Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 8, Number 1, pp. 13–23 (2013)
http://campus.mst.edu/adsa

A General Formulation for Time-Harmonic Maxwell Equations in 3D Cavities and Approximation by a Spectral Method

Mohammed El Rhabi and **Abdelilah Hakim**
University Cadi Ayyad
Faculty of Sciences and Technology
LAMAI, Po box 549, Marrakech, Morocco
elrhabi@gmail.com and abdelilah.hakim@gmail.com

Zouhir Mahani and **Sahar Saoud**
University Ibn Zohr
High School of Technology
MSTI, PO Box 33/S, Agadir, Morocco
z.mahani@uiz.ac.ma and s.saoud@uiz.ac.ma

Abstract

The first purpose of this paper is to investigate time-harmonic Maxwell's equations in Lipschitz and multiply connected cavities of $\mathbb{R}^3$. We prove the wellposedness of the current source problem by means of a new formulation. Our starting point is the curl second order equation satisfied by the magnetic field. The use of an appropriate compact operator is at the heart of the proof. Secondly, we propose a discretization relying on spectral elements and numerical integration. Then we prove the convergence of the discrete solutions to the exact one and we derive error estimates. Examples of numerical solutions are given and compared with those obtained by a finite element method in the case of a simple geometry.

AMS Subject Classifications: 35P10, 47B25, 35Q61.
Keywords: Maxwell's equations, spectral method, numerical approximation.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^3$ with a boundary $\partial\Omega$ and a unit outward normal $\mathbf{n}$. The electromagnetic field in the cavity is described by time harmonic Maxwell's

Received May 22, 2012; Accepted November 15, 2012
Communicated by Agnieszka B. Malinowska

equation:

$$\begin{array}{rcl} \operatorname{curl} \mathbf{E} + i\omega\mu\mathbf{H} & = & \mathbf{0}, \\ \operatorname{curl} \mathbf{H} - i\omega\epsilon\mathbf{E} & = & \mathbf{j}, \\ \operatorname{div}\,(\mu\mathbf{H}) & = & 0, \\ \operatorname{div}\,(\epsilon\mathbf{E}) & = & i\omega^{-1}\operatorname{div}\mathbf{j}, \end{array} \tag{1.1}$$

where $\mathbf{E}$ and $\mathbf{H}$ are respectively the electric and the magnetic intensities, and $\mathbf{j}$ is the imposed source of electric current density.Parameters ϵ and μ refer to the permittivity and the permeability of the medium. For a perfect conducting boundary $\partial\Omega$, the electric fields satisfy the boundary conditions

$$\begin{array}{rcl} \mathbf{E} \times n|_{\partial\Omega} & = & \mathbf{0}, \\ \mu\mathbf{H} \cdot n|_{\partial\Omega} & = & \mathbf{0}. \end{array} \tag{1.2}$$

The system of equations (1.1) and (1.2) involves several theoretical and numerical problems which have been emphasized in the litterature on the subject. When the domain is smooth, the analysis of time-harmonic Maxwell equation has been carried through successfully by means of the Maxwell operator (see, e. g., [9, 14]):

$$\mathcal{A} = \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix}.$$

However, when the domain is nonsmooth, (Ω contains inward edges and corners), the treatment of time-harmonic Maxwell's equations involves some serious complications. This is due mainly to the appearance of singularities near these corners and edges. As far as we know, this fact was first underlined by Weck [18]. Then [5] studies the eigenvalue problem for nonsmooth cavities and clarified the nature of singularities.

The first subject in this work is to treat the current source problem (1.1) and (1.2) in a nonsmooth and multiply connected domains of $\mathbb{R}^3$. We propose a new approach based on the $\operatorname{curl} - \operatorname{curl}$ second order equations satisfied by the magnetic field $\mathbf{H}$. Namely $\mathbf{H}$ is the solution of the system:

$$\begin{array}{rcl} \operatorname{curl}(\epsilon^{-1}\operatorname{curl}\mathbf{u}) - \omega^2\mu\mathbf{u} & = & \operatorname{curl}(\epsilon^{-1}\mathbf{j}), \\ \operatorname{div}\,(\mu\mathbf{u}) & = & 0. \end{array} \tag{1.3}$$

The electric fields is given by $\mathbf{E} = (i\omega\epsilon)^{-1}(\operatorname{curl}\mathbf{H} - \mathbf{j})$.

This paper is organized as follows: in Section 2, we firstly give a geometrical description of the domain and a review of some functional spaces. Then the well posedness of the current source problem (1.3), with appropriated boundary conditions, is given. The regularity of solutions is also discussed. In Section 3, we propose a spectral method for solving (1.3). We provide some convergence result and we derive error estimates. Section 4 presents a conforming Finite Element Method. Finally, Section 5 illustrates this work by some numerical results in a simple 3D geometry.

