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Two-Parameter Eigenvalues Steklov Problem involving the p-Laplacian

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Abstract

We study the existence of eigenvalues for a two parameter Steklov eigenvalues problem with weights. Moreover, we prove the simplicity and the isolation results of the principal eigenvalue. Finally, we obtain the continuity and the differentiability of this principal eigenvalue.

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1 Introduction

Consider the two parameter Steklov eigenvalues problem

$$\begin{cases} \Delta_p u = \lambda m_1(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \mu m_2(x)|u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where Ω is a bounded domain in $\mathbb{R}^N (N \geq 2)$ with smooth boundary $\partial\Omega$, ν is the unit outward normal to $\partial\Omega$, $\lambda \in \mathbb{R}^+$ is a parameter, μ is a number and the operator $\Delta_p := \text{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian with $1 < p < \infty$. The weight function m_1 satisfies the following assumption:

$$m_1 \in L^\infty(\Omega) \text{ and } m_1(x) \geq const > 0. \tag{1.2}$$

We also assume that the weight function m_2 is indefinite, which satisfies the following assumption:

$$m_2 \in L^q(\partial\Omega) \text{ and } m_2^+ \not\equiv 0 \text{ on } \partial\Omega, \tag{1.3}$$

such that $q > (N-1)/(p-1)$ if $1 < p \leq N$ and $q \geq 1$ if $p > N$.

The growing attention in the study of the p-Laplacian operator is motivated by the fact that it arises in various applications, for example, non Newtonian fluids, reaction diffusion problems, flow through porus media, glacial sliding, theory of superconductors, biology, and so forth — see [4, 9] and the references therein. The case $\lambda = 0$ was considered by Torné in [8]: he showed, using the infinite dimensional Ljusternik–Schnirelman theory [7], that problem (1.1) admits a sequence of eigenvalues and he investigated some nodal properties of eigenfunctions associated to the first and second eigenvalues. Amongst other results he proved that if $\max(m_2, 0) \neq 0$ and $\displaystyle\int_{\partial\Omega} m_2 d\sigma < 0$, then the first positive eigenvalue is the only eigenvalue associated to an eigenfunction of definite sign and any eigenfunction associated to the second positive eigenvalue has exactly two nodal domains. In this case we have shown in [1] the existence of another nondecreasing unbounded sequence of positive eigenvalue of the problem (1.1) by using a deformation lemma. We also established some properties for the first and the second positive eigenvalues. Bonder and Rossi studied the case $\lambda = 1$ and $m_1 \equiv 1$: they proved that there exists a sequence of variational eigenvalues and that the first eigenvalue is isolated, simple and monotone with respect to the weight [5].

The plan of this paper is the following. In Section 2 we use a variational method to prove the existence of a sequence of eigenvalues for the problem (1.1). In Section 3, we establish the simplicity and the isolation results of the principal eigenvalue. Finally, in Section 4 we establish the continuity of the eigenpair $(\mu_1(\lambda), u(\lambda))$ in λ and the differentiability of the principal eigenvalue $\mu_1(\lambda)$.

2 Existence of Eigenvalues

A sequence of eigenvalues of problem (1.1) can be obtained as follows: the space $W^{1,p}(\Omega)$ will be endowed with the usual norm

$$\|u\| := \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |u|^p dx\right)^{1/p}$$

and the weak solutions of (1.1) are defined by

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx + \lambda \int_\Omega m_1(x)|u|^{p-2}u\varphi dx = \mu \int_{\partial\Omega} m_2(x)|u|^{p-2}u\varphi d\sigma \qquad (2.1)$$

for all $\varphi \in W^{1,p}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure. We can introduce the equivalent norm

$$\|u\|_\lambda := \left(\int_\Omega |\nabla u|^p dx + \lambda \int_\Omega m_1|u|^p dx\right)^{1/p}$$

and consider the even functional

$$\phi_\lambda(u) = \frac{1}{p}\|u\|_\lambda^p \quad \forall u \in \Sigma,$$

where

$$\Sigma := \left\{u \in W^{1,p}(\Omega); \frac{1}{p}\int_{\partial\Omega} m_2|u|^p d\sigma = 1\right\}.$$

