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Landesman–Lazer Conditions for the Steklov Problem

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Abstract

We prove existence of weak solutions to an eigenvalue Steklov problem defined in a bounded domain with a Lipschitz continuous boundary.

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1 Introduction

In a previous work [1], the solvability of the problem

$$\begin{cases} \triangle_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \mu_1 m(x)|u|^{p-2}u + f(x,u) - h & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

was investigated, where Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$), with a Lipschitz continuous boundary, $1 < p < \infty$, $m \in L^q(\partial\Omega)$, such that $m^+ = \max(m,0) \not\equiv 0$ and $\displaystyle\int_{\partial\Omega} m d\sigma < 0$, $(N-1)/(p-1) < q < \infty$ if $p < N$ and $q \geq 1$ if $p \geq N$. Let $f : \partial\Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the growth condition

$$|f(x,s)| \leq a|s|^{r-1} + b(x) \tag{1.2}$$

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for all $s \in \mathbb{R}$ and a.e. $x \in \partial\Omega$. Here a is a positive constant, $b \in L^{r'}(\partial\Omega)$ and r' is the conjugate of $r = pq/(q-1)$. We also assume that the function f satisfies the Landesman–Lazer conditions

$$\lim_{s\to-\infty} f(x,s) = \alpha(x), \quad \lim_{s\to+\infty} f(x,s) = \beta(x) \quad \text{a.e. } x \in \partial\Omega, \tag{1.3}$$

$$\int_{\partial\Omega} \beta(x)\varphi_1 d\sigma < \int_{\partial\Omega} h(x)\varphi_1 d\sigma < \int_{\partial\Omega} \alpha(x)\varphi_1 d\sigma, \tag{1.4}$$

where $h \in L^{r'}(\partial\Omega)$ and φ_1 is the normalized positive eigenfunction associated to μ_1, which is the first positive eigenvalue of the following Steklov problem:

$$\left\{ \begin{array}{rcll} \text{Find } (u,\mu) & \in & (W^{1,p}(\Omega) \setminus \{0\}) \times \mathbb{R}^+ & \text{such that} \\ \triangle_p u & = & 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} & = & \mu m(x)|u|^{p-2}u & \text{on } \partial\Omega. \end{array} \right. \tag{1.5}$$

Under the conditions (1.2), (1.3) and (1.4), it was proved that the problem (1.1) admits at least a weak solution in $W^{1,p}(\Omega)$. Our purpose in this work is to study existence for the Steklov problem

$$\left\{ \begin{array}{ll} \triangle_p u = m_1(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} = \mu_1 m_2(x)|u|^{p-2}u + f(x,u) - h & \text{on } \partial\Omega, \end{array} \right. \tag{1.6}$$

where Ω, p, q, f and h are assumed to satisfy the conditions indicated at the beginning of the introduction. We further assume that the weight function m_1 satisfies the following assumption:

$$m_1 \in L^\infty(\Omega) \text{ and } m_1(x) \geq cst > 0, \tag{1.7}$$

where cst is a real positive number. We also assume that m_2 is an indefinite weight satisfying

$$m_2 \in L^q(\partial\Omega) \text{ and } m_2^+ \not\equiv 0 \text{ on } \partial\Omega, \tag{1.8}$$

where μ_1 denotes the first positive eigenvalue of the Steklov problem

$$\left\{ \begin{array}{rcll} \text{Find } (u,\mu) & \in & (W^{1,p}(\Omega) \setminus \{0\}) \times \mathbb{R}^+ & \text{such that} \\ \triangle_p u & = & m_1(x)|u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} & = & \mu m_2(x)|u|^{p-2}u & \text{on } \partial\Omega. \end{array} \right. \tag{1.9}$$

It is well-known that

$$\mu_1 := \inf_{u\in W^{1,p}(\Omega)} \left\{ \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{p}\int_\Omega m_1|u|^p dx : \frac{1}{p}\int_{\partial\Omega} m_2(x)|u|^p d\sigma = 1 \right\}.$$

Recall that μ_1 is simple (see [10]). Moreover, there exists a unique positive eigenfunction φ_1 whose norm $\|u\| := \left(\int_\Omega |\nabla u|^p dx + \int_\Omega m_1|u|^p dx\right)^{1/p}$ in $W^{1,p}(\Omega)$ equals to one. We say that $u \in W^{1,p}(\Omega)$ is a weak solution of (1.6) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi + \int_\Omega m_1|u|^{p-2}u\varphi dx$$
$$= \mu_1 \int_{\partial\Omega} m_2|u|^{p-2}u\varphi d\sigma + \int_{\partial\Omega} f(x,u)\varphi d\sigma - \int_{\partial\Omega} h\varphi d\sigma$$

for all $\varphi \in W^{1,p}(\Omega)$, where $d\sigma$ is the $N-1$ dimensional Hausdorff measure.

