_{p,q} = \dfrac{\|u - u^{(n)}_{p,q}\|_2}{\|u\|_2}$ is the relative error between u and the solution of the algorithm after n iterations.

We investigate the influence of some parameters that control the convergence of the numerical scheme. The parameters p ($\dim V_p = 2^p$) and m (spline order) control the quality of the approximation of H^1 by V_p. The parameter q ($\dim V_q = 2^q$) and the norm $\|D\nu\|_\infty$ control the norm of $U^2_{p,q}$ and consequently the convergence of $u^{(n)}_{p,q}$ towards u_p. Figure 6.1 demonstrates the influence of the spline order m on the error E_p for $p = 8, q = 2$ and for the exact solution $u(x) = \sin(2\pi x) + 0.5\sin(4\pi x)$ on $[0, 1]$ and the operator coefficient $\nu(x) = x(x-1) + 1$. Figure 6.2 illustrates clearly the influence of the number p on the error $E^n_{p,q}$. We present also some numerical results related to the choice of a family of functions $\nu(x)$ that influences the accuracy of the algorithm: $\nu(x) = 2 + \cos^k(2\pi x)$, $1 \le k \le 256$, and $u(x) = \sin(2\pi x) + 0.5\sin(4\pi x)$. For this family, $\|D\nu\|_\infty$ is variable but $\|\nu\|_\infty$ and $\|\nu^{-1}\|_\infty$ are constants. Figures 6.3, 6.4, 6.2 and 6.6 illustrate these results. In these figures, the number of iterations to get the numerical convergence remains very reasonable (less than 20) in any case and for every value of q and leads to a convergent algorithm. Moreover, when k increases, the number of iterations to get the convergence decreases; it is due to the localization of ν derivatives.

7 Concluding Remarks

We proposed an efficient and fast algorithm to invert elliptic operators associated to Lipschitz coefficients. The numerical inverse of the operator was constructed through the coupling between a Galerkin approximation and a preconditioner based on wavelets. However, the construction can be extended to multi-dimension, the difficulty arises when one deals with the derivation of the related algorithms. The numerical results presented in this paper were connected to various choices of the operator coefficient and demonstrated the feasibility of the iterative algorithm. The main advantages are its simplicity and small computation costs: this is due to the sparsity of the transform matrices and to the small number of significant wavelet coefficients. The algorithm is wholly competitive and efficient in comparison with the Galerkin method.

Our approach may be extended theoretically to boundary value problems. However, in such cases the difficulty is connected with the construction of the functions τ_{jk} and the related biorthogonal multiresolution analysis.

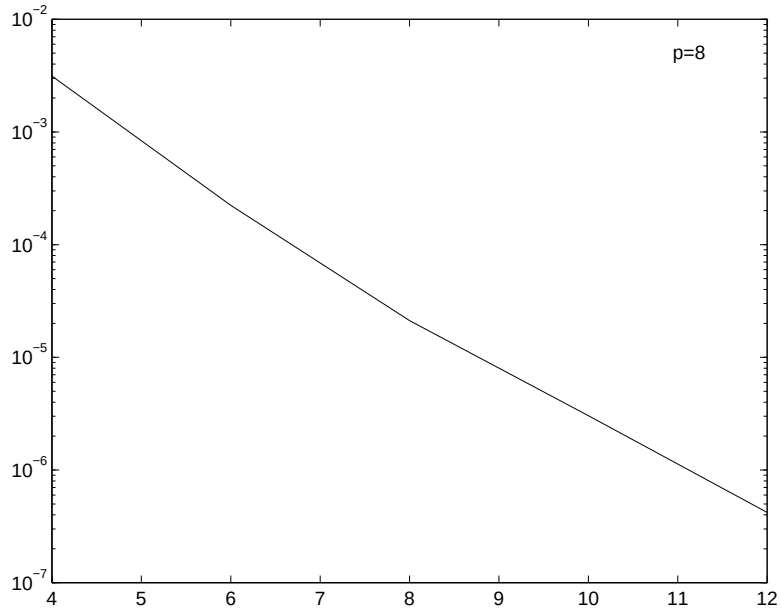

Figure 6.1: Evolution of E_p versus the spline order m.

