Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 8, Number 1, pp. 37–49 (2013)
http://campus.mst.edu/adsa

Nonresonance under and between the First two Eigenvalues of a Steklov Problem

Belhadj Karim
Faculté des Sciences et Techniques, Errachidia, Maroc
karembelf@gmail.com

Abdellah Zerouali
Centre Pédagogique Régional, Fès, Maroc
abdellahzerouali@yahoo.fr

Aomar Anane and **Omar Chakrone**
Faculté des Sciences, Oujda, Maroc
ananeomar@yahoo.fr and chakrone@yahoo.fr

Abstract

We study the existence of p-harmonic solutions for the Steklov problem

$$\Delta_p u = 0 \text{ in } \Omega, \quad |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = f(x,u) \text{ on } \partial\Omega,$$

under assumptions on the asymptotic behavior of the quotients $f(x,s)/|s|^{p-2}s$ and $pF(x,s)/|s|^p$, where the limits at infinity of these quotients lie between the first two eigenvalues. Finally we establish, in a certain sense, the solvability of the problem under the first eigenvalue.

AMS Subject Classifications: 35J70, 35P30.
Keywords: Nonresonance, Sobolev trace embedding, p-harmonic solutions, Steklov problem, first nonprincipal eigenvalue.

1 Introduction

In a previous work [2], we investigated the solvability of the following problem:

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = f(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

Received May 14, 2012; Accepted November 23, 2012
Communicated by Delfim F. M. Torres

under assumptions on the asymptotic behaviour of the quotients $f(x,s)/|s|^{p-2}s$ and $pF(x,s)/|s|^p$ with $F(x,s) = \int_0^s f(x,t)dt$, where the limits at infinity of these quotients lie between the first principal and nonprincipal eigenvalues for the asymmetric Steklov problem

$$\begin{cases} \Delta_p u = |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

with the weights m, $n \in M^+ = \{m \in L^q(\partial\Omega); m^+ \not\equiv 0 \text{ in } \partial\Omega\}$, where Δ_p is the p-Laplacian, $1 < p < +\infty$, $q > (N-1)/(p-1)$ if $1 < p < N$ and $q \geq 1$ if $p \geq N$, and Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 1$. In the present paper, we are interested in studying the existence of the p-harmonic solutions for the following Steklov problem:

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = f(x,u) & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying the growth condition

$$|f(x,s)| \leq a(x)|s|^{p-1} + b(x) \tag{1.4}$$

for a.e. $x \in \partial\Omega$ and all $s \in \mathbb{R}$. Here $a \in L^q(\partial\Omega)$ and $b \in L^{p'}(\partial\Omega)$, where p' its the conjugate of p, $q > (N-1)/(p-1)$ if $1 < p < N$ and $q \geq 1$ if $p \geq N$, with $N \geq 2$. We assume that the inequalities

$$\gamma_\pm(x) := \liminf_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} \leq \limsup_{s\to\pm\infty} \frac{f(x,s)}{|s|^{p-2}s} := \Gamma_\pm(x) \tag{1.5}$$

hold uniformly with respect to $x \in \partial\Omega$, where $\gamma_\pm$ and $\Gamma_\pm$ are in M_q with

$$M_q = \left\{ m \in L^q(\partial\Omega), m^+ \not\equiv 0 \text{ and } \int_{\partial\Omega} m d\sigma < 0 \right\}$$

and satisfy

$$\lambda_1(\gamma_+) \leq 1, \;\; \lambda_1(\gamma_-) \leq 1, \;\; c(\Gamma_+, \Gamma_-) \geq 1. \tag{1.6}$$

We also assume that the inequalities

$$\delta_\pm(x) := \liminf_{s\to\pm\infty} \frac{pF(x,s)}{|s|^p} \leq \limsup_{s\to\pm\infty} \frac{pF(x,s)}{|s|^p} := \Delta_\pm(x) \tag{1.7}$$

hold uniformly with respect to $x \in \partial\Omega$, where $\delta_\pm$ and $\Delta_\pm$ are in M_q and satisfy

$$\lambda_1(\delta_+) < 1, \;\; \lambda_1(\delta_-) < 1, \;\; c(\Delta_+, \Delta_-) > 1. \tag{1.8}$$

Here $\lambda_1(m)$ and $c(m,n)$ are respectively the first principal and nonprincipal eigenvalues of the following asymmetric Steklov problem:

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \lambda[m(x)(u^+)^{p-1} - n(x)(u^-)^{p-1}] & \text{on } \partial\Omega, \end{cases} \tag{1.9}$$

with the weights m,$n \in M_q$ (see [1,3]). The problem (1.3) can be a limit situation of the problem (1.1) because we can replace $|u|^{p-2}u$ by $\epsilon|u|^{p-2}u$ in (1.1), where ϵ is small enough. Problem (1.3) appears naturally in several branches of pure and applied mathematics, such as the theory of quasiregular and quasiconformal mappings in Riemannian manifolds with boundary (see [11, 13]), non Newtonian fluids, reaction diffusion problems, flow through porus media, nonlinear elasticity, glaciology, etc. (see [6, 8, 9]).