2 The Continuous Problem

Section 2.1 provides the functional framework.

2.1 Geometrical Description of the Domain

Let Ω be a bounded open set of $\mathbb{R}^3$ and denote by $\partial\Omega$ its boundary. We assume that

1. The domain Ω is Lipschitz-continuous. The boundary $\partial\Omega$ is the union of $p+1$ connected components $\Gamma_0, \dots, \Gamma_p$, where Γ_0 is the boundary of the only unbounded connected component of $\mathbb{R}^3/\Omega$. Note that $p = 0$ when $\partial\Omega$ is connected.

2. Ω is connected but not necessarily simply-connected, we suppose that there exists m smooth surfaces $\Sigma_1, \dots, \Sigma_m$ ("cuts") such that: for any $i \in \{1, \dots, m\}$, Σ_i is an open part of a smooth manifold $\mathcal{M}_i$. For any $i \in \{1, \dots, m\}$, the boundary of Σ_i is contained in $\partial\Omega$. The intersection $\overline{\Sigma_i} \cap \overline{\Sigma_j}$, is empty if $i \neq j$. The open set $\mathring{\Omega} = \Omega / \bigcup_{i=1}^{m} \Sigma_i$ is simply connected and pseudo-Lipschitz [2].

By convention, we set $m = 0$ when Ω is simply-connected.

In the following we denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)$ or in $L^2(\Omega)^3$. For any $s \geq 0$, $H^s(\Omega)$ is the classical Sobolev space defined on Ω and $H_0^s(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$.

In addition, for any $i \in \{1, \dots, p\}$, $H^{\frac{1}{2}}(\Gamma)$ denotes the space of traces on Γ_i of distributions in $H^1(\Omega)$ and $H^{-\frac{1}{2}}(\Gamma_i)$ denotes its dual space. The duality product between $H^{-\frac{1}{2}}(\Gamma_i)$ and $H^{\frac{1}{2}}(\Gamma_i)$ is denoted by $\langle \cdot, \cdot \rangle_{\Gamma_i}$.

Similarly, for $i \in \{1, \dots, m\}$, $H^{\frac{1}{2}}(\Sigma_i)$ is the space of restrictions to Σ_i of the distributions belonging to $H^{\frac{1}{2}}(\mathcal{M}_i)$ and $H^{\frac{1}{2}}(\Sigma_i)'$ is the dual space.

Now, we consider the spaces [12]:

$$\mathbf{H}(\mathrm{div}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 \mid \mathrm{div}\, \mathbf{v} \in L^2(\Omega)\},$$

$$\mathbf{H}(\mathbf{curl}; \Omega) = \{\mathbf{v} \in L^2(\Omega)^3 \mid \mathbf{curl}\, \mathbf{v} \in L^2(\Omega)\}$$

equipped respectively with the norms

$$||\mathbf{v}||_{\mathbf{H}(\mathrm{div};\Omega)} = (||\mathbf{v}||_{0,\Omega}^2 + ||\mathrm{div}\ \mathbf{v}||_{0,\Omega}^2)^{\frac{1}{2}},\ ||\mathbf{v}||_{\mathbf{H}(\mathbf{curl};\Omega)} = (||\mathbf{v}||_{0,\Omega}^2 + ||\mathbf{curl}\, \mathbf{v}||_{0,\Omega}^2)^{\frac{1}{2}}.$$

We consider also the following subspaces of $\mathbf{H}(\mathrm{div}; \Omega)$ and $\mathbf{H}(\mathbf{curl}; \Omega)$:

$$\mathbf{H}_0(\mathrm{div}; \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{div}; \Omega) \mid \mathbf{v}.\mathbf{n} = \mathbf{0}\ on\ \Gamma\},$$

$$\mathbf{H}_0(\mathbf{curl}; \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathbf{curl}; \Omega) \mid \mathbf{v} \times \mathbf{n} = \mathbf{0}\ on\ \Gamma\}.$$

