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Strongly Nonlinear Elliptic Problems in Musielak–Orlicz–Sobolev Spaces

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Abstract

We prove existence of solutions for strongly nonlinear elliptic equations of the form $A(u)+g(x,u) = f$, where $A(u) = -$ div $a(x, u, \nabla u)$ is a Leray–Lions operator defined on $D(A) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ with φ and ψ two complementary Musielak–Orlicz functions, f is a distribution of the dual, and g a nonlinearity with the sign condition but without any restriction on its growth.

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1 Introduction

Let Ω be a bounded subset of $\mathbb{R}^n$. We consider the strongly nonlinear elliptic problem

$$A(u) + g(x, u) = f(x \in \Omega), \tag{1.1}$$

where $A(u) = -$ div $a(x, u, \nabla u)$ is an operator of Leray–Lions type, g is a nonlinearity with the sign condition but any restriction on its growth. When $g \equiv 0$ we say that the problem is quasilinear. Quasilinear problems have been extensively studied by Browder and others in the context of the theory of mappings of monotone type from a reflexive Banach space to its dual and in the case where $a(\cdot)$ has polynomial growth in u and its gradient [2, 3, 13]. In the same context, an existence result for (1.1) has been provided by Hess [12]. From 1970, these results have been extended by Donaldson [4] and

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Gossez [6, 7] for quasilinear problems, and by Gossez [8] and Gossez and Mustonen in [9] for the strongly nonlinear problems, to the case where the function a do not necessarily have polynomial growth in u and its gradient. The Banach spaces in which the problems are formulated (the Orlicz–Sobolev spaces) are not reflexive and the corresponding mappings of monotone type are not bounded nor everywhere defined and are not generally a priori globally bounded (and consequently are not generally coercive).

In the last decade several works have been concerned to extend the classical polynomial growth to the nonstandard growth case in the so-called variable exponent Sobolev spaces (see [11] and references within, and also [16]). Recently, M. Mihăilescu and V. Rădulescu in [14] and X. L. Fan and C. X. Guan in [5], obtained new results which improved the already known existence results for the $p(x)$-Laplacian operator in the Musielak–Orlicz–Sobolev spaces $W^1L_\varphi(\Omega)$ under some assumptions such as the condition Δ_2 on φ and also the uniform convexity of φ which assure that the space $L_\varphi(\Omega)$ is reflexive. The study of variational boundary value problems for quasilinear elliptic equations in the general case, when the Musielak–Orlicz–Sobolev spaces $W^mL_\varphi(\Omega)$ are not reflexive, was initiated by the author et al. in [1], with the assumption that the conjugate function ψ of φ has the Δ_2 property.

Our purpose in this paper is to study the problem (1.1) in the context of Musielak–Orlicz–Sobolev spaces $W^mL_\varphi(\Omega)$ with same assumptions as in [1]. The study of nonlinear partial differential equations in this type of spaces is strongly motivated by numerous phenomena of physics, namely the problems related to non Newtonian fluids of strongly inhomogeneous behavior with a high ability of increasing their viscosity under a different stimulus, like the shear rate, magnetic or electric field [10]. Note that the Δ_2 condition on ψ in this paper is only used for building the suitable complementary system with nonattendance of the analogous of [6, Theorem 1.3] in the context of Musielak–Orlicz–Sobolev spaces. This result can be applied, for example, for finding a weak solution for the following equation:

$$- \operatorname{div} \left(\frac{m(x, |\nabla u|)}{|\nabla u|} \cdot \nabla u \right) + u \sin^2(u) = f,$$