The paper is organized as follows. In Section 2, which has a preliminary character, we collect some results relative to the asymmetric Steklov problem (1.9). In Section 3 we study, as in [2], the case of nonresonance of the problem (1.3) between the first principal and nonprincipal eigenvalues of the asymmetric Steklov problem. Finally, in Section 4 we study the solvability of problem (1.3) under the first eigenvalue.

2 Preliminaries

Our main purpose in this preliminary section is to collect some results relative to the asymmetric Steklov problem (1.9). For any integer $k \geq 1$, let

$$\Gamma_k := \{K \subset S; K \text{ is symmetric, compact and } \gamma(K) \geq k\}$$

with $S := \left\{u \in W^{1,p}(\Omega); \dfrac{1}{p}\displaystyle\int_{\partial\Omega} m|u|^p d\sigma = 1\right\}$ and $\gamma(K)$ be the Krasnoselski genus of K. Let

$$\lambda_k(m) := \inf_{K \in \Gamma_k} \sup_{u \in K} \frac{1}{p}\int_\Omega |\nabla u|^p dx. \tag{2.1}$$

In [14], Torné proved the following proposition using infinite dimensional Ljusternik–Schnirelman theory.

Proposition 2.1 (See [14]). *Let $m \in M_q$. Then $\lambda_k(m)$ given by* (2.1) *is a sequence of eigenvalues of the problem* (1.9) *with $m = n$ such that $\lambda_k \to +\infty$ as $k \to +\infty$.*

The author established the simplicity and isolation of the first eigenvalue $\lambda_1(m)$ of the Steklov eigenvalue problem (1.9) with $m = n$. The strict monotonicity and the continuity of $\lambda_1(m)$ respect to the weight are proved respectively in [3, 4].

Let us conclude this section with some results concerning $c(m,n)$ the first nonprincipal positive eigenvalue of (1.9). Let m, $n \in M_q$ and let A, $B_{m,n} : W^{1,p}(\Omega) \to \mathbb{R}$, defined by $A(u) = \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p dx$ and $B_{m,n}(u) = \dfrac{1}{p}\displaystyle\int_{\partial\Omega} [m(u^+)^p + n(u^-)^p] d\sigma$. At this point let us introduce the set $M_{m,n} := \{u \in W^{1,p}(\Omega); B_{m,n}(u) = 1\}$. The condition $m^+ \not\equiv 0$ implies that $M_{m,n} \neq \emptyset$. Moreover, the set $M_{m,n}$ is a C^1 manifold in $W^{1,p}(\Omega)$. Let $\tilde{A}$ denote the restriction of A to the manifold $M_{m,n}$. In [1], we showed the following proposition concerning the first nonprincipal positive eigenvalue $c(m,n)$ for (1.9), where

$$c(m,n) = \inf_{\gamma \in \Gamma} \max_{u \in \gamma[0,1]} \tilde{A}(u) \tag{2.2}$$

and

$$\Gamma = \{\gamma \in C([0,1], M_{m,n}) : \gamma(0) = -\varphi_n \text{ and } \gamma(1) = \varphi_m\}$$

with φ_m the normalized positive first eigenvalue of $\lambda_1(m)$.

Proposition 2.2. *Assume m, $n \in M_q$. Then $c(m,n)$ is an eigenvalue of* (1.9) *which satisfies*

$$\max\{\lambda_1(m), \lambda_1(n)\} < c(m,n).$$

Moreover, there is no eigenvalue of (1.9) *between* $\max\{\lambda_1(m), \lambda_1(n)\}$ *and* $c(m,n)$.

The continuity and the monotonicity of the nonprincipal eigenvalue $c(m,n)$ with respect to the weights m and n are proved in [5].

Proposition 2.3 (See [5])**.** *Assume m_k, n_k, m, n, $\hat{m}$, $\hat{n} \in M_q$.*

1. *If $(m_k, n_k) \to (m,n)$ in $L^q(\partial\Omega) \times L^q(\partial\Omega)$, then $c(m_k, n_k) \to c(m,n)$.*
2. *If $m \leq \hat{m}$ and $n \leq \hat{n}$ in $\partial\Omega$, then $c(m,n) \geq c(\hat{m}, \hat{n})$.*

The monotonicity provided by Proposition 2.3 is generally not strict, as in [2, Example 3.1]. The following proposition guarantees, in a certain sense, the strict monotonicity.

Proposition 2.4. *Assume m, n, $\hat{m}$, $\hat{n} \in M_q$. If $m \leq \hat{m}$, $n \leq \hat{n}$ in $\partial\Omega$, and*

$$\int_{\partial\Omega} (\hat{m} - m)(u^+)^p d\sigma + \int_{\partial\Omega} (\hat{n} - n)(u^-)^p d\sigma > 0 \tag{2.3}$$

for at least one eigenfunction u associated to $c(m,n)$, then $c(m,n) > c(\hat{m}, \hat{n})$.