Now, we introduce the spaces

$$Y_T(\Omega) = \mathbf{H}_0(\text{div};\Omega) \cap \mathbf{H}(\mathtt{c}\text{url};\Omega),\ Y_N(\Omega) = \mathbf{H}(\text{div};\Omega) \cap \mathbf{H}_0(\mathtt{c}\text{url};\Omega)$$

equipped with the norm:

$$||\mathbf{v}||_Y = (||\mathbf{v}||^2_{0,\Omega} + ||\text{div }\mathbf{v}||^2_{0,\Omega} + ||\mathtt{c}\text{url }\mathbf{v}||^2_{0,\Omega})^{\frac{1}{2}}, \tag{2.1}$$

and we set

$$G_T = \{\mathbf{v} \in Y_T(\Omega) \mid \text{div }\mathbf{v} = 0,\ \mathtt{c}\text{url }\mathbf{v} = \mathbf{0}\},$$
$$G_N = \{\mathbf{v} \in Y_N(\Omega) \mid \text{div }\mathbf{v} = 0,\ \mathtt{c}\text{url }\mathbf{v} = \mathbf{0}\}.$$

Lemma 2.1. *The space G_T and G_N are finite dimensional such that* $\dim\ G_T = m$, $\dim\ G_N = p$. *Moreover, there exists a basis $(\mathbf{q}_i)_{i=1,\ldots,m}$ (resp. $(\mathbf{f}_i)_{i=1,\ldots,p}$) of G_T (resp. of G_N) such that:*

$$\forall i,j \in \{1,\ldots,m\}\ \ \langle \mathbf{q}_i \cdot \mathbf{n}, 1\rangle_{\Sigma_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{2.2}$$

$$\forall i,j \in \{1,\ldots,p\}\ \ \langle \mathbf{f}_i \cdot \mathbf{n}, 1\rangle_{\Gamma_j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

We denote by $\mathcal{P}_T$ (resp. $\mathcal{P}_N$) the orthogonal projection from $Y_T(\Omega)$ (resp. $Y_N(\Omega)$) on G_T (resp. on G_N) with respect to inner product associated with the norm $||\cdot||_Y$. We note that $\mathcal{P}_N(\mathbf{v}) = \sum_{i=1}^{m} \langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{\Sigma_i}\mathbf{q}_i$, for any $\mathbf{v} \in L^2(\Omega)^3$ such that $\text{div }\mathbf{v} = 0$ (see [2, 10]).

The following lemmas are due to Dominguez [10] when the domain is smooth and to Amrouche *et al.* [2] when it is nonsmooth.

Lemma 2.2. *The mapping*

$$\mathbf{v} \longrightarrow |\mathbf{v}|_{Y_T(\Omega)} = \left(||\text{div }\mathbf{v}||^2_{0,\Omega} + ||\mathtt{c}\text{url }\mathbf{v}||^2_{0,\Omega} + \sum_{i=1}^{m} |\langle \mathbf{v}\cdot\mathbf{n}, 1\rangle_{\Sigma_i}|^2 \right)^{\frac{1}{2}}$$

is a norm on the space $Y_T(\Omega)$, equivalent to the norm $||\cdot||_Y$.

Lemma 2.3. *The mapping*

$$\mathbf{v} \longrightarrow |\mathbf{v}|_{Y_N(\Omega)} = \left(||\text{div }\mathbf{v}||^2_{0,\Omega} + ||\mathtt{c}\text{url }\mathbf{v}||^2_{0,\Omega} + \sum_{i=1}^{m} |\langle \mathbf{v}.\mathbf{n}, 1\rangle_{\Gamma_i}|^2 \right)^{\frac{1}{2}}$$

is a norm on the space $Y_N(\Omega)$, equivalent to the norm $||\cdot||_Y$.

In the sequel, we set

$$\alpha_0 = \inf_{\mathbf{v}\in Y_T(\Omega),\mathbf{v}\neq\mathbf{0}} \frac{|\mathbf{v}|_{Y_T(\Omega)}}{||\mathbf{v}||_{0,\Omega}}. \tag{2.4}$$

Then, according to Lemma 2.2, we have $\alpha_0 > 0$.

2.2 Statement of the Problem: A Weak Formulation

Let us consider the system: *given* $\mathbf{j} \in L^2(\Omega)^3$*, we look for* $\mathbf{u} \in Y_T(\Omega)$ satisfying:

$$\operatorname{curl} \operatorname{curl} \mathbf{u} - k^2 \mathbf{u} = \operatorname{curl} \mathbf{j}, \tag{2.5}$$

$$\operatorname{div} \mathbf{u} = \mathbf{0} \tag{2.6}$$

$$\operatorname{curl} \mathbf{u} \times \mathbf{n}|_{\partial\Omega} = \mathbf{j} \times \mathbf{n}, \tag{2.7}$$