The growing attention in the study of the p-Laplacian operator is motivated by the fact that it arises in various applications, for example, in non Newtonian fluids, reaction diffusion problems, flow through porous media, glacial sliding, theory of superconductors, biology, etc. Classical Dirichlet problems involving the p-Laplacian have been studied by various authors. We cite here the works [2–8]. However, nonlinear boundary conditions have only been considered in recent years. For previous works for the p-Laplacian with nonlinear boundary conditions of different type we refer to [1, 9, 10]. Here we extend some of the results for the Dirichlet p-Laplacian problem. We prove existence of solutions for problem (1.6) under Landesman–Lazer conditions (see Theorem 2.2). Our main tool is the minimum principle combined with variational arguments.

2 Existence of Solutions for a Steklov Problem

Throughout this section the weights m_1, m_2 are assumed to satisfy respectively the conditions (1.7) and (1.8). Our purpose is, by using the minimum principle, to study the solvability of the Steklov problem (1.6) under Landesman–Lazer conditions.

Theorem 2.1 (Minimum principle)**.** *Let X be a Banach space and $\Phi \in C^1(X, \mathbb{R})$. Assume that Φ satisfies the Palais–Smale condition and bounded from below. Then $c = \inf_X \Phi$ is a critical point.*

The following theorem is the main result in this work.

Theorem 2.2. *Assume that* (1.2)*,* (1.3) *and* (1.4) *are fulfilled. Then the problem* (1.6) *admits at least a weak solution in $W^{1,p}(\Omega)$.*

The following lemmas will be used in the proof of Theorem 2.2. They enable us to prove the existence of a critical point. The functional energy associated to the problem (1.6) is given by

$$\Phi(u) = \frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{p}\int_\Omega m_1|u|^p dx - \frac{\mu_1}{p}\int_{\partial\Omega} m_2|u|^p d\sigma - \int_{\partial\Omega} F(x,u)d\sigma + \int_{\partial\Omega} hud\sigma,$$

where

$$F(x,t) := \int_0^t f(x,s)ds.$$

Lemma 2.3. *Assume that* (1.2), (1.3) *and* (1.4) *hold. Then* Φ *satisfies the Palais–Smale condition (PS) on* $W^{1,p}(\Omega)$.

Proof. Let (u_n) be a sequence in $W^{1,p}(\Omega)$ and c be a real positive number such that $|\Phi(u_n)| \leq c$ for all n and $\Phi'(u_n) \to 0$. We prove that (u_n) is bounded in $W^{1,p}(\Omega)$. Indeed, let us assume, by contradiction, that $||u_n|| \to +\infty$ as $n \to +\infty$. Let $v_n = \dfrac{u_n}{||u_n||}$. Then v_n is bounded and, for a subsequence still denoted by (v_n), we have $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$, $v_n \to v$ strongly in $L^p(\Omega)$ and $v_n \to v$ strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$. The hypothesis $|\Phi(u_n)| \leq c$ implies

$$\lim_{n\to+\infty}\left(\frac{1}{p}\int_\Omega |\nabla v_n|^p dx + \frac{1}{p}\int_\Omega m_1|v_n|^p dx - \frac{\mu_1}{p}\int_{\partial\Omega} m_2|v_n|^p d\sigma \right.$$
$$\left. - \int_{\partial\Omega}\frac{F(x,u_n)}{||u_n||^p}d\sigma + \int_{\partial\Omega} h\frac{u_n}{||u_n||^p}d\sigma\right) = 0.$$

Since, by hypotheses on p, h, u_n and using (1.3),

$$\lim_{n\to+\infty}\left(-\int_{\partial\Omega}\frac{F(x,u_n)}{||u_n||^p}d\sigma + \int_{\partial\Omega} h\frac{u_n}{||u_n||^p}d\sigma\right) = 0,$$

while

$$\lim_{n\to+\infty}\frac{1}{p}\int_{\partial\Omega} m_2|v_n|^p d\sigma = \frac{1}{p}\int_{\partial\Omega} m_2|v|^p d\sigma,$$

we have

$$\lim_{n\to+\infty}\left(\int_\Omega |\nabla v_n|^p dx + \int_\Omega m_1|v_n|^p dx\right) = \mu_1\int_{\partial\Omega} m_2|v|^p d\sigma.$$