Proof. This is an easy adaptation of proof of [2, Proposition 3.2]. □

The lemma below guarantees that in a mountain pass situation, any minimizing path contains a critical point at the mountain pass level.

Lemma 2.5 (See [7])**.** *Let E be a real Banach space and let $M := \{u \in E; g(u) = 1\}$, where $g \in C^1(E, \mathbb{R})$ and 1 is a regular value of g. Let $f \in C^1(E, \mathbb{R})$. Consider the restriction $\tilde{f}$ of f to M. Let u, $v \in M$ with $u \neq v$ and assume that*

$$H := \{h \in C([0,1], M); h(0) = u \text{ and } h(1) = v\}$$

is nonempty and that

$$c := \inf_{h\in H} \max_{w\in h([0,1])} f(w) > \max\{f(u), f(v)\}.$$

Suppose that $h \in H$ *is such that* $\max\limits_{u\in h([0,1])} \tilde{f}(u) = c$. *Then there exists* $u \in h([0,1])$ *with* $\tilde{f}(u) = c$ *which is a critical point of* $\tilde{f}$.

3 Nonresonance Between the First Two Eigenvalues

In this section, we study the existence of the p-harmonic solutions for the Steklov problem (1.3), under assumptions (1.4), (1.5), (1.6), (1.7) and (1.8). We can apply a version of the mountain pass theorem in a Banach space as given for instance in [10]. The following theorem is the main result in this section.

Theorem 3.1. *Assume* (1.4)–(1.8). *Then the problem* (1.3) *admits at least one solution* u *in* $W^{1,p}(\Omega)$.

Remark 3.2. Let us recall the precise meaning of the fact that the limits in (1.7) are uniform with respect to x: for any $\epsilon > 0$, there exists $a_\epsilon \in L^1(\partial\Omega)$ such that

$$\begin{aligned} &\frac{1}{p}\delta_+(x)|s^+|^p + \frac{1}{p}\delta_-(x)|s^-|^p - \frac{\epsilon}{p}|s|^p - a_\epsilon(x) \\ &\qquad\le F(x,s) \le \frac{1}{p}\Delta_+(x)|s^+|^p + \frac{1}{p}\Delta_-(x)|s^-|^p + \frac{\epsilon}{p}|s|^p + a_\epsilon(x). \end{aligned} \tag{3.1}$$

Note also that one clearly has

$$\gamma_\pm(x) \le \delta_\pm(x) \le \Delta_\pm(x) \le \Gamma_\pm(x) \text{ a.e. in } \partial\Omega. \tag{3.2}$$

We consider now the functional

$$\phi(u) := \frac{1}{p}\int_\Omega |\nabla u|^p dx - \int_{\partial\Omega} F(x,u)d\sigma.$$

Assumption (1.4) implies that ϕ is a C^1 functional on $W^{1,p}(\Omega)$. Its critical points are exactly the solutions of problem (1.3).

Lemma 3.3. *Functional* ϕ *satisfies the* (PS) *condition on* $W^{1,p}(\Omega)$.

Proof. Let u_k be a (PS) sequence, i.e.,

$$|\phi(u_k)| \le c, \tag{3.3}$$

$$|\langle\phi'(u_k), w\rangle| \le \varepsilon_k\|w\| \;\; \forall w \in W^{1,p}(\Omega), \tag{3.4}$$

where c is constant and $\varepsilon_k \to 0$. It suffices to prove that u_k remains bounded in $W^{1,p}(\Omega)$. Assume by contradiction that, for a subsequence, $\|u_k\| \to +\infty$. Put $v_k := u_k/\|u_k\|$. For a further subsequence, $v_k \to v$ weakly in $W^{1,p}(\Omega)$, $v_k \to v$ a.e. in $\partial\Omega$, $v_k \to v$ strongly in $L^p(\Omega)$ and by the Sobolev trace embedding $W^{1,p}(\Omega) \to L^p(\partial\Omega)$, $v_k \to v$ strongly in $L^p(\partial\Omega)$. Using (1.4), we deduce that $f(x,u_k)/\|u_k\|^{p-1}$ remains bounded in $L^{p'}(\partial\Omega)$. Thus $f(x,u_k)/\|u_k\|^{p-1} \to f_0(x)$ weakly in $L^{p'}(\partial\Omega)$. We first take $w = v - v_k$ in (3.4) and divide by $\|u_k\|^{p-1}$ to deduce $\displaystyle\int_\Omega |\nabla v_k|^{p-2}\nabla v_k \nabla(v - v_k)dx \to 0$. Since $v_k \to v$ strongly in $L^p(\Omega)$, we have

$$\int_\Omega |\nabla v_k|^{p-2}\nabla v_k \nabla(v - v_k)dx + \int_\Omega |v_k|^{p-2}(v - v_k)dx \to 0.$$

Thus by the (S^+) type property of the operator $-\Delta_p u + |u|^{p-2}u$ on $W^{1,p}(\Omega)$, we have $v_k \to v$ strongly in $W^{1,p}(\Omega)$. In particular $\|v\| = 1$. One also deduces in a similar manner from (3.4) that

$$\int_\Omega |\nabla v|^{p-2}\nabla v \nabla\varphi dx = \int_{\partial\Omega} f_0(x)\varphi d\sigma \;\; \forall\varphi \in W^{1,p}(\Omega). \tag{3.5}$$