where k is the wave number given by

$$k = \sqrt{\epsilon\mu}\omega \tag{2.8}$$

with ϵ and μ are supposed nonnegative and constants. Observe that the boundary condition (2.7) is meaning full if $\mathbf{j} \in \mathbf{H}(\operatorname{curl}, \Omega)$ (thus $\operatorname{curl} \mathbf{u} \in \mathbf{H}(\operatorname{curl}, \Omega)$). If $\mathbf{j}$ belongs only to $L^2(\Omega)^3$, then we interpret the problem (2.5)–(2.7) in a weaker from; a vector field $\mathbf{u}$ in $Y_T(\Omega)$ is called a *generalized* or a *weak* solution of (2.5)–(2.7) if it satisfies:

$$\begin{aligned}(\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}) + \gamma(\operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v}) + \delta(\mathcal{P}_T(\mathbf{u}), \mathcal{P}_T(\mathbf{v})) - k^2(\mathbf{u}, \mathbf{v})& \\ = (\mathbf{j}, \operatorname{curl} \mathbf{v})\,,\ \forall \mathbf{v} \in Y_T(\Omega),&\end{aligned} \tag{2.9}$$

where δ and γ are two nonnegative real constants.

Proposition 2.4. *Let* $\mathbf{j} \in L^2(\Omega)^3$ *and suppose that* $k > 0$ *and that* γ *and* δ *are such that:* $\gamma > 0$*,* $\delta > 0$ *and*

$$\frac{k^2}{\gamma} \notin EV(\Delta^{neu}), \quad \frac{k^2}{\delta} \neq 1, \tag{2.10}$$

where $EV(\Delta^{neu})$ *is the set of eigenvalues of the Laplace operator with an homogenous Neumann condition. Then*

- *Any solution of* (2.9) *satisfies* (2.5) *and* (2.6) *in the sense of distributions.*
- *If* $\mathbf{j}$ *belongs to* $\mathbf{H}(\operatorname{curl}; \Omega)$*, then problems* (2.9) *and* (2.5)–(2.7) *are equivalent.*

2.3 Well-Posedness of the Problem: The Case of Low Frequencies

When the wave number k is smaller than the parameter α_0 defined by (2.4), existence and uniqueness of solutions of (2.9) stem immediately from the Lax–Milgram theorem and Lemma 2.2.

Proposition 2.5. *Assume that* $\gamma \geq 1$*,* $\delta \geq 1$ *and* $k < \alpha_0$*. Then, the problem* (2.9) *admits one and only one solution* $\mathbf{u} \in Y_T(\Omega)$*. Furthermore, we have*

$$|\mathbf{u}|_{Y_T(\Omega)} \leq c\frac{\alpha_0^2}{\alpha_0^2 - k^2}||\mathbf{j}||_{0,\Omega}, \tag{2.11}$$

where c *is a constant depending only on* Ω*.*

Remark 2.6. Note that if $\operatorname{curl} \mathbf{j} \in L^2(\Omega)^3$ and $\mathbf{j} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$, then the estimate (2.11) can be replaced by the following $|\mathbf{u}|_{Y_T(\Omega)} \leq c\frac{\alpha_0^2}{\alpha_0^2 - k^2}||\operatorname{curl} \mathbf{j}||_{0,\Omega}$.

2.4 Well-Posedness of the Problem: The General Case

The aim of this section is to treat the problem (2.9) when k is not necessarily small. We state the following theorem.

Theorem 2.7. *Assume that* $\mathbf{j} \in L^2(\Omega)^3$ *and that* (2.10) *is fulfilled. Then, there exists a countable sequence of real values* $\{\alpha_i, i \in \mathbb{N}\}$*, tending to* $+\infty$ *such that*

1. *If* $k \notin \{\alpha_i, i \in \mathbb{N}\}$*, then the problem* (2.10) *admits one and only one solution* $\mathbf{u} \in Y_T(\Omega)$.
2. *If* $k = \alpha_m$ *for some* $m \in \mathbb{N}$*, then the homogeneous problem (when* $\mathbf{j} = \mathbf{0}$*) admits a finite dimensional space* E_m *of solutions, and the problem* (2.9) *is solvable in* $Y_T(\Omega)$ *if, and only if,* $\mathbf{j}$ *satisfies* $(\mathbf{j}, \mathbf{curl}\,\phi) = \mathbf{0}$ *for any* ϕ *in* E_m*. If this condition is fulfilled, then the solution of* (2.9) *is unique up to elements of* E_m.

2.5 Regularity of Solutions

The purpose of this section is to show some regularity properties of the solution when the domain has a smooth boundary and when it is a parallelepiped (as involved by pseudo-spectral and spectral methods). Note that the general case of a polygonal domain contains some technical complications, due to the appearance of the singularities, and which are beyond the scope of this paper (that the inclusion $Y_T(\Omega) \subset H^1(\Omega)^3$ does not hold in general case, see, e.g., [8]).