For any $k \in \mathbb{N}^\star$ let

$$\mathcal{F}_k := \left\{\mathcal{A} \subset \Sigma;\ \text{there exists a continuous odd surjection } h : S^{k-1} \to \mathcal{A}\right\},$$

where S^{k-1} represents the unit sphere in $\mathbb{R}^k$. Next we define

$$\mu_k(\lambda) := \inf_{\mathcal{A}\in\mathcal{F}_k} \max_{u\in\mathcal{A}} \phi_\lambda(u). \qquad (2.2)$$

The set $\mathcal{F}_k$ is nonempty (see [1]), thus $\mu_k(\lambda)$ is well defined.

The following theorem studies the particular case $\lambda = 0$, where the eigenfunctions of problem (1.1) are p-harmonics. This theorem was proved in [1].

Theorem 2.1 (See [1])**.** *In the case $\lambda = 0$, we have the following assertions.*

1. *Assume that* (1.3) *holds. Then $\mu_k(0)$ given by* (2.1) *is a nondecreasing and unbounded sequence of positive eigenvalues.*

2. *Assume that* (1.3) *holds and $\displaystyle\int_{\partial\Omega} m_2\, d\sigma < 0$. Then $\mu_1(0) > 0$ is the first positive eigenvalue of problem* (1.1)*. Moreover, $\mu_1(0)$ is simple and isolated and it is the only nonzero eigenvalue associated to an eigenfunction of definite sign. Also, $\mu_2(0)$ is the second positive eigenvalue of problem* (1.1) *and any eigenfunction associated to $\mu_2(0)$ has exactly two nodal domains.*

Now we study the case $\lambda > 0$, where the eigenfunctions of the problem are no longer p-harmonic. The proof is an adaptation of the variational method of [1], which is based on the deformation lemma. The following theorem is the main result of this section.

Theorem 2.2. *Let m_1 and m_2 be two weight functions satisfying assumptions* (1.2) *and* (1.3)*, respectively. If $\lambda > 0$, then $\mu_k(\lambda)$ given by* (2.2) *is a nondecreasing and unbounded sequence of positive eigenvalues.*

Lemma 2.3. *Functional ϕ_λ satisfies the Palais–Smale condition on Σ.*

Proof. Let $(u_k) \subset \Sigma$ and $c > 0$ such that $|\phi_\lambda(u_k)| \leq c$ and $\phi'_\lambda(u_k) = A'_\lambda(u_k) - A_\lambda(u_k)B'(u_k) \to 0$ in $(W^{1,p}(\Omega))'$, where $A_\lambda(u) = \frac{1}{p}\|u\|^p_\lambda$ and $B(u) = \frac{1}{p}\int_{\partial\Omega} m_2|u|^p$. The sequence (u_k) is bounded since ϕ_λ is coercive for a subsequence $u_k \to u$ weakly in $W^{1,p}(\Omega)$. It remains to show that $u_k \to u$ strongly in $W^{1,p}(\Omega)$. Using the compactness property of the trace mapping, it follows that $A'_\lambda(u_k) \to cB'(u)$ strongly in $(W^{1,p}(\Omega))'$, where $c = \lim\limits_{k\to+\infty} A_\lambda(u_k)$. As $A'_\lambda : W^{1,p}(\Omega) \to (W^{1,p}(\Omega))'$ is an homeomorphism, then $u_k \to u = (A'_\lambda)^{-1}[cB'(u)]$ strongly in $W^{1,p}(\Omega)$. □

Since ϕ_λ is of class $C^1(\Omega)$ and the Palais–Smale condition has been verified, we can apply a deformation lemma which plays a fundamental role in proving that ϕ_λ has a critical value. Observe that, since ϕ_λ is even on Σ, the deformation preserves symmetries (see [6, p. 79]).