Now, by standard arguments based on assumption (1.6) (cf., e.g., [12]), the function $f_0(x)$ can be written as $\alpha(x)(v^+)^{p-1} - \beta(x)(v^-)^{p-1}$ for some $L^q(\partial\Omega)$ functions α, β satisfying

$$\gamma_+(x) \le \alpha(x) \le \Gamma_+(x), \;\; \gamma_-(x) \le \beta(x) \le \Gamma_-(x) \text{ a.e. in } \partial\Omega. \tag{3.6}$$

Since the values of $\alpha(x)$ (resp. $\beta(x)$) on $\{x \in \partial\Omega; v(x) \le 0\}$ (resp. $\{x \in \partial\Omega; v(x) \ge 0\}$) are irrelevant in the above expression of $f_0(x)$ as $\alpha(x)(v^+)^{p-1} - \beta(x)(v^-)^{p-1}$, we can assume that

$$\alpha(x) = \Delta_+(x) \text{ on } \{x \in \partial\Omega; v(x) \le 0\}, \;\; \beta(x) = \Delta_-(x) \text{ on } \{x \in \partial\Omega; v(x) \ge 0\}. \tag{3.7}$$

We now distinguish three cases: (i) $v \ge 0$ a.e. in $\partial\Omega$, (ii) $v \le 0$ a.e. in $\partial\Omega$ and (iii) v changes sign in $\partial\Omega$. We will see that each case leads to a contradiction.
In case (i), (3.5) implies $\lambda_1(\alpha) = 1$ and $v(x) > 0$ in $\partial\Omega$. Using the monotonicity of $\lambda_1(\cdot)$ with respect to the weight, it follows from (3.6) and (1.6) that $\lambda_1(\gamma_+) = 1$ and also, by the strict monotonicity of $\lambda_1(\cdot)$, we have $\alpha = \gamma_+$ a.e. in $\partial\Omega$. Dividing (3.3) by $\|u_k\|^p$ and going to the limit, using (1.7) and Fatou's lemma, one gets

$$\int_{\partial\Omega} \alpha v^p d\sigma = \int_\Omega |\nabla v|^p dx = \lim_{k\to+\infty} \int_{\partial\Omega} \frac{pF(x,u_k)}{\|u_k\|^p} d\sigma \ge \int_{\partial\Omega} \delta_+ v^p d\sigma.$$

Since $\alpha = \gamma_+ \le \delta_+$ a.e. in $\partial\Omega$ and $v > 0$, we deduce $\alpha = \delta_+$ a.e. in $\partial\Omega$. Consequently $\lambda_1(\delta_+) = 1$, which contradicts (1.8). Case (ii) can be treated similarly. In case (iii), (3.5) shows that v is a solution of the following problem which changes sign

$$\begin{cases} \Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \alpha(u^+)^{p-1} - \beta(u^-)^{p-1} & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

and consequently $c(\alpha, \beta) \leq 1$. By (1.6), (3.6) and using the monotonicity of $c(\cdot, \cdot)$ with respect to the weights,we have $c(\alpha, \beta) = c(\Gamma_+, \Gamma_-) = 1$. Dividing (3.5) by $\|u_k\|^p$ and going to the limit, using (1.7) and Fatou's lemma, one gets

$$\begin{aligned} \int_{\partial\Omega} \big(\alpha(v^+)^p + \beta(v^-)^p\big)\, d\sigma = \int_\Omega |\nabla v|^p dx &= \lim_{k\to+\infty} \int_{\partial\Omega} \frac{pF(x,u_k)}{\|u_k\|^p} d\sigma \\ &\leq \int_{\partial\Omega} \big(\Delta_+(v^+)^p + \Delta_-(v^-)^p\big)\, d\sigma \qquad (3.9) \\ &\leq \int_{\partial\Omega} \big(\Gamma_+(v^+)^p + \Gamma_-(v^-)^p\big)\, d\sigma. \end{aligned}$$

The first integral and the last integral in (3.9) are equal. Indeed, if

$$\int_{\partial\Omega} \big(\alpha(v^+)^p + \beta(v^-)^p\big)\, d\sigma < \int_{\partial\Omega} \big(\Gamma_+(v^+)^p + \Gamma_-(v^-)^p\big)\, d\sigma,$$

then

$$\int_{\partial\Omega} \big((\Gamma_+ - \alpha)(v^+)^p + (\Gamma_- - \beta)(v^-)^p\big)\, d\sigma > 0.$$

Thus, Proposition 2.4 yields that $c(\alpha, \beta) > c(\Gamma_+, \Gamma_-)$. This contradicts the fact that $c(\alpha, \beta) = c(\Gamma_+, \Gamma_-) = 1$. We conclude that all the terms are equal in (3.9) and we deduce, using (2.1), that $\Delta_+ = \Gamma_+$ on $\{x \in \partial\Omega; v(x) > 0\}$, $\Delta_- = \Gamma_-$ on $\{x \in \partial\Omega; v(x) < 0\}$, and using (3.6), that $\alpha = \Gamma_+$ on $\{x \in \partial\Omega; v(x) > 0\}$, $\beta = \Gamma_-$ on $\{x \in \partial\Omega; v(x) < 0\}$. Combining with (3.7), we finally get $\alpha = \Delta_+$ and $\beta = \Delta_-$ a.e. in $\partial\Omega$. Therefore, $c(\Delta_+, \Delta_-) = 1$, which contradicts (1.8). This concludes the proof of Lemma 3.3. □

We now turn to the study of the geometry of ϕ; and first look for directions along which ϕ goes to $-\infty$.