where m is the derivative of φ with respect to t. In the particular case when $\varphi(x,t) = \dfrac{1}{p(x)} t^{p(x)}$, our result gives an essential improvements of the known existence results for (1.1) in the statement of the variable exponent Sobolev spaces $W^{1,p(x)}$. Simply put, we avoid assuming any conditions of Log–Hölder continuity type on $p(\cdot)$. Furthermore, the essential supreme of the function $p(\cdot)$ takes the value of infinity. The following are some examples of exponents for which our existence result is valid while the previously results fail. (i) *One dimensional case.* Take $\Omega =]-1, 1[$ and $p(x) = \dfrac{1}{1-x^2} + \dfrac{1}{2}$. (ii) *Two dimensional case.* Consider the famous Zhikov's example in which the Lavrentiev phenomenon occurs: let $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : |x| < 1\}$,

$$p(x) = \begin{cases} a_1 & \text{if } x_1x_2 > 0, \\ a_2 & \text{if } x_1x_2 < 0, \end{cases}$$

where $1 < a_1 < 2 < a_2$.

The paper is organized as follows. In Section 2 we introduce some basic definitions and properties in Musielak–Orlicz–Sobolev spaces as well as an abstract theorem valid in any complementary system. Section 3 contains the main result of this paper.

2 Preliminaries

In this section we list briefly some definitions and facts about Musielak–Orlicz–Sobolev spaces. For further definitions and properties we refer the reader to [1, 15]. We also include an abstract surjectivity result.

2.1 Musielak–Orlicz–Sobolev Spaces

Let Ω be an open subset of $\mathbb{R}^n$ and let φ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

a) $\varphi(x, \cdot)$ is an N-function, i.e., convex, nondecreasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$ for all $t > 0$, and

$$\lim_{t \to 0} \frac{\varphi(x,t)}{t} = 0 \quad \text{for almost all } x \in \Omega,$$

$$\lim_{t \to \infty} \frac{\varphi(x,t)}{t} = \infty \quad \text{for almost all } x \in \Omega.$$

b) $\varphi(\cdot, t)$ is a measurable function.

A function $\varphi(x, t)$, which satisfies the conditions a) and b), is called a Musielak–Orlicz function. For a Musielak–Orlicz function $\varphi(x, t)$ we put $\varphi_x(t) = \varphi(x, t)$ and we associate its nonnegative reciprocal function with respect to t and φ_x^{-1}, that is,

$$\varphi_x^{-1}(\varphi(x,t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$

For any two Musielak–Orlicz functions φ and γ we introduce the following ordering:

c) If there exists two positives constants c and T such that for almost all $x \in \Omega$

$$\varphi(x,t) \le \gamma(x, ct) \text{ for } t \ge T,$$

then we write $\varphi \prec \gamma$ and we say that γ dominate φ globally if $T = 0$ and near infinity if $T > 0$.

d) If for every positive constant c and almost everywhere $x \in \Omega$ we have

$$\lim_{t\to 0}\left(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\right)=0 \quad \text{or} \quad \lim_{t\to\infty}\left(\sup_{x\in\Omega}\frac{\varphi(x,ct)}{\gamma(x,t)}\right)=0,$$

then we write $\varphi \prec\prec \gamma$ at 0 or near ∞ respectively, and we say that φ increases essentially more slowly than γ at 0 or near infinity, respectively.

In the following the measurability of a function $u : \Omega \mapsto \mathbb{R}$ means the Lebesgue measurability. We define the functional

$$\varrho_{\varphi,\Omega}(u)=\int_\Omega \varphi(x,|u(x)|)dx,$$

where $u : \Omega \mapsto \mathbb{R}$ is a measurable function. The set

$$K_\varphi(\Omega)=\{u:\Omega\to\mathbb{R} \text{ measurable } /\varrho_{\varphi,\Omega}(u)<+\infty\}$$

is called the Musielak–Orlicz class (the generalized Orlicz class). The Musielak–Orlicz space (the generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently,

$$L_\varphi(\Omega)=\left\{u:\Omega\to\mathbb{R} \text{ measurable } /\varrho_{\varphi,\Omega}\left(\frac{|u(x)|}{\lambda}\right)<+\infty, \text{ for some } \lambda>0\right\}.$$