Lemma 2.4 (Deformation lemma)**.** *Let $\beta \in \mathbb{R}$ be a regular value of ϕ_λ on Σ and let $\bar{\epsilon} > 0$. Then there exists $\epsilon \in (0, \bar{\epsilon})$ and a continuous one-parameter family of homeomorphisms, $\psi : S \times [0,1] \to S$, with the following properties:*

1. *$\psi(u,t) = u$, if $t = 0$ or if $|\phi_\lambda(u) - \beta| \geq \bar{\epsilon}$ for all $u \in \Sigma$;*
2. *$\phi_\lambda(\psi(u,t))$ is nonincreasing in t for any $u \in S$;*
3. *if $\phi_\lambda(u) \leq \beta + \epsilon$ for all $u \in \Sigma$, then $\phi_\lambda(\psi(u,1)) \leq \beta - \epsilon$;*
4. *$\psi(-u,t) = -\psi(u,t)$ for any $t \geq 0$ and any $u \in \Sigma$.*

Proof of Theorem 2.2. Let $\lambda > 0$ and suppose, by contradiction, that $\mu_k(\lambda)$ is a regular value. Using $\bar{\epsilon} = 1$ and $\beta = \mu_k(\lambda)$ (in Lemma 2.4), let $\epsilon \in (0,1)$ and ψ_λ be the objects guaranteed by the deformation lemma above. By definition, there is an $\mathcal{A} \in \mathcal{F}_k$ such that $\sup\limits_{u\in\mathcal{A}} \phi_\lambda(u) \leq \mu_k(\lambda) + \epsilon$. But if $h : S^{k-1} \to \mathcal{A}$ is a continuous odd surjection, then so is $\psi(h(\cdot),1) : S^{k-1} \to \psi(\mathcal{A},1)$. Thus, $\psi(\mathcal{A},1) \in \mathcal{F}_k$ such that $\sup\limits_{u\in\psi(\mathcal{A},1)} \phi_\lambda(u) \leq \mu_k(\lambda) - \epsilon$, which contradicts the definition of $\mu_k(\lambda)$.

Now we prove that $\mu_k(\lambda)$ is nondecreasing. Let $\epsilon > 0$, there exist $\mathcal{A} \in \mathcal{F}_{k+1}$ such that $\mu_{k+1}(\lambda) + \epsilon \geq \max\limits_{u\in\mathcal{A}} \phi_\lambda(u)$ and there exist $h : S^k \to \mathcal{A}$ a continuous odd surjection. Put $\mathcal{A}' = h\left(S^{k-1} \times \{0\}\right)$, we have $\mathcal{A}' \subset \mathcal{A}$ and $\mathcal{A}' \in \mathcal{F}_k$; indeed $\overline{h} : S^{k-1} \to \mathcal{A}'$ such that $\overline{h} = h \circ b$ where $b : S^{k-1} \to S^{k-1} \times \{0\} : (x_1, x_2, \ldots, x_k) \to (x_1, x_2, \ldots, x_k, 0)$. Thus $\mu_k(\lambda) \leq \max\limits_{u\in\mathcal{A}'} \phi_\lambda(u) \leq \max\limits_{u\in\mathcal{A}} \phi_\lambda(u) \leq \mu_{k+1}(\lambda) + \epsilon$.

It remains to prove that the sequence $\mu_k(\lambda)$ is unbounded. For all k and $\lambda \geq 0$ we have $\mu_k(\lambda) \geq \mu_k(0)$. Under Theorem 2.1, the sequence $\mu_k(0)$ is unbounded, thus the sequence $\mu_k(\lambda)$ is also unbounded. $\square$

3 Qualitative Properties of the Principal Eigenvalue

Assume in this section that the weights m_1 and m_2 satisfy, respectively, (1.2) and (1.3). Now, we are concerned with the study of the principal eigenvalue $\mu_1(\lambda)$ defined by the following variational characterization:

$$\mu_1(\lambda) = \inf_{u \in \Sigma} \left\{ \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{\lambda}{p} \int_\Omega m_1 |u|^p dx \right\}. \tag{3.1}$$

Let us note that all solutions of problem (1.1) are of class $C^{1,\alpha}(\Omega)$ in the case $\lambda = 0$ (see [8]) and are of class $C^{1,\alpha}(\bar{\Omega})$ in the case $\lambda > 0$ (see [2]).