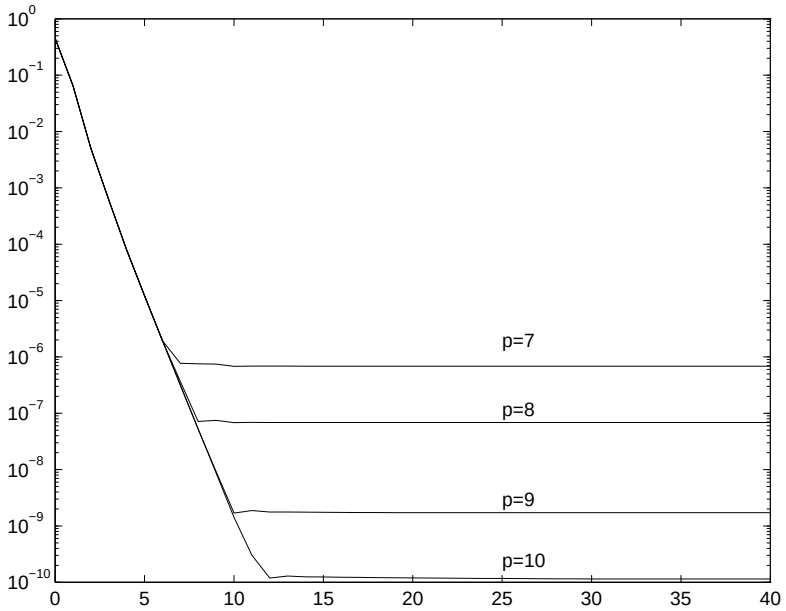

Figure 6.2: Evolution of $E^n_{p,q}$ versus the number n of iterations, $m = 4$, $q = 2$, $\nu(x) = x(x-1)+1$.

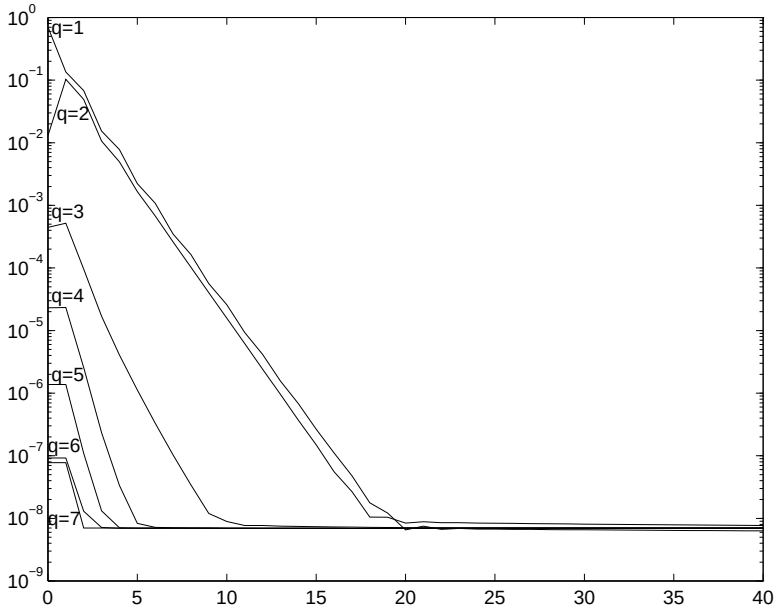

Figure 6.3: Evolution of $E_{p,q}^n$ versus the number n of iterations: $\nu(x) = 2 + \cos(2\pi x)$, $E_p = 1.139693 \times 10^{-9}$, $E_G = 8.836547 \times 10^{-9}$.