Let

$$\psi(x,s)=\sup_{t\geq 0}\{st-\varphi(x,t)\},$$

that is, ψ is the Musielak–Orlicz function complementary to (or conjugate of) $\varphi(x,t)$ in the sense of Young with respect to the variable s. In the space $L_\varphi(\Omega)$ we define the following two norms:

$$||u||_{\varphi,\Omega}=\inf\left\{\lambda>0/\int_\Omega\varphi\left(x,\frac{|u(x)|}{\lambda}\right)dx\leq 1\right\},$$

which is called the Luxemburg norm, and the so-called Orlicz norm by

$$|||u|||_{\varphi,\Omega}=\sup_{||v||_\psi\leq 1}\int_\Omega |u(x)v(x)|dx,$$

where ψ is the Musielak–Orlicz function complementary (or conjugate) to φ. These two norms are equivalent [15].

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\overline{\Omega}$ is denoted by $E_\varphi(\Omega)$. It is a separable space and $E_\psi(\Omega)^*=L_\varphi(\Omega)$ [15]. We have $E_\varphi(\Omega)=K_\varphi(\Omega)$ if and only if $K_\varphi(\Omega)=L_\varphi(\Omega)$ if and only if φ has the Δ_2 property for large values of t or for all values of t, according to whether Ω has finite measure or not, i.e., there exists $k > 0$ independent of $x \in \Omega$ and a nonnegative function h, integrable in Ω, such that $\varphi(x, 2t) \leq k\varphi(x,t) + h(x)$ for large values of t, or for all values of t.

We say that a sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n\to\infty} \varrho_{\varphi,\Omega}\left(\frac{u_n - u}{k}\right) = 0.$$

For any fixed nonnegative integer m we define

$$W^m L_\varphi(\Omega) = \{u \in L_\varphi(\Omega) : \forall |\alpha| \leq m \; D^\alpha u \in L_\varphi(\Omega)\},$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ with nonnegative integers α_i $|\alpha| = |\alpha_1| + |\alpha_2| + \cdots + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^m L_\varphi(\Omega)$ is called the Musielak–Orlicz–Sobolev space. Let

$$\overline{\varrho}_{\varphi,\Omega}(u) = \sum_{|\alpha|\leq m} \varrho_{\varphi,\Omega}(D^\alpha u) \text{ and } ||u||^m_{\varphi,\Omega} = \inf\left\{\lambda > 0 : \overline{\varrho}_{\varphi,\Omega}\left(\frac{u}{\lambda}\right) \leq 1\right\}$$

for $u \in W^m L_\varphi(\Omega)$. These functionals are convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $\langle W^m L_\varphi(\Omega), ||u||^m_{\varphi,\Omega}\rangle$ is a Banach space if φ satisfies the following condition [15]:

$$\text{there exist a constant } c > 0 \text{ such that } \inf_{x\in\Omega} \varphi(x, 1) \geq c. \tag{2.1}$$

The space $W^m L_\varphi(\Omega)$ is identified to a subspace of the product $\Pi_{|\alpha|\leq m} L_\varphi(\Omega) = \Pi L_\varphi$; this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed. Let $W_0^m L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$. Let $W^m E_\varphi(\Omega)$ be the space of functions u such that u and its distribution derivatives up to order m lie in $E_\varphi(\Omega)$, and $W_0^m E_\varphi(\Omega)$ is the (norm) closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$. The following spaces of distributions will also be used:

$$W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\}$$

and

$$W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha|\leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}.$$

In the particular case when $\varphi(x,t) = \dfrac{1}{p(x)} t^p(x)$, we use the notations $L^{p(x)}(\Omega) = L_\varphi(\Omega)$ and $W^{m,p(x)}(\Omega) = W^m L_\varphi(\Omega)$. These spaces are called the variable exponent Lebesgue and Sobolev spaces.

