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First Level's Connection-to-Stokes Formulae for Meromorphic Linear Differential Systems

Pascal Remy
6 rue Chantal Mauduit
F-78 420 Carrières-sur-Seine, France
pascal.remy07@orange.fr

Abstract

Given a multi-leveled meromorphic linear differential system, we deduce from the factorization theorem explicit formulæ allowing to express *all* the first level's Stokes multipliers in terms of connection constants in the Borel plane, generalizing thus the formulæ displayed by M. Loday–Richaud and the author in the case of single-leveled systems. As an illustration, we develop three examples. No assumption of genericity is made.

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1 Introduction

All along the article, we are given a linear differential system (in short, a differential system or a system) of dimension $n \geqslant 2$ with meromorphic coefficients of order $r+1$ at 0 in $\mathbb{C}$, $r \in \mathbb{N}^*$, of the form

$$x^{r+1}\frac{dY}{dx} = A(x)Y,\ A(x) \in M_n(\mathbb{C}\{x\}),\ A(0) \neq 0 \tag{1.1}$$

together with a formal fundamental solution at 0

$$\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$$

normalized as follows:

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- $\widetilde{F}(x) \in M_n(\mathbb{C}[[x]])$ is a formal power series in x satisfying $\widetilde{F}(x) = I_n + O(x^{r_1})$, where I_n is the identity matrix of size n and where r_1 is an integer $\geqslant 1$ fixed below,
- $L = \bigoplus_{j=1}^{J}(\lambda_j I_{n_j} + J_{n_j})$ where J is an integer $\geqslant 2$, the eigenvalues λ_j verify $0 \leqslant \mathrm{Re}(\lambda_j) < 1$ and where

$$J_{n_j} = \begin{cases} 0 & \text{if } n_j = 1 \\ \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix} & \text{if } n_j \geqslant 2 \end{cases}$$

is an irreducible Jordan block of size n_j,

- $Q(1/x)$ is a diagonal matrix with polynomial entries in $1/x$ of the form

$$Q\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J} q_j\left(\frac{1}{x}\right) I_{n_j}, \quad q_j\left(\frac{1}{x}\right) \in \frac{1}{x}\mathbb{C}\left[\frac{1}{x}\right].$$

Recall that any meromorphic linear differential system with an irregular singular point at 0 can always be reduced to System (1.1) by means of a finite algebraic extension $x \longmapsto x^{\nu}$, $\nu \in \mathbb{N}^*$, of the variable x and a meromorphic gauge transformation $Y \longmapsto T(x)Y$ where $T(x)$ has explicit computable polynomial entries in x and $1/x$ [2].

In addition, we suppose that there exist j and ℓ such that $q_j \not\equiv q_\ell$, otherwise $\widetilde{F}(x)$ is a convergent series and System (1.1) has no Stokes phenomenon.

Under the hypothesis that System (1.1) has the *unique level* $r \geqslant 1$ (see Def. 2.1 below for the exact definition of levels), M. Loday–Richaud and the author displayed in [9] (case $r = 1$) and [16] (case $r \geqslant 2$) formulæ making explicit the Stokes multipliers of $\widetilde{F}(x)$ in terms of connection constants in the Borel plane. More precisely, these constants are given by the singularities of the Borel transforms $\widehat{\boldsymbol{F}}^{[u]}(\tau)$ of the sub-series $\widetilde{\boldsymbol{F}}^{[u]}(t)$, $u = 0, \ldots, r-1$ and $t = x^r$, of terms r by r of $\widetilde{F}(x)$, also called *r-reduced series* of $\widetilde{F}(x)$.

In the present paper, we suppose that System (1.1) is a *multi-leveled system*. Our aim is to make explicit formulæ similar to those in [9, 16] for the *first level's Stokes multipliers of* $\widetilde{F}(x)$ (Section 3.6, Theorem 3.12), i.e., the Stokes multipliers of $\widetilde{F}(x)$ associated with the smallest level $r_1 \geqslant 1$ of System (1.1).

Such formulæ, obtained by various integral methods such as Cauchy-Heine integral and Laplace transform, were already given by many authors under sufficiently generic hypothesis (see [1, 3, 4] for instance).