Proposition 2.8. *Assume that* Ω *is of class* $\mathcal{C}^{m,1}$ *with* $m \geq 2$ *and let* $\mathbf{j} \in L^2(\Omega)^3$ *such that* $\mathbf{curl}\,\mathbf{j} \in H^{m-2}(\Omega)^3, \mathbf{j} \times \mathbf{n} \in H^{m-3/2}(\partial\Omega)^3$*. Then, the solution* $\mathbf{u}$ *of* (2.9) *belongs to* $H^m(\Omega)^3$.

When the domain is a parallelepiped, we have the following.

Proposition 2.9. *Assume that* Ω *is a rectangular parallelepiped of* $\mathbb{R}^3$*, namely* $\Omega =]0, a_1[\times]0, a_2[\times]0, a_3[$*. Suppose that* $\mathbf{j} \in \mathbf{H}(\mathbf{curl}; \Omega)$ *and satisfies* $\mathbf{j} \times \mathbf{n} = \mathbf{0}$ on $\partial\Omega$*. Then, the solution of the problem* (2.9) *belongs to* $H^2(\Omega)^3$*. Moreover, if* $k < \alpha_0$*, then*

$$||\mathbf{u}||_{H^2(\Omega)} \leq \frac{c(\Omega)}{\alpha_0^2 - k^2} ||\mathbf{curl}\,\mathbf{j}||_{0,\Omega}.$$

Remark 2.10. Note that in both propositions, $\mathbf{u}$ is also solution of the classical problem (2.5)–(2.7).

3 Approximation by a Spectral Method

The aim of this section is to solve time harmonic Maxwell's equation using spectral method. Even if a number finite element discretization exists for this type of problem, with and without domain decomposition (see, e.g., [1, 13] and the references therein, it seems that only a few works deal with the spectral element method (see [3,7]). Here, we suppose that Ω is a polyhedral domain of $\mathbb{R}^3$, which is not necessarily convex. More precisely, we assume that there exists d open rectangular parallelepipeds $\Omega_1, \dots, \Omega_d$ such that the decomposition is conform. For simplicity, we suppose also that Ω is simply-connected.

Now, let $N \geq 1$ be an integer. For any open set $\mathcal{O}$ in $\mathbb{R}^3$, we denote by $P_N(\mathcal{O})$ the space of polynomial functions of degree less than or equal to N with respect to each variable. We recall the Gauss–Lobatto formula in $[-1,1]$: $\forall p \in P_{2N-1}([-1,1])$, $\displaystyle\int_{-1}^{1} p(x)dx = \sum_{i=0}^{m} \rho_i p(\xi_i)$, where $\xi_0 = -1 < \xi_1 < \cdots < \xi_N = 1$ are the zeros of $(1-x^2)L'_N(x)$ (L_N is the Legendre polynomial of degree N) and $\rho_0, \dots, \rho_N$ are positive real weights. For any $m \in \{1, \dots, d\}$, we denote by $\Xi_N^{(m)} = \big\{\xi_\alpha^{(m)} = (\xi_i^{(m)}, \xi_j^{(m)}, \xi_l^{(m)});$ $\alpha = (i,j,l)\text{with}|\alpha| \leq N\big\}$, $\Xi_N = \displaystyle\bigcup_{m=1}^{1} \Xi_N^{(m)}$. The grid of Ω_m obtained by translation and homothety in each direction of the nodes $\xi_0, \xi_1, \dots, \xi_N$ and by $\rho_\alpha, |\alpha| \leq N$, the correponding weights (note that $\Xi_N^{(m)} \cap \Xi_N^{(j)}$ is a grid of $\Omega_m \cap \Omega_j$ when the latter is not empty). Now, $(u,v)_N = \displaystyle\sum_{m=1}^{d} \sum_{|\alpha| \leq N} \rho_\alpha^{(m)} u(\xi_\alpha^{(m)}) v(\xi_\alpha^{(m)})$ define the discrete product on the space of continuous functions in $\bar{\Omega}$. Thus, if u and v are such that $uv_{|\Omega_m} \in P_{2N-1(\Omega_m)}$, $m = 1, 2, \dots, d$, then $(u,v)_N = \displaystyle\int_\Omega u(x)v(x)dx$. Similarly, $(\mathbf{u}, \mathbf{v})_N = \displaystyle\sum_{i=1}^{3} (u_i, v_i)_N$ defines the scalar product for any continuous vector functions $\mathbf{u}$ and $\mathbf{v}$. Moreover, for any continuous ϕ_N such that $\phi_{N|\Omega_m} \in P_N(\Omega_m), m = 1, \dots, d$, we have [4]

$$||\phi_N||_{0,\Omega}^2 \leq (\phi_N, \phi_N)_N \leq 27||\phi_N||_{0,\Omega}^2. \tag{3.1}$$

Now, we introduce the interpolation operator $\mathcal{I}_N$ on the grid Ξ_N defined as follows: for any $m \in \{1, \dots, d\}, \mathcal{I}_N f|_{\Omega_m} \in P_N(\Omega_m)$ and $\forall \xi \in \Xi_N, \mathcal{I}_N f(\xi) = f(\xi)$.