2.2 An Abstract Result

Definition 2.1. Let Y and Z be two real Banach spaces in duality with respect to a continuous pairing $< \cdot, \cdot >$ and let Y_0 and Z_0 be subspaces of Y and Z, respectively. Then $(Y, Y_0; Z, Z_0)$ is called a complementary system if, by means of $\langle \cdot, \cdot \rangle$, Y_0^* can be identified (i.e., is linearly homeomorphic) to Z and Z_0^* to Y.

Let $(Y, Y_0; Z, Z_0)$ be a complementary system and T be a mapping from the domain $D(T)$ in Y to Z which satisfy the following conditions, with respect to some element $\bar{y} \in Y_0$ and $f \in Z_0$:

(i) (finite continuity) $D(T) \supset Y_0$ and T is continuous from each finite dimensional subspaces of Y_0 to the $\sigma(Z, Y_0)$ topology of Z,

(ii) (sequential pseudo-monotonicity) for any sequence $\{y_i\}$ with $y_i \to y \in Y$ for $\sigma(Y, Z_0)$, $T(y_i) \to z \in Z$ for $\sigma(Z, Y_0)$ and $\limsup \langle T(y_i), y_i \rangle \leq \langle z, y \rangle$, it follows that $T(y) = z$ and $< T(y_i), y_i > \to \langle z, y \rangle$,

(iii) $T(y)$ remains bounded in Z whenever $y \in D(T)$ remains bounded in Y and $\langle y - \bar{y}, Ty \rangle$ remains bounded from above,

(iv) $\langle y - \overline{y}, Ty - f \rangle \to +\infty$ as $||y||_Y \to +\infty$ in $D(T)$.

Given a convex set $K \subset Y$ and an element $f \in Z_0$, we are interested in finding a solution y of the variational inequality

$$\begin{cases} y \in K \cap D(T), \\ \langle y - z, Ty \rangle \leq \langle y - z, f \rangle \text{ for all } z \in K. \end{cases}$$

Theorem 2.2 (See [9]). *Let $(Y, Y_0; Z, Z_0)$ be a complementary system with Y_0 and Z_0 separable. Let $K \subset Y$ be convex, $\sigma(Y, Z_0)$ sequentially closed and such that $K \cap Y_0$ is $\sigma(Y, Z)$ dense in K. Let $f \in Z_0$ and let $T : D(T) \subset Y \to Z$ satisfy (i) to (iv) with respect to some $\overline{y} \in K \cap Y_0$ and the given f. Then the variational inequality (10) has at least one solution y.*

3 Main Result

Let Ω be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$. Let φ and γ be two Musielak–Orlicz functions such that $\gamma \ll \varphi$. Let $A : D(T) \subset W_0^1 L_\varphi(\Omega) \to W^{-1} L_\psi(\Omega)$ be a mapping (not everywhere defined) given by

$$A(u) = -\text{ div } a(x, u, \nabla u),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function satisfying, for a.e. $x \in \Omega$, and for all $s \in \mathbb{R}$ and all $\xi, \xi_* \in \mathbb{R}^n, \xi \neq \xi_*$,

$$|a(x,s,\xi)| \leq k_1 \left[c(x) + \psi_x^{-1}\left(\gamma\left(x, k_2|s|\right)\right) + \psi_x^{-1}\left(\varphi(x, k_2|\xi|)\right)\right], \tag{3.1}$$

$$[a(x,s,\xi) - a(x,s,\xi_*)]\,[\xi - \xi_*] > 0, \tag{3.2}$$

$$k_3\varphi(x,|\xi|) \leq a(x,s,\xi)\xi, \tag{3.3}$$

where $c(x)$ belongs to $E_\psi(\Omega)$, $c \geq 0$ and $k_i > 0$ for $i = 1, 2, 3$. We define the mapping T by the formula

$$\langle v, Tu\rangle = \int_\Omega a(x,u,\nabla u)\nabla v dx$$

for $v \in W_0^1 L_\varphi(\Omega)$. For the convex set $K \subset W_0^1 L_\varphi(\Omega)$ we need the following two approximation properties:

(K_1) For each $u \in K \cap L^\infty(\Omega)$ there exists a sequence $u_n \in K \cap L^\infty(\Omega) \cap W_0^1 E_\varphi(\Omega)$ such that $u_n \to u$ for $\sigma(\Pi L_\varphi(\Omega), \Pi L_\psi(\Omega))$ with $||u_n||_\infty$ bounded.