Theorem 3.1. *For any $\lambda > 0$, the eigenvalue $\mu_1(\lambda)$ defined by* (3.1) *is simple and the eigenfunctions associated to $\mu_1(\lambda)$ are either positive or negative in $\overline{\Omega}$.*

The proof is a straightforward adaptation of our work in [1], so we only make a sketch in order to make the paper self contained. The following lemma derives from Picone's identity.

Lemma 3.2. *Let u and v be two nonnegative eigenfunctions associated to some eigenvalues μ and $\tilde{\mu}$, respectively. Then*

$$0 \leq (\mu - \tilde{\mu}) \int_{\partial\Omega} m_2(x) u^p d\sigma \tag{3.2}$$

and equality holds if, and only if, v is multiple of u.

Proof of Theorem 3.1. By Theorem 2.2, it is clear that $\mu_1(\lambda)$ is an eigenvalue of problem (1.1) for any $\lambda > 0$. Let u be an eigenfunction associated to $\mu_1(\lambda)$ so that $|u|$ is a minimiser for (3.1) and is thus an eigenfunction associated to $\mu_1(\lambda)$. It follows from the maximum principle of Vazquez that $|u| > 0$ in Ω and we conclude that u has constant sign. Taking $\mu = \tilde{\mu} = \mu_1(\lambda)$ in (3.2), we see that any eigenfunction v associated of $\mu_1(\lambda)$ must be a multiple of u, so that $\mu_1(\lambda)$ is simple. $\square$

To prove the isolation of $\mu_1(\lambda)$, we need the two following lemmas.

Lemma 3.3. *Let $(k, q) \in \mathbb{N}^* \times \mathbb{N}$ and let $\lambda \in \mathbb{R}^+$. If $\mu_k(\lambda) = \mu_{k+1}(\lambda) = \cdots = \mu_{k+q}(\lambda)$, then $\gamma(K) \geq q+1$ where $K := \{u \in \Sigma; u$ is an eigenfunction associated to $\mu_k(\lambda)\}$ and $\gamma(K)$ is the Krasnoselski genus of K.*

The Lemma 3.3 is proved by applying a general result from infinite dimensional Ljusternik–Schnirelmann theory.

Lemma 3.4. *For each $\lambda > 0$, $\mu_1(\lambda)$ is the only positive eigenvalue, having an eigenfunction that does not change sign on the boundary $\partial\Omega$.*

Proof. For the proof, we use Lemma 3.2. Taking $\mu = \mu_1(\lambda)$ in (3.2), we see that no eigenvalue $\tilde{\mu} > \mu_1(\lambda)$ can be associated to a positive eigenfunction. Thus $\mu_1(\lambda)$ is the only positive eigenvalue associated to an eigenfunction of definite sign. □

Theorem 3.5. *For each $\lambda > 0$, $\mu_1(\lambda)$ is isolated.*

Proof. It suffices to prove that $\mu_2(\lambda)$ is indeed the second positive eigenvalue of problem (1.1), i.e., $\mu_1(\lambda) < \mu_2(\lambda)$ for all $\lambda > 0$ and if $\mu_1(\lambda) < \mu < \mu_2(\lambda)$, then μ is not an eigenvalue of problem (1.1). By Theorem 3.1, $\gamma(K_1) = 1$ where the set K_1 is defined by $K_1 := \{u \in \Sigma; u \text{ is an eigenfunction associated to } \mu_1(\lambda)\}$. Thus, by Lemma 3.3, $\mu_1(\lambda) < \mu_2(\lambda)$. By contradiction, we suppose that μ is an eigenvalue of problem (1.1). Let u be an eigenfunction associated to μ. Since $\mu \neq \mu_1(\lambda)$, we deduce by Lemma 3.4 that $u^+ = \max(u, 0) \neq 0$ and $u^- = \min(u, 0) \neq 0$. It follows from (2.1) that