The discrete space is: $Y_T^N(\Omega) = \{\mathbf{u} \in Y_T(\Omega); u|_{\Omega_m} \in P_N(\Omega_m)^3, \forall m = 1, \dots, d\}$. We set

$$\alpha_{0,N} = \inf_{\phi \in Y_T^N(\Omega), \phi \neq 0} \frac{\{(\mathbf{curl}\, \phi, \mathbf{curl}\, \phi)_N + (\operatorname{div} \phi, \operatorname{div} \phi)_N\}^{\frac{1}{2}}}{(\phi, \phi)_N^{\frac{1}{2}}}. \tag{3.2}$$

The following lemma is an immediate consequence of inequality (3.1) and Lemma 2.2.

Lemma 3.1. *There exists a constant $c_0 > 0$, not depending on N, such that*

$$c_0 \leq \alpha_{0,N} \leq \frac{1}{c_0}$$

for all N.

We suppose now that the vector function $\mathbf{j}$ is continuous in $\bar{\Omega}$ and we consider the discrete problem

$$a_N(\mathbf{u}_N, \mathbf{v}_N) = l_n(\mathbf{v}_N), \forall \mathbf{v}_N \in Y_T^N(\Omega), \tag{3.3}$$

where $a_N(\cdot,\cdot)$ and $l_N(\cdot)$ are the discrete version of $a(\cdot,\cdot)$ and $l(\cdot)$ (the continuous scalar product is replaced by the discrete one).

Theorem 3.2. *Assume that $k < \alpha_{0,N}$. Then, the discrete problem* (3.3) *admits one and only one solution. Furthermore, if the solution $\mathbf{u}$ of the continuous problem* (2.5)–(2.7) *belongs to $H^s(\Omega)^3$ with $s \geq 1$ if and only if $\mathbf{j} \in H^r(\Omega)^3$ with $r > 3/2$, then*

$$|\mathbf{u} - \mathbf{u}_N|_{Y_T(\Omega)} \leq c(\Omega) \left(N^{1-s} ||\mathbf{u}||_{H^s(\Omega)} + N^{-r} ||\mathbf{j}||_{H^r(\Omega)} \right). \tag{3.4}$$

Moreover, if Ω is a rectangular parallelepiped and if in addition $k < \alpha_0$, then

$$||\mathbf{u} - \mathbf{u}_N||_{0,\Omega} \leq \frac{c(\Omega)}{(\alpha_0^2 - k^2)(\alpha_{0,N}^2 - k^2)} \left(N^{-s} ||\mathbf{u}||_{H^s(\Omega)} + N^{-r} ||\mathbf{j}||_{H^r(\Omega)} \right). \tag{3.5}$$

Remark 3.3. Existence and uniqueness of the discrete solution is in fact guaranteed even though $k > \alpha_{0,N}$ provided that k is not an eigenvalue of the finite-dimensional problem (3.3).

Now, let us describe briefly the implementation of the method. Suppose that $\mathbf{j}$ is given at the nodes of the grid Ξ_N. For any $\xi \in \Xi_N^{(m)} (m = 1, \ldots, d)$, we denote by $l_\xi^{(m)}$ the Lagrange polynomials associated to the node ξ in the Ω_m, thus the discrete solution can be decompressed into the form: $\mathbf{u}_{N_{|\Omega_m}} = \sum_{i=1}^{3} \sum_{\xi \in \Xi_N^{(m)}} \mathbf{u}_{i,\xi} l_\xi^{(m)} e_i$, where the coefficients $\mathbf{u}_{i,\xi}$ are unknown real numbers. The boundary condition $\mathbf{u}_N \cdot \mathbf{n} = 0$ on the boundary implies $\mathbf{u}_{i,\xi} = 0$ if $\xi \in \partial\Omega$ and $|\mathbf{e}_i \cdot \mathbf{n}| = 1$.

4 A Finite Element Method

The approximation of time harmonic maxwell equations by finite element methods has been widely studied in the literature (see, e.g., [6, 11, 15–17]). Here we describe briefly a direct method based on nonconvergence free elements [6, 11].