(K_2) For each $u \in K$ there exists a sequence $u_n \in K \cap L^\infty(\Omega)$ and a constant c such that $u_n \to u$ for $\sigma(\Pi L_\varphi(\Omega), \Pi L_\psi(\Omega))$ and $|u_n(x)| \leq c|u(x)|$ for a.e. $x \in \Omega$ and all u in R.

Note that (K_1) and (K_2) together imply that $K \cap W_0^1 E_\varphi(\Omega)$ is $\sigma(\Pi L_\varphi(\Omega), \Pi L_\psi(\Omega))$ dense in K.

Theorem 3.1. *Assume that* (3.1), (3.2) *and* (3.3) *hold true. Let K be a convex set of $W_0^1 L_\varphi(\Omega)$ satisfying the condition K_1 and K_2. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that for each $r \in \mathbb{R}$ there exists $h_r \in L^1(\Omega)$ with*

$$g(x,u) \leq h_r(x) \tag{3.4}$$

for a.a. $x \in \Omega$ and all $u \in \mathbb{R}$ with $|u| < r$. Assume that

$$g(x,u)u \geq 0, \tag{3.5}$$

for a.a. $x \in \Omega$ and all $u \in \mathbb{R}$. Then, given $f \in W^{-1}E_\psi(\Omega)$, there exists $u \in W_0^1 L_\varphi(\Omega)$ such that $g(x,u) \in L^1(\Omega)$, $g(x,u)u \in L^1(\Omega)$ and

$$\langle u - v, T(u)\rangle + \int_\Omega g(x,u)(u-v) \leq \langle u - v, f\rangle \tag{3.6}$$

for all $v \in K \cap L^\infty(\Omega)$.

Proof. The quadruple $(W_0^1 L_\varphi(\Omega), W^m E_\varphi(\Omega); W^{-1} L_\psi(\Omega), W^{-1} E_\psi(\Omega))$ constitutes a complementary system, by the statement of [1]. For this complementary system we use the notation $(Y, Y_0; Z, Z_0)$. Also by [1] we can deduce that the corresponding mapping T of A satisfies the conditions (i)–(iv) of Theorem 2.2. We truncate g by letting:

$$g_n(x,u) = \begin{cases} g(x,u) & \text{if } |g(x,u)| \leq n, \\ n \text{ sgn } g(x,u) & \text{if } |g(x,u)| > n. \end{cases}$$

By (3.4), (3.5) and the fact that $\bar{u} \in L^\infty(\Omega)$, we have

$$\begin{aligned} &\int_\Omega g_n(x,u)(u-\bar{u})dx \\ &\qquad = \int_{|u|>||\bar{u}||_\infty} g_n(x,u)(u-\bar{u})dx + \int_{|u|\le||\bar{u}||_\infty} g_n(x,u)(u-\bar{u})dx \\ &\qquad \ge \int_{|u|\le||\bar{u}||_\infty} g_n(x,u)(u-\bar{u})dx \ge -2\int_{|u|\le||\bar{u}||_\infty} |g_n(x,u)||\bar{u}|dx \\ &\qquad \ge -2||\bar{u}||_\infty \int_\Omega h_{||\bar{u}||_\infty} dx \end{aligned} \tag{3.7}$$

for all $u \in Y$. For $u \in Y$, put $(G_n u)(x) = g_n(x,u)$. Then the mapping $T + G_n : D(T) \subset Y \to Z$ satisfies (i) and (ii) (by [6, Proposition 2.2], (iii) (since for each $n, (G_n u)$ is bounded) and (vi) (by (3.7)). Consequently, by Theorem 2.2, there exists $u_n \in K \cap D(T)$ such that