Lemma 3.4. *Let w_+ (resp. w_-) be a positive eigenfunction associated to $\lambda_1(\delta_+)$ (resp. $\lambda_1(\delta_-)$). Then $\phi(Rw_+) \to -\infty$ and $\phi(-Rw_-) \to -\infty$ as $R \to +\infty$.*

Proof. We will prove the assertion relative to $\phi(Rw_+)$, the other one is proved similarly. (3.1) implies, for $R > 0$, that

$$\begin{aligned} \phi(Rw_+) &\leq \frac{R^p}{p} \int_\Omega |\nabla w_+|^p dx - \frac{R^p}{p} \int_{\partial\Omega} (\delta_+ w_+^p - \epsilon w_+^p) d\sigma + \int_{\partial\Omega} a_\epsilon d\sigma \\ &\leq \frac{R^p}{p} \left(1 - \frac{1}{\lambda_1(\delta_+)}\right) \int_\Omega |\nabla w_+|^p dx + \frac{R^p}{p} \epsilon \int_{\partial\Omega} w_+^p d\sigma + \int_{\partial\Omega} a_\epsilon d\sigma. \\ &\leq \frac{R^p}{p} \left(1 - \frac{1}{\lambda_1(\delta_+)} + \varepsilon k\right) \int_\Omega |\nabla w_+|^p dx + \int_{\partial\Omega} a_\epsilon d\sigma \end{aligned}$$

with $k = \dfrac{\int_{\partial\Omega} w_+^p d\sigma}{\int_\Omega |\nabla w_+|^p dx} > 0$. Choosing $\epsilon > 0$ such that $1 - \dfrac{1}{\lambda_1(\delta_+)} + k\epsilon < 0$, which is possible by assumption (1.8), we get that $\phi(Rw_+) \to -\infty$ as $R \to +\infty$. □

Lemma 3.5. *There exists R_0 such that for all $R \geq R_0$ and for all $h \in H_R := \{h \in C([0,1], W^{1,p}(\Omega)); h(0) = Rw_+$ and $h(1) = -Rw_-\}$, we have*

$$\max_{u \in h([0,1])} \phi(u) > \max\{\phi(Rw_+), \phi(-Rw_-)\}. \tag{3.10}$$

Proof. We take a_ϵ according to (3.1) and use Lemma 3.3 to choose $R_0 > 0$ such that

$$-\int_{\partial\Omega} a_\epsilon d\sigma > \max\{\phi(Rw_+), \phi(-Rw_-)\} \tag{3.11}$$

for all $R \geq R_0$. Take such a value R and let $h \in H_R$. To prove (3.10), we distinguish two cases: either (i) $B_{\Delta_+,\Delta_-}(h(t_0)) \leq 0$ for some $t_0 \in [0,1]$, or (ii) $B_{\Delta_+,\Delta_-}(h(t)) > 0$ for all $t \in [0,1]$. We recall here that B_{Δ_+,Δ_-} is the function which defines the manifold M_{Δ_+,Δ_-} (cf. Section 2).

Case (i). We first use (3.1) to obtain

$$\begin{aligned} \phi(u) \geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_{\partial\Omega} \left(\Delta_+(u^+)^p + \Delta_-(u^-)^p\right) d\sigma \\ - \frac{\epsilon}{p}\int_{\partial\Omega} |u|^p d\sigma - \int_{\partial\Omega} a_\epsilon d\sigma. \end{aligned} \tag{3.12}$$

This implies, since we are in case (i),

$$\max_{u \in h([0,1])} \phi(u) \geq \phi(h(t_0)) \geq \frac{1}{p}\int_\Omega |\nabla h(t_0)|^p dx - \frac{\epsilon}{p}\int_{\partial\Omega} |h(t_0)|^p d\sigma - \int_{\partial\Omega} a_\epsilon d\sigma. \tag{3.13}$$

If $\int_{\partial\Omega} |h(t_0)|^p d\sigma = 0$, then $\max_{u \in h([0,1])} \phi(u) \geq \phi(h(t_0)) \geq -\int_{\partial\Omega} a_\epsilon d\sigma.$