$$\int_\Omega |\nabla u^+|^p dx + \lambda \int_\Omega m_1(x)|u^+|^p dx = \mu \int_{\partial\Omega} m_2(x)|u^+|^p d\sigma,$$

$$\int_\Omega |\nabla u^-|^p dx + \lambda \int_\Omega m_1(x)|u^-|^p dx = \mu \int_{\partial\Omega} m_2(x)|u^-|^p d\sigma.$$

Assume that u is normalized in such a way that

$$\frac{1}{p}\int_{\partial\Omega} m_2(x)|u^+|^p d\sigma = \frac{1}{p}\int_{\partial\Omega} m_2(x)|u^-|^p d\sigma = 1.$$

Let $K := \{\alpha u^+ + \beta u^-; \alpha, \beta \in \mathbb{R} \text{ with } |\alpha|^p + |\beta|^p = 1\}$ and $h : S^1 \to K : (a, b) \to |a|^{\frac{2}{p}-1} a u^+ + |b|^{\frac{2}{p}-1} b u^-$, where $S^1 := \{(x, y) \in \mathbb{R}^2; x^2 + y^2 = 1\}$. We have that h is a continuous odd surjection. Consequently, $K \in \mathcal{F}_2$ and therefore

$$\begin{aligned}\mu_2(\lambda) &\leq \max_{|\alpha|^p+|\beta|^p=1} \left(\frac{1}{p}\int_\Omega \left|\nabla\left(\alpha u^+ + \beta u^-\right)\right|^p dx + \frac{\lambda}{p}\int_\Omega m_1(x)|\alpha u^+ + \beta u^-|^p dx \right) \\ &= \mu.\end{aligned}$$

This is a contradiction. The proof of the isolation of $\mu_1(\lambda)$ is complete. □

4 Continuity and Differentiability in λ

The assumptions on weights m_1 and m_2 in this section are those of Section 3. We assume again that $\int_{\partial\Omega} m_2 d\sigma < 0$. The following investigation adopts the scheme of Binding and Huang [3]. Let $\lambda \in \mathbb{R}^+$ and $(\mu_1(\lambda), u(\lambda))$ be the corresponding eigenpair. Henceforth we normalize the eigenfunction $u(\lambda)$ to $u(\lambda) \in \Sigma$ with $u(\lambda) > 0$. In the following theorem, we consider continuity of the eigenpair in λ and differentiability of the principal eigenvalue $\mu_1(\lambda)$ in λ.

Theorem 4.1. *For any bounded domain Ω, the function $\lambda \to \mu_1(\lambda)$ is differentiable on $\mathbb{R}^+$ and the function $\lambda \to u(\lambda)$ is continuous from $\mathbb{R}^+$ into $W^{1,p}(\Omega)$. More precisely,*

$$\mu_1'(\lambda_0) = \frac{1}{p}\int_\Omega m_1(x)(u(\lambda_0))^p dx \quad \forall \lambda_0 \geq 0. \tag{4.1}$$

Proof. By (3.1), it is easy to see that $\lambda \to \mu_1(\lambda)$ is a concave function in $\mathbb{R}^+_*$. Continuity of $\lambda \to \mu_1(\lambda)$ in $\mathbb{R}^+_*$ follows from the concavity. It remains to show right continuity from zero. Let φ be an eigenfunction associated to $\mu_1(0)$. For all $\lambda \geq 0$

$$\mu_1(0) \leq \mu_1(\lambda) \leq \frac{1}{p}\left(\int_\Omega |\nabla \varphi|^p dx + \lambda \int_\Omega m_1|\varphi|^p dx\right).$$

Passing to the limit as $\lambda \to 0^+$, we have $\mu_1(\lambda) \to \mu_1(0)$ as $\lambda \to 0^+$.