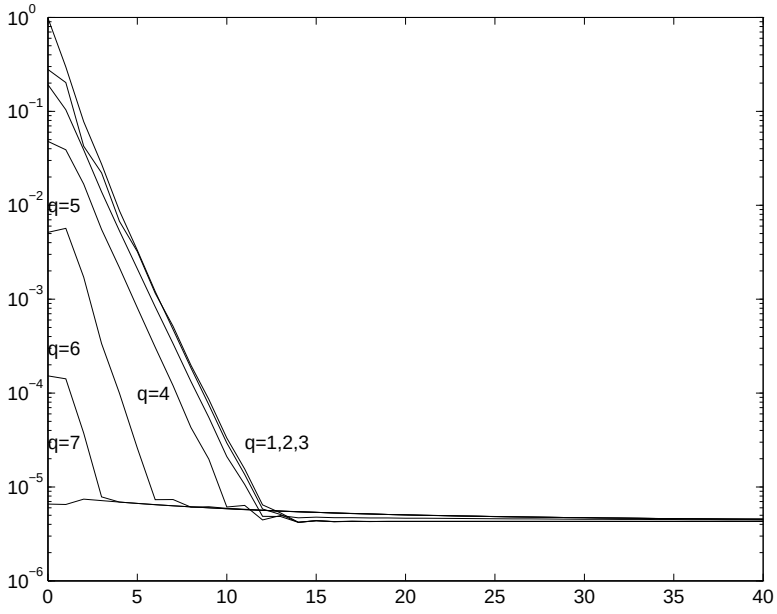

Figure 6.4: Evolution of $E_{p,q}^n$ versus the number n of iterations: $\nu(x) = 2 + \cos^{32}(2\pi x)$, $E_p = 7.316891 \times 10^{-7}$, $E_G = 4.015059 \times 10^{-7}$.

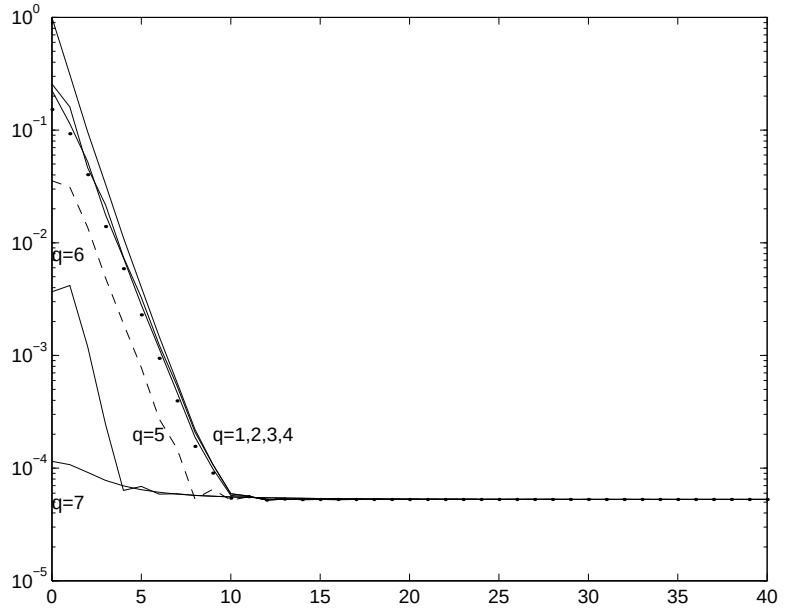

Figure 6.5: Evolution of $E^n_{p,q}$ versus the number n of iterations: $\nu(x) = 2+\cos^{128}(2\pi x)$, $E_p = 9.204909 \times 10^{-6}$, $E_G = 4.872407 \times 10^{-6}$.

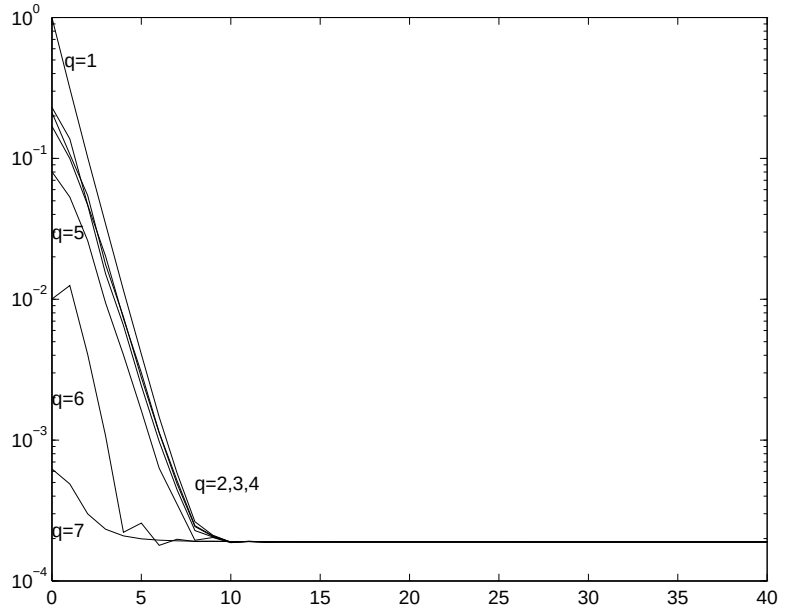

Figure 6.6: Evolution of $E^n_{p,q}$ versus the number n of iterations: $\nu(x) = 2+\cos^{256}(2\pi x)$, $E_p = 3.422981 \times 10^{-5}$, $E_G = 1.906790 \times 10^{-5}$.

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