If $\int_{\partial\Omega} |h(t_0)|^p d\sigma > 0$ and $\int_\Omega |\nabla h(t_0)|^p dx = 0$, then $h(t_0) = c \neq 0$. The sign of c gives $B_{\Delta_+,\Delta_-}(h(t_0)) = \frac{|c|^p}{p}\int_{\partial\Omega} \Delta_+ d\sigma < 0$ or $B_{\Delta_+,\Delta_-}(h(t_0)) = \frac{|c|^p}{p}\int_{\partial\Omega} \Delta_- d\sigma < 0$. Thus by (3.12), we have

$$\max_{u \in h([0,1])} \phi(u) \geq \phi(h(t_0)) \geq \frac{|c|^p}{p}\left(-\int_{\partial\Omega} \Delta_+ d\sigma - \epsilon|\partial\Omega|\right) - \int_{\partial\Omega} a_\epsilon d\sigma \tag{3.14}$$

or

$$\max_{u \in h([0,1])} \phi(u) \geq \phi(h(t_0)) \geq \frac{|c|^p}{p}\left(-\int_{\partial\Omega} \Delta_- d\sigma - \epsilon|\partial\Omega|\right) - \int_{\partial\Omega} a_\epsilon d\sigma. \tag{3.15}$$

If $\int_{\partial\Omega} |h(t_0)|^p d\sigma > 0$ and $\int_\Omega |\nabla h(t_0)|^p dx > 0$, then by (3.13) we obtain

$$\max_{u \in h([0,1])} \phi(u) \geq \phi(h(t_0)) \geq \frac{1}{p}(1 - \epsilon k)\int_\Omega |\nabla h(t_0)|^p dx - \int_{\partial\Omega} a_\epsilon d\sigma \tag{3.16}$$

with $k = \dfrac{\int_{\partial\Omega} |h(t_0)|^p d\sigma}{\int_\Omega |\nabla h(t_0)|^p dx} > 0$. Now, by the choice of ϵ in (3.14), (3.15) and (3.16), one has

$$\max_{u\in h([0,1])} \phi(u) \geq -\int_{\partial\Omega} a_\epsilon d\sigma > \max\{\phi(Rw_+), \phi(-Rw_-)\},$$

which implies the inequality (3.10) of Lemma 3.5.

Case (ii). In this case we can normalize the path $h(t)$ to get a path

$$\tilde{h}(t) := h(t)/B_{\Delta_+,\Delta_-}(h(t))^{1/p}$$

on the manifold M_{Δ_+,Δ_-} which satisfies, by (2.2) for $m = \Delta_+$ and $n = \Delta_-$,

$$\max_{u\in\tilde{h}([0,1])} \frac{1}{p}\int_\Omega |\nabla u|^p dx \geq c(\Delta_+, \Delta_-). \tag{3.17}$$

We now use (3.1) to get

$$\phi(u) \geq \frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{1}{p}\int_{\partial\Omega} \left(\Delta_+(u^+)^p + \Delta_-(u^-)^p\right) d\sigma - \frac{\epsilon}{p}\int_{\partial\Omega} |u|^p d\sigma - \int_{\partial\Omega} a_\epsilon d\sigma$$

which implies, by (3.17),

$$\max_{u\in h([0,1])} \frac{1}{B_{\Delta_+,\Delta_-}(u)} \left\{\phi(u) + B_{\Delta_+,\Delta_-}(u) + \frac{\epsilon}{p}\int_{\partial\Omega} |u|^p d\sigma + \int_{\partial\Omega} a_\epsilon d\sigma\right\} \geq c(\Delta_+, \Delta_-).$$

Hence there exists $u_\epsilon \in h([0,1])$ such that

$$\phi(u_\epsilon) \geq (c(\Delta_+, \Delta_-) - 1)\, B_{\Delta_+,\Delta_-}(u_\epsilon) - \frac{\epsilon}{p}\int_{\partial\Omega} |u_\epsilon|^p d\sigma - \int_{\partial\Omega} a_\epsilon d\sigma.$$

If there exists $\epsilon_0 > 0$ such that $\displaystyle\int_{\partial\Omega} |u_{\epsilon_0}|^p d\sigma = 0$, then

$$\max_{u\in h([0,1])} \phi(u) \geq \phi(u_{\epsilon_0}) \geq -\int_{\partial\Omega} a_{\epsilon_0} d\sigma.$$

If for all $\epsilon > 0$ $\displaystyle\int_{\partial\Omega} |u_\epsilon|^p d\sigma > 0$, then we have the following claim.