To prove the continuity of $\lambda \to u(\lambda)$, we proceed as follows. Let $\Lambda \subset \mathbb{R}^+$ be bounded. For $\lambda \in \Lambda$, since

$$\mu_1(\lambda) = \frac{1}{p}\int_\Omega |\nabla u(\lambda)|^p dx + \frac{\lambda}{p}\int_\Omega m_1(x)|u(\lambda)|^p dx \leq const,$$

$u(\lambda)$ is bounded in $W^{1,p}(\Omega)$, $u(\lambda) \to u_0$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$ and strongly in $L^p(\partial\Omega)$ as $\lambda \to \lambda_0 \in \overline{\Lambda}$. Passing to the limit in the equality

$$\begin{aligned}\int_\Omega |\nabla u(\lambda)|^{p-2}\nabla u(\lambda)\nabla\varphi dx &+ \lambda\int_\Omega m_1|u(\lambda)|^{p-2}u\varphi dx \\ &= \mu_1(\lambda)\int_{\partial\Omega} m_2|u(\lambda)|^{p-2}u(\lambda)\varphi d\sigma,\end{aligned} \tag{4.2}$$

we have

$$\begin{aligned}\int_\Omega |\nabla u_0|^{p-2}\nabla u_0\nabla\varphi dx &+ \lambda_0\int_\Omega m_1|u_0|^{p-2}u_0\varphi dx \\ &= \mu_1(\lambda_0)\int_{\partial\Omega} m_2|u_0|^{p-2}u_0\varphi d\sigma.\end{aligned} \tag{4.3}$$

On the other hand, $u_0 \not\equiv 0$ (since $u_0 \in \Sigma$). Thus u_0 is an eigenfunction associated to $\mu_1(\lambda_0)$. By simplicity of $\mu_1(\lambda_0)$, we have $u_0 = u(\lambda_0)$. Taking $\varphi = u_0$ in (4.3),

$$\frac{1}{p}\int_\Omega |\nabla u_0|^p dx + \frac{\lambda_0}{p}\int_\Omega m_1(x)|u_0|^p dx = \mu_1(\lambda_0). \tag{4.4}$$

For $\varphi = u(\lambda)$ in (4.2), we get

$$\frac{1}{p}\int_\Omega |\nabla u(\lambda)|^p dx + \frac{\lambda}{p}\int_\Omega m_1(x)|u(\lambda)|^p dx = \mu_1(\lambda). \tag{4.5}$$

Letting $\lambda \to \lambda_0$ in (4.5), we have

$$\lim_{\lambda\to\lambda_0}\frac{1}{p}\int_\Omega |\nabla u(\lambda)|^p dx = -\frac{\lambda_0}{p}\int_\Omega m_1(x)|u_0|^p dx + \mu_1(\lambda_0) = \frac{1}{p}\int_\Omega |\nabla u_0|^p dx.$$

Since $u(\lambda) \to u_0$ strongly in $L^p(\Omega)$, $\|u(\lambda)\| \to \|u_0\|$ as $\lambda \to \lambda_0$. Finally, by the uniform convexity of $W^{1,p}(\Omega)$, we conclude that $u(\lambda) \to u_0 = u(\lambda_0)$ strongly in $W^{1,p}(\Omega)$ as $\lambda \to \lambda_0$. For the differentiability of $\lambda \to \mu_1(\lambda)$, it suffices to use the variational characterization of $\mu_1(\lambda)$ and of $\mu_1(\lambda_0)$, so that

$$\frac{\lambda_0 - \lambda}{p} \int_\Omega m_1(x)(u(\lambda))^p dx \le \mu_1(\lambda_0) - \mu_1(\lambda) \le \frac{\lambda_0 - \lambda}{p} \int_\Omega m_1(x)(u(\lambda_0))^p dx$$

for all $\lambda, \lambda_0 \in \mathbb{R}$. Dividing by $\lambda - \lambda_0$ and letting $\lambda \to \lambda_0$, we obtain (4.1). □

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