Here, besides no assumption of genericity is made, our approach is quite different and is based on the factorization theorem of $\widetilde{F}(x)$ (see [7, 14, 15], Section 2.3 below) and on the results of [9, 16].

More precisely, we proceed in two steps. First, we show that a "good normalization" of the r_1-summable factor of $\widetilde{F}(x)$ allows to see the first level's Stokes multipliers of $\widetilde{F}(x)$ as Stokes multipliers of convenient systems with a single level equal to r_1 (Section 3.2). Thus, according to [9, 16], the first level's Stokes multipliers of $\widetilde{F}(x)$ are expressed in terms of connection constants in the Borel plane relative to these single-leveled systems.

Second, we prove that these connection constants are actually given by the singularities of the Borel transforms $\widehat{\boldsymbol{F}}^{[u]}(\tau)$, $u = 0, \ldots, r_1 - 1$, of the r_1-reduced series of $\widetilde{F}(x)$ (Sections 3.4 and 3.5). To this end, we prove a resurgence theorem for the r_1-reduced series $\widetilde{\boldsymbol{F}}^{[u]}(t)$ of $\widetilde{F}(x)$ (Theorem 3.7) and we display a precise description of the singularities of the Borel transforms $\widehat{\boldsymbol{F}}^{[u]}(\tau)$ (Theorem 3.9).

In Section 4, as an illustration of the first level's connection-to-Stokes formulæ, we develop three examples.

2 Preliminaries

2.1 Some Definitions and Notations

We recall here below some definitions about levels and singular directions $-$also called anti-Stokes directions$-$ of System (1.1).

- Given a pair (q_j, q_ℓ) such that $q_j \not\equiv q_\ell$, we denote

$$(q_j - q_\ell)\left(\frac{1}{x}\right) = -\frac{\alpha_{j,\ell}}{x^{r_{j,\ell}}} + o\left(\frac{1}{x^{r_{j,\ell}}}\right), \quad \alpha_{j,\ell} \neq 0.$$

Definition 2.1 (Levels of System (1.1))**.** All the degrees $r_{j,\ell}$ of polynomials $q_j - q_\ell \not\equiv 0$ are called *levels of System* (1.1). Notice that, according to normalizations of System (1.1), levels are integers. One sometimes refers to this case as *the unramified case*.

We denote by $R := \{r_1 < r_2 < \ldots < r_p\}$, $p \in \mathbb{N}^*$, the set of all levels of System (1.1). Notice that $r_1 \geqslant 1$ and $r_p \leqslant r$ the rank of System (1.1). Actually, if $r_p < r$, all the polynomials q_j, $j = 1, \ldots, J$, have the same degree r and the terms of highest degree coincide. One then reduces this case to the case $r_p = r$ by means of a change of unknown vector of the form $Y = Ze^{q(1/x)}$ with a convenient polynomial $q(1/x) \in x^{-1}\mathbb{C}[x^{-1}]$. Recall that such a change does not affect levels or Stokes–Ramis matrices of System (1.1).

When $p = 1$, System (1.1) is said to be *with the unique level* r_1. Recall that, for such a system, the connection-to-Stokes formulæ were already displayed in [9] (case $r_1 = 1$)

and [16] (case $r_1 \geqslant 2$). Henceforth, we suppose $p \geqslant 2$, i.e., System (1.1) has at least two levels.

• Let us now split the matrix $\widetilde{F}(x)$ into J column-blocks

$$\widetilde{F}(x) = \left[\widetilde{F}^{\bullet;1}(x) \quad \widetilde{F}^{\bullet;2}(x) \quad \cdots \quad \widetilde{F}^{\bullet;J}(x)\right]$$

fitting to the Jordan structure of L (the size of $\widetilde{F}^{\bullet;\ell}(x)$ is $n \times n_\ell$ for all ℓ).

Definition 2.2 (Anti-Stokes directions, Stokes values)**.**

1. The *anti-Stokes directions* of System (1.1) (or $\widetilde{F}(x)$) are the directions of maximal decay of the exponentials $e^{(q_j-q_\ell)(1/x)}$ with $q_j - q_\ell \not\equiv 0$. The coefficients $\alpha_{j,\ell}$ generating these directions are called *Stokes values* of System (1.1). The k^{th} *level's anti-Stokes directions of System* (1.1) *(or* $\widetilde{F}(x)$*)* are the anti-Stokes directions of System (1.1) given by the exponentials $e^{(q_j-q_\ell)(1/x)}$ with $r_{j,\ell} = r_k$. In this case, $\alpha_{j,\ell}$ is called k^{th} *level's Stokes value of System* (1.1).