Claim. There exists $\epsilon_0 > 0$ such that

$$(c(\Delta_+, \Delta_-) - 1)\, B_{\Delta_+,\Delta_-}(u_{\epsilon_0}) - \frac{\epsilon_0}{p}\int_{\partial\Omega} |u_{\epsilon_0}|^p d\sigma \geq 0.$$

Proof of the Claim. Suppose, by contradition, that for all $\epsilon > 0$

$$(c(\Delta_+, \Delta_-) - 1)\, B_{\Delta_+,\Delta_-}(u_\epsilon) - \frac{\epsilon}{p}\int_{\partial\Omega} |u_\epsilon|^p d\sigma < 0.$$

Then,

$$\frac{(c(\Delta_+, \Delta_-) - 1)\, B_{\Delta_+,\Delta_-}(u_\epsilon)}{\frac{1}{p}\int_{\partial\Omega} |u_\epsilon|^p d\sigma} < \epsilon \ \text{ for all } \ \epsilon > 0. \tag{3.18}$$

Let u_0 and u_1 be such that

$$B(u_0) = \min_{u \in h([0,1]} \{B(u); B(u) > 0\};$$

$$\int_{\partial\Omega} |u_1|^p d\sigma = \max_{u \in h([0,1]} \left\{ \int_{\partial\Omega} |u|^p d\sigma; \int_{\partial\Omega} |u|^p d\sigma > 0 \right\}.$$

Thus, by (3.18), we have

$$\frac{(c(\Delta_+, \Delta_-) - 1)\, B_{\Delta_+,\Delta_-}(u_0)}{\frac{1}{p}\int_{\partial\Omega} |u_1|^p d\sigma} < \epsilon \ \text{ for all } \ \epsilon > 0. \tag{3.19}$$

This implies that $(c(\Delta_+, \Delta_-) - 1)B_{\Delta_+,\Delta_-}(u_0) = 0$, which contradicts the fact that

$$c(\Delta_+, \Delta_-) > 1 \text{ and } B_{\Delta_+,\Delta_-}(u_0) > 0.$$

Finally, by the claim, one has

$$\max_{u \in h([0,1])} \phi(u) \geq -\int_{\partial\Omega} a_{\epsilon_0} d\sigma > \max\{\phi(Rw_+), \phi(-Rw_-)\},$$

which implies the inequality (3.10) of Lemma 3.5. □

Proof of Theorem 3.1. Now, we can apply a version of the mountain pass theorem in a Banach space as given for instance in [10] to conclude that

$$\inf_{h \in H_R} \max_{u \in h([0,1])} \phi(u)$$

is a critical value of ϕ. Theorem 3.1 is proved. □

4 Nonresonance under the First Eigenvalue

In this section we are interested at nonresonance for Steklov problem (1.1), under the first eigenvalue for the problem (1.9) (with $m = n$). Suppose that f satisfies the condition (1.4) and there exists $m(x) \in M_q$ such that

$$\limsup_{|s| \to +\infty} \frac{pF(x, s)}{|s|^p} := m(x). \tag{4.1}$$

Theorem 4.1. *Assume that* (1.4) *and* (4.1) *hold. Then the problem* (1.3) *admits at least a solution for* $\lambda_1(m) > 1$.

Remark 4.2. We can have $\lambda_1(m) > 1$, since $\lambda_1(m)$ is homogeneous with respect to the weight in the sense $\lambda_1(\alpha m) = \dfrac{\lambda_1(m)}{\alpha}$ for all $\alpha > 0$.

Remark 4.3. The condition (4.1) implies that for all $\varepsilon > 0$, there exists $d_\varepsilon \in L^1(\partial\Omega)$ such that a.e. $x \in \partial\Omega$ and $\forall s \in \mathbb{R}$, we have $F(x, s) \leq (m(x) + \varepsilon)\dfrac{|s|^p}{p} + d_\varepsilon(x)$.

Lemma 4.4. *Assume* (4.1) *holds. Then the functional* ϕ *is coercive for* $\lambda_1(m) > 1$.

Proof. Suppose by contradiction that there exist a sequence $u_n \in W^{1,p}(\Omega)$ and $c \geq 0$ such that $||u_n|| \to +\infty$ and $|\Phi(u_n)| \leq c$. The condition $c \geq |\Phi(u_n)|$ implies that

$$c \geq \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \int_{\partial\Omega} F(x, u_n)d\sigma. \tag{4.2}$$

From Remark 4.3, we have

$$c \geq \frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \frac{1}{p}\int_{\partial\Omega}(m(x) + \varepsilon)|u_n|^p d\sigma - \int_{\partial\Omega} d_\varepsilon(x)d\sigma. \tag{4.3}$$

Let $\varepsilon > 0$ be such that $\lambda_1(m + \varepsilon) > 1$ (the continuity of $m \to \lambda_1(m)$ is used here, see [4, Proposition 3.3]). Thus

$$c \geq \left(1 - \frac{1}{\lambda_1(m + \varepsilon)}\right)\frac{1}{p}\int_\Omega |\nabla u_n|^p dx - \int_{\partial\Omega} d_\varepsilon(x)d\sigma. \tag{4.4}$$

Put $v_n = \dfrac{u_n}{||u_n||}$. Dividing (4.4) by $||u_n||^p$, we obtain

$$\frac{c}{||u_n||^p} \geq \left(1 - \frac{1}{\lambda_1(m + \varepsilon, p)}\right)\frac{1}{p}\int_\Omega |\nabla v_n|^p dx - \frac{1}{||u_n||^p}\int_{\partial\Omega} d_\varepsilon(x)d\sigma. \tag{4.5}$$