Let $\mathcal{T}_h = \cup \mathcal{K}$ be a triangulation of Ω satisfying the following standard regularity assumptions (conform decomposition). We define $Y_T^h(\Omega)$ by: $Y_T^h(\Omega) = \{\mathbf{v} \in C^0(\bar{\Omega})^3; \mathbf{v}|_{\mathcal{K}}$ is affine, $\forall \mathcal{K}; \mathbf{v}(M_j) \cdot \mathbf{n} = 0, \forall M_j \in \Gamma\}$. A basis of $Y_Y^h(\Omega)$ may be easily constructed using the basis ω_j of P^1 elements: ω_j is a continuous piecewise affine function on $\mathcal{T}_h$ which verifies: $\omega_j(M_m) = \delta_{m,j}$. If we assume that the first I vertices are internals, then a basis $\{\omega_j\}$ of $Y_Y^h(\Omega)$ is given by:

$$\mathbf{w}_{3(m-1)+i} = \mathbf{w}_m \mathbf{e}_i, \ 1 \leq m \leq I, \ i = 1, 2, 3, \tag{4.1}$$

$$\mathbf{w}_{3I+2(m-I-1)} = \mathbf{w}_m \mathcal{T}_i,\ m > I,\ i = 1, 2, \tag{4.2}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an orthonormal basis of $\mathbb{R}^3$ and $(\mathcal{T}_1, \mathcal{T}_2)$ an orthogonal set of tangent vectors. We set $\alpha_{0,h} = \inf\limits_{\mathbf{v}_h \in Y_T^h(\omega), \mathbf{v}_h \neq 0} \dfrac{(||\mathbf{curl} \mathbf{v}_h||_{0,\Omega} + ||\text{div } \mathbf{v}_h||_{0,\Omega})^{\frac{1}{2}}}{||\mathbf{v}_h||_{0,\Omega}^{\frac{1}{2}}}$. It is quite obvious that $\alpha_{0,h} \geq \alpha_0$ thanks to the inclusion $Y_T^h(\omega) \subset Y_T(\Omega)$. We consider also the discrete problem: find $\mathbf{u}_h \in Y_T^h(\Omega)$ solution of

$$a(\mathbf{u}_h, \mathbf{v}_h) = (\mathbf{j}, \mathbf{curl}\ \mathbf{v}_h), \forall \mathbf{v}_h \in Y_T^h. \tag{4.3}$$

We have the following (see, e.g., [6, 11]).

Proposition 4.1. *Suppose that Ω is convex and that $0 \leq k \leq \alpha_{0,h}$, the discrete problem* (4.3) *admits one and only one solution $\mathbf{u}_h$. If, in addition, $k < \alpha_0$ and the solution $\mathbf{u}$ of the continuous problem* (2.9) *belong to $H^2(\Omega)^3$, then there exists two constants C_1 and C_2, depending only on Ω, such that*

$$||\mathbf{u} - \mathbf{u}_h||_{Y_T(\Omega)} \leq \frac{C_1}{\alpha_0^2 - k^2} h ||\mathbf{u}||_{2,\Omega}, \tag{4.4}$$

$$||\mathbf{u} - \mathbf{u}_h||_{0,\Omega} \leq \frac{C_2}{\alpha_0^2 - k^2} h ||\mathbf{u}||_{2,\Omega}. \tag{4.5}$$

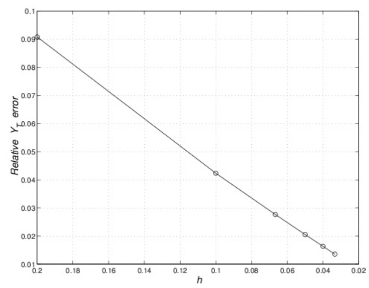
5 Implementation and Numerical Results

In this part, we have to show and compare some numerical results obtained by the Spectral Method (SM) and the Finite Element Method (FEM) exposed above. All the tests are done in the box $\Omega =]-1, 1[^3$. We note that in the both of methods, we are lead to solve a square linear system of the form

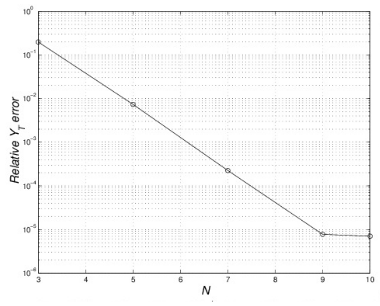
$$AU = F, \tag{5.1}$$

where the matrix A is very large since its size is of order $3N^3$, where N is the number of nodes or vertices. A classical iterative algorithm (the Conjugate Gradient Algorithm) is used for solving (5.1). Is well known that such an algorithm avoids the storage of the matrix since it requires only the matrix-vector product. We use an analytical example in the tests.