Using the weak lower semi-continuity of the norm and the definition of μ_1, we get

$$\begin{aligned}\mu_1\int_{\partial\Omega} m_2|v|^p d\sigma &\leq \int_\Omega |\nabla v|^p dx + \int_\Omega m_1|v|^p dx\\ &\leq \liminf_{n\to+\infty}\left(\int_\Omega |\nabla v_n|^p dx + \int_\Omega m_1|v_n|^p dx\right)\\ &= \mu_1\int_{\partial\Omega} m_2|v|^p d\sigma.\end{aligned}$$

Because $\|u\| := \left(\int_\Omega |\nabla u|^p dx + \int_\Omega m_1|u|^p dx\right)^{1/p}$ is a norm on $W^{1,p}(\Omega)$ equivalent to the usual norm, we have $v_n \to v$ strongly in $W^{1,p}(\Omega)$ and

$$\mu_1\int_{\partial\Omega} m_2|v|^p d\sigma = \int_\Omega |\nabla v|^p dx + \int_\Omega m_1|v|^p dx.$$

This implies, by the definition of φ_1 and $\int_{\partial\Omega} m_2 d\sigma < 0$, that $v = \pm\varphi_1$. Let

$$g(x,s) = \begin{cases} \dfrac{F(x,s)}{s}, & \text{if } s \neq 0; \\ f(x,0), & \text{if } s = 0. \end{cases}$$

Case 1: Suppose that $v_n \to \varphi_1$. Then we have $u_n(x) \to +\infty$ and

$$f(x, u_n(x)) \to \beta(x) \text{ a.e. } x \in \partial\Omega,$$

$$g(x, u_n(x)) \to \beta(x) \text{ a.e. } x \in \partial\Omega.$$

Therefore, the Lebesgue theorem implies that

$$\lim_{n+\infty} \int_{\partial\Omega} \left(pg\left(x, u_n(x)\right) - f\left(x, u_n(x)\right)\right) v_n d\sigma = (p-1) \int_{\partial\Omega} \beta(x)\varphi_1(x) d\sigma.$$

On the other hand, $|\Phi(u_n)| \leq c$ implies that

$$\begin{aligned} -cp \leq \int_\Omega |\nabla v_n|^p dx + \int_\Omega m_1 |v_n|^p dx - \mu_1 \int_{\partial\Omega} m_2 |v_n|^p d\sigma \\ - \int_{\partial\Omega} pF(x, u_n) d\sigma + \int_{\partial\Omega} h u_n d\sigma \leq cp, \quad (2.1) \end{aligned}$$

and $\Phi'(u_n) \to 0$ implies that for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$\begin{aligned} -\varepsilon \leq -\int_\Omega |\nabla u_n|^p dx - \int_\Omega m_1 |u_n|^p dx + \mu_1 \int_{\partial\Omega} m_2 |u_n|^p d\sigma \\ + \int_{\partial\Omega} f(x, u_n(x)) u_n(x) d\sigma - \int_{\partial\Omega} h(x) u_n(x) d\sigma \leq \varepsilon \quad (2.2) \end{aligned}$$

for all $n \geq n_0$. By summing up (2.1) and (2.2), we get

$$\int_{\partial\Omega} f(x, u_n(x)) u_n(x) d\sigma - \int_{\partial\Omega} pF(x, u_n) d\sigma + (p-1) \int_{\partial\Omega} h(x) u_n(x) d\sigma \geq -cp - \varepsilon.$$

Dividing by $||u_n||$, we obtain

$$\int_{\partial\Omega} f(x, u_n(x)) v_n(x) d\sigma - \int_{\partial\Omega} pg(x, u_n) v_n(x) d\sigma + (p-1) \int_{\partial\Omega} h(x) v_n(x) d\sigma \geq \frac{-cp - \varepsilon}{||u_n||}.$$

Passing to the limit, we obtain

$$\int_{\partial\Omega} h(x)\varphi_1(x) d\sigma \geq \int_{\partial\Omega} \beta(x)\varphi_1(x) d\sigma,$$

which contradicts (1.4).