Since v_n is a bounded, for a further subsequence still denoted by $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and $v_n \to v$ strongly in $L^p(\Omega)$, on the other hand, we have

$$\int_\Omega |\nabla v|^p dx + \int_\Omega |v|^p dx \leq \liminf_{n\to+\infty}\left(\int_\Omega |\nabla v_n|^p dx + \int_\Omega |v_n|^p dx\right).$$

Passing to the limit in (4.5), we obtain $0 = \displaystyle\int_\Omega |\nabla v|^p dx$. Thus $v = c_1 = const$ and $||v_n||_{1,p} \to ||v||_{1,p}$. Since $W^{1,p}(\Omega)$ is uniformly convex (then reflexive), $v_n \to c_1$ strongly in $W^{1,p}(\Omega)$. Dividing (4.3) by $||u_n||^p$ and passing to the limit, we have $0 \geq -\dfrac{|c_1|^p}{p}\displaystyle\int_{\partial\Omega}(m(x) + \varepsilon)d\sigma$. As ε is arbitrary, we obtain $\dfrac{|c_1|^p}{p}\displaystyle\int_{\partial\Omega} m(x)d\sigma \geq 0$. Since $\displaystyle\int_{\partial\Omega} m(x)d\sigma < 0$, we have $c_1 = 0$ and, consequently, $||v_n|| \to 0$. This contradicts $||v_n|| = 1$. We conclude that ϕ is coercive. □

Lemma 4.5. *Assume that* (1.4) *and* (4.1) *hold. Then the energy functional* ϕ *is weakly lower semicontinuous.*

Proof. It suffices to see that the trace mapping $W^{1,p}(\Omega) \to L^{\frac{pq}{q-1}}(\partial\Omega)$ is compact. □

Proof of Theorem 4.1. From Lemma 4.5, we know that ϕ is weakly lower semicontinuous, while by Lemma 4.4 ϕ is coercive. Thus, ϕ is continuously differentiable and the proof is complete. □

Corollary 4.6. *Suppose that* $m \in M_q$. *If* $\lambda_1(m) > \lambda > 0$. *Then problem* (1.3) *admits at least a solution for* $f(x,u) = \lambda m|u|^{p-2}u + g(x)$ *with* $g \in L^{p'}(\partial\Omega)$.

References

[1] A. Anane, O. Chakrone, B. Karim and A. Zerouali, An asymmetric Steklov problem with weights, Nonlinear Anal. **71** (2009), no. 1-2, 614–621.

[2] A. Anane, O. Chakrone, B. Karim and A. Zerouali, Nonresonance between the first two eigenvalues for a Steklov problem, Nonlinear Anal. **72** (2010), no. 6, 2974–2981.

[3] A. Anane, O. Chakrone, B. Karim and A. Zerouali, Eigenvalues for the Steklov problem, Int. J. Math. Stat. **7** (2010), no. W10, 67–72.

[4] A. Anane, O. Chakrone, B. Karim and A. Zerouali, Existence for an elliptic system with nonlinear boundary conditions, Bol. Soc. Parana. Mat. (3) **28** (2010), no. 2, 49–56.

[5] A. Anane, O. Chakrone, B. Karim and A. Zerouali, Fuc̃ik spectrum for the Steklov problem, Adv. Appl. Math. Sci. **7** (2010), no. 2, 111–118.

[6] D. Arcoya, J. I. Diaz and L. Tello, S-shaped bifurcation branch in a quasilinear multivalued model arising in climatology, J. Differential Equations **150** (1998), no. 1, 215–225.

[7] M. Arias, J. Campos, M. Cuesta and J. P. Gossez, Asymmetric elliptic problems with indefinite weights, Ann. Inst. H. Poincaré Anal. Non Linéaire **19** (2002), no. 5, 581–616.

[8] C. Atkinson and C. R. Champion, On some boundary value problems for the equation $\nabla \cdot (F(|\nabla w|)\nabla w) = 0$, Proc. Roy. Soc. London Ser. A **448** (1995), no. 1933, 269–279.

[9] C. Atkinson and K. El Kalli, Some boundary value problems for the Bingham model, J. Non-Newtonian Fluid Mech. **41** (1992), 339–363.

[10] D. G. de Figueiredo, *Lectures on the Ekeland variational principle with applications and detours*, Tata Institute of Fundamental Research Lectures on Mathematics and Physics, 81, Published for the Tata Institute of Fundamental Research, Bombay, 1989.

[11] J. F. Escobar, Uniqueness theorems on conformal deformation of metrics, Sobolev inequalities, and an eigenvalue estimate, Comm. Pure Appl. Math. **43** (1990), no. 7, 857–883.

[12] J.-P. Gossez and P. Omari, Nonresonance with respect to the Fučik spectrum for periodic solutions of second order ordinary differential equations, Nonlinear Anal. **14** (1990), no. 12, 1079–1104.

[13] P. Tolksdorf, Regularity for a more general class of quasilinear elliptic equations, J. Differential Equations **51** (1984), no. 1, 126–150.

[14] O. Torné, Steklov problem with an indefinite weight for the p-Laplacian, Electron. J. Differential Equations **2005** (2005), no. 87, 8 pp.