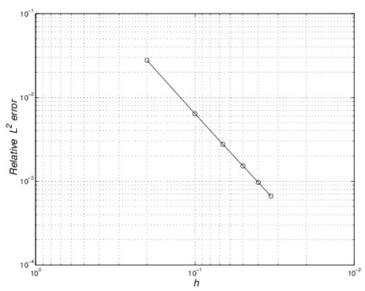
Figure 5.1 shows the behavior of the relative $Y_T(\Omega)$ and $L^2(\Omega)^3$ errors versus the discretization parameters h (FEM) and N (SM). The behavior of these errors confirms the error estimates above. Note that the SM errors are quite smaller than FEM ones when the number of the nodes and vertices is the same.



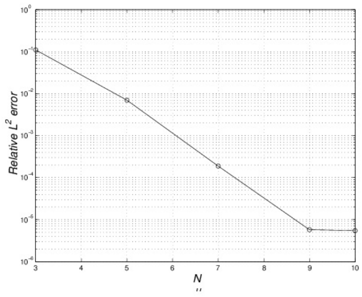
(a) (FEM): The relative error $Y_T(\Omega)$ versus h



(b) (SM): The relative error $Y_T(\Omega)$ versus N (left) and N (right) ($k = 0.3\pi$)



(c) (FEM): The relative error $L^2(\Omega)^3$ versus h



(d) (SM): The relative error $L^2(\Omega)^3$ versus N

Figure 5.1: The relative error versus h (left) and N (right).

References

[1] A. Ben Abdallah, F. Ben Belgacem, Y. Maday and F. Rapetti, Mortaring the two-dimensional edge finite elements for the discretization of some electromagnetic models, Math. Models Methods Appl. Sci. **14** (2004), no. 11, 1635–1656.

[2] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains, Math. Methods Appl. Sci. **21** (1998), no. 9, 823–864.

[3] M. Azaïez, F. Ben Belgacem, C. Bernardi and M. El Rhabi, The mortar spectral element method in domains of operators. II. The curl operator and the vector potential problem, IMA J. Numer. Anal. **28** (2008), no. 1, 106–120.

[4] C. Bernardi and Y. Maday, Spectral methods, in *Handbook of numerical analysis, Vol. V*, 209–485, Handb. Numer. Anal., V North-Holland, Amsterdam, 1997.

[5] M. Sh. Birman and M. Z. Solomyak, On main singularities of the electric component of the electro-magnetic field in regions with screens, St. Petersburg Math. J. **5** (1994), no. 1, 125–139.

[6] T. Z. Boulmezaoud and T. Amari, Approximation of linear force-free fields in bounded 3-D domains, Math. Comput. Modelling **31** (2000), no. 2-3, 109–129.

[7] T. Z. Boulmezaoud and M. El Rhabi, A mortar spectral element method for 3D Maxwell's equations, IMA J. Numer. Anal. **25** (2005), no. 3, 577–610.

[8] M. Costabel and M. Dauge, Singularities of electromagnetic fields in polyhedral domains, Arch. Ration. Mech. Anal. **151** (2000), no. 3, 221–276.

[9] R. Dautray and J. L. Lions, Analyse mathématique et calcul numérique, Vol. 5, Masson, Paris, 1988.

[10] J. M. Dominguez, Etude des équations de la Magnétohydrodynamique stationnaires et leur approximation par él'ements finis, PhD thesis, Pierre and Marie Curie University, 1982.

[11] G. J. Fix and M. E. Rose, A comparative study of finite element and finite difference methods for Cauchy-Riemann type equations, SIAM J. Numer. Anal. **22** (1985), no. 2, 250–261.

[12] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer Series in Computational Mathematics, 5, Springer, Berlin, 1986.

[13] Q. Hu and J. Zou, Substructuring preconditioners for saddle-point problems arising from Maxwell's equations in three dimensions, Math. Comp. **73** (2004), no. 245, 35–61.

[14] R. Leis, *Initial-boundary value problems in mathematical physics*, Teubner, Stuttgart, 1986.

[15] J.-C. Nédélec, Mixed finite elements in $\mathbf{R}^3$, Numer. Math. **35** (1980), no. 3, 315–341.

[16] J.-C. Nédélec, A new family of mixed finite elements in $\mathbf{R}^3$, Numer. Math. **50** (1986), no. 1, 57–81.

[17] R. Verfürth, Mixed finite element approximation of the vector potential, Numer. Math. **50** (1987), no. 6, 685–695.

[18] N. Weck, Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries, J. Math. Anal. Appl. **46** (1974), 410–437.