$$\langle u_n - v, Tu_n\rangle + \int_\Omega g_n(x,u_n)(u_n - v) \le \langle f, u_n - v\rangle \tag{3.8}$$

for all $v \in K$. Take $v = \bar{u}$ in (3.8). Then, using (3.7), we obtain:

$$\langle u_n - \bar{u}, Tu_n - f\rangle \le -\int_\Omega g_n(x,u_n)(u_n-\bar{u})dx \le ||\bar{u}||_\infty \int_\Omega h_{||\bar{u}||_\infty} dx.$$

By this estimate we conclude that u_n remains bounded in Y, that Tu_n remains bounded in Z and that $\int_\Omega g_n(x,u_n)(u_n - \bar{u})dx$ also remains bounded. Thus, passing to a subsequence, we can assume that $u_n \to u \in Y$ for $\sigma(Y, Z_0)$ and a.e. in Ω and that $u_n \to \chi \in Z$ for $\sigma(Z, Y_0)$. Therefore, $g_n(x,u_n) \to g(x,u)$ a.e. in Ω. Moreover,

$$\begin{aligned} \int_\Omega |g_n(x,u_n)(u_n-\bar{u})|dx &= \int_\Omega g_n(x,u_n)(u_n-\bar{u}) - 2\int_{\bar{u}>u_n>0} g_n(x,u_n)(u_n-\bar{u}) \\ &- 2\int_{\bar{u}<u_n<0} g_n(x,u_n)(u_n-\bar{u})dx \\ &\le C \end{aligned}$$

with C a constant independent of n. On the other hand, for all $r > 0$ we have

$$\begin{aligned} |g(x,u_n)| &\le \sup_{|u_n|\le r+||\bar{u}||_\infty} |g(x,u_n)| + \frac{1}{r}|g(x,u_n)(u_n-\bar{u})| \\ &\le h_{r+||\bar{u}||_\infty} + \frac{1}{r}|g(x,u_n)(u_n-\bar{u})|. \end{aligned}$$

Hence,

$$\int_E |g_n(x,u_n)|dx \le \int_E h_{r+||\bar{u}||_\infty} dx + \frac{C}{r}$$

for some measurable subset E of Ω. For $|E|$ sufficiently small and $r = \dfrac{2C}{\varepsilon}$, we obtain

$$\int_E |g_n(x,u_n)|dx < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} < \varepsilon,$$

since $h_{r+||\bar{u}||_\infty} \in L^1(\Omega)$. Thus, by Vitali's theorem, we get $g_n(x,u_n) \to g(x,u)$ in $L^1(\Omega)$. Using Fatou's lemma, we obtain:

$$\int_\Omega g(x,u)udx \leq \liminf \left[\int_\Omega g_n(x,u_n)(u_n - \bar{u})dx + \int_\Omega g_n(x,u_n)\bar{u}dx\right] < +\infty.$$

This implies that $g(x,u)u \in L^1(\Omega)$. We turn now to (3.8). Fatou's lemma gives that

$$\limsup\langle u_n, Tu_n\rangle \leq \langle v, \chi\rangle + \int_\Omega g(x,u)(v-u) + \langle f, u-v\rangle \tag{3.9}$$

for all $v \in K \cap Y_0 \cap L^\infty(\Omega)$. By K_1 and K_2 we conclude that (3.9) holds also for $v = u$, which implies

$$\limsup\langle u_n, Tu_n\rangle \leq \langle u, \chi\rangle.$$

Hence, by (ii), $u \in D(T), \chi = Tu$ and $\langle u_n, Tu_n\rangle \to \langle u, Tu\rangle$. Therefore,

$$\langle u - v, Tu\rangle + \int_\Omega g(x,u)(u-v) \leq \langle u-v, f\rangle,$$

which completes the proof. □

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