Case 2: Suppose that $v_n \to -\varphi_1$. Then we have $u_n(x) \to -\infty$ and

$$f(x, u_n(x)) \to \alpha(x) \text{ a.e. } x \in \partial\Omega,$$

$$g(x, u_n(x)) \to \alpha(x) \text{ a.e. } x \in \partial\Omega.$$

By summing up (2.1) and (2.2), we get

$$\int_{\partial\Omega} f(x, u_n(x))u_n(x)d\sigma - \int_{\partial\Omega} pF(x, u_n)d\sigma + (p-1)\int_{\partial\Omega} h(x)u_n(x)d\sigma \le cp + \varepsilon.$$

Dividing by $||u_n||$, we obtain

$$\int_{\partial\Omega} f(x, u_n(x))v_n(x)d\sigma - \int_{\partial\Omega} pg(x, u_n)v_n(x)d\sigma + (p-1)\int_{\partial\Omega} h(x)v_n(x)d\sigma \le \frac{cp+\varepsilon}{||u_n||}.$$

Passing to the limit, we get

$$\int_{\partial\Omega} \alpha(x)\varphi_1(x)d\sigma \le \int_{\partial\Omega} h(x)\varphi_1(x)d\sigma,$$

which contradicts (1.4). Finally, (u_n) is bounded in $W^{1,p}(\Omega)$, and for a subsequence still denoted by (u_n), there exists $u \in W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and $u_n \to u$ strongly in $L^{\frac{pq}{q-1}}(\partial\Omega)$. By the hypotheses on m_2, h, u_n and using (1.3), we deduce that

$$\lim_{n\to+\infty} \int_{\partial\Omega} m_2|u_n|^{p-2}u_n(u_n - u)d\sigma = 0,$$

$$\lim_{n\to+\infty} \int_{\partial\Omega} f(x, u_n(x))(u_n - u)d\sigma = 0,$$

$$\lim_{n\to+\infty} \int_{\partial\Omega} h(u_n - u)d\sigma = 0.$$

On the other hand, we have

$$\lim_{n\to+\infty} \Phi'(u_n)(u_n - u) = 0.$$

Therefore,

$$\lim_{n\to+\infty} \int_{\Omega} |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx = 0,$$

and $u_n \to u$ strongly in $L^p(\Omega)$. Thus

$$\lim_{n\to+\infty} \int_{\Omega} |u_n|^{p-2}u_n(u_n - u)dx = 0$$

and it follows from the (S^+) property that $u_n \to u$ strongly in $W^{1,p}(\Omega)$. $\square$

Lemma 2.4. *Assume that* (1.2), (1.3) *and* (1.4) *are satisfied. Then* Φ *is bounded from below.*

Proof. It suffices to show that Φ is coercive. Suppose, by contradiction, that there exists a sequence (u_n) such that $||u_n|| \to +\infty$ and $\Phi(u_n) \leq c$. As in the proof of Lemma 2.3, we can show that $v_n = \dfrac{u_n}{||u_n||} \to \pm\varphi_1$. By the definition of μ_1, we have

$$0 \leq \int_\Omega |\nabla u_n|^p dx + \int_\Omega m_1 |u_n|^p dx - \mu_1 \int_{\partial\Omega} m_2 |u_n|^p d\sigma.$$

Thus

$$-\int_{\partial\Omega} F(x, u_n(x)) d\sigma + \int_{\partial\Omega} h u_n d\sigma \leq \Phi(u_n) \leq c. \tag{2.3}$$

Case 1: Suppose that $v_n \to \varphi_1$. Dividing (2.3) by $||u_n||$, we obtain

$$-\int_{\partial\Omega} \frac{F(x, u_n(x))}{||u_n||} d\sigma + \int_{\partial\Omega} \frac{h u_n}{||u_n||} d\sigma \leq \frac{\Phi(u_n)}{||u_n||} \leq \frac{c}{||u_n||}.$$

Passing to the limit, we get

$$-\int_{\partial\Omega} \beta(x)\varphi_1 d\sigma + \int_{\partial\Omega} h(x)\varphi_1 d\sigma \leq 0,$$

which contradicts (1.4).

Case 2: Assume that $v_n \to -\varphi_1$. Dividing (2.3) by $||u_n||$, we obtain

$$-\int_{\partial\Omega} \frac{F(x, u_n(x))}{||u_n||} d\sigma + \int_{\partial\Omega} \frac{h u_n}{||u_n||} d\sigma \leq \frac{\Phi(u_n)}{||u_n||} \leq \frac{c}{||u_n||}.$$

Passing to the limit, we get

$$\int_{\partial\Omega} \alpha(x)\varphi_1 d\sigma - \int_{\partial\Omega} h(x)\varphi_1 d\sigma \leq 0,$$

which contradicts (1.4). □

Proof of Theorem 2.2. Assumption (1.2) implies that Φ is a C^1-functional on $W^{1,p}(\Omega)$. By Lemma 2.3, Φ satisfies the Palais–Smale condition and it is bounded from below by Lemma 2.4. Furthermore, we proved in Theorem 2.1 that Φ attains its proper infimum in $W^{1,p}(\Omega)$. We conclude that problem (1.6) admits at least a weak solution. □

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