2. Let $\ell \in \{1, \ldots, J\}$. The *anti-Stokes directions associated with* $\widetilde{F}^{\bullet;\ell}(x)$ are the anti-Stokes directions of $\widetilde{F}(x)$ given by the exponentials $e^{(q_j-q_\ell)(1/x)}$ for all j such that $q_j - q_\ell \not\equiv 0$. The k^{th} *level's anti-Stokes directions associated with* $\widetilde{F}^{\bullet;\ell}(x)$ are the anti-Stokes directions of $\widetilde{F}(x)$ given by the exponentials $e^{(q_j-q_\ell)(1/x)}$ for all j such that $q_j - q_\ell \not\equiv 0$ and $r_{j,\ell} = r_k$. In this case, $\alpha_{j,\ell}$ is called k^{th} *level's Stokes value of System* (1.1) *associated with* $\widetilde{F}^{\bullet;\ell}(x)$.

Notice that a given anti-Stokes direction of System (1.1) or of $\widetilde{F}^{\bullet;\ell}(x)$ may be with several levels. Notice also that the denomination "anti-Stokes directions" is not universal. Indeed, such directions are called sometimes "Stokes directions".

2.2 Stokes–Ramis Automorphisms

Given a non anti-Stokes direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of System (1.1) and a choice of an argument of θ, say its principal determination $\theta^\star \in]-2\pi, 0]$[1], we consider the sum of $\widetilde{Y}$ in the direction θ given by

$$Y_\theta(x) = s_{r_1,r_2,\ldots,r_p;\theta}(\widetilde{F})(x) Y_{0,\theta^\star}(x)$$

where $s_{r_1,r_2,\ldots,r_p;\theta}(\widetilde{F})$ is the uniquely determined $(r_1, r_2, \ldots, r_p)$-sum of $\widetilde{F}$ at θ and where $Y_{0,\theta^\star}(x)$ is the actual analytic function $Y_{0,\theta^\star}(x) := x^L e^{Q(1/x)}$ defined by the choice $\arg(x)$ close to $\theta^\star$ (denoted below $\arg(x) \simeq \theta^\star$). Recall that $s_{r_1,r_2,\ldots,r_p;\theta}(\widetilde{F})$ is an analytic function defined on a sector bisected by θ with opening larger than π/r_p [12].

[1] Any choice is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leqslant \arg(z = 1/x) < 2\pi$ of the principal determination at infinity, we suggest to choose $-2\pi < \arg(x) \leqslant 0$ as principal determination about 0 as well as about any ω at finite distance.

When $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is an anti-Stokes direction of System (1.1), we consider the two lateral sums $s_{r_1,r_2,\ldots,r_p;\theta^-}(\widetilde{F})$ and $s_{r_1,r_2,\ldots,r_p;\theta^+}(\widetilde{F})$ respectively obtained as analytic continuations of $s_{r_1,r_2,\ldots,r_p;\theta-\varepsilon}(\widetilde{F})$ and $s_{r_1,r_2,\ldots,r_p;\theta+\varepsilon}(\widetilde{F})$ to a sector with vertex 0, bisected by θ and opening π/r_p. Notice that such analytic continuations exist without ambiguity when $\varepsilon > 0$ is small enough. We denote by Y_{θ^-} and Y_{θ^+} the two sums of $\widetilde{Y}$ respectively defined for $\arg(x) \simeq \theta^\star$ by $Y_{\theta^-}(x) := s_{r_1,r_2,\ldots,r_p;\theta^-}(\widetilde{F})(x)Y_{0,\theta^\star}(x)$ and $Y_{\theta^+}(x) := s_{r_1,r_2,\ldots,r_p;\theta^+}(\widetilde{F})(x)Y_{0,\theta^\star}(x)$.

The two lateral sums $s_{r_1,r_2,\ldots,r_p;\theta^-}(\widetilde{F})$ and $s_{r_1,r_2,\ldots,r_p;\theta^+}(\widetilde{F})$ of $\widetilde{F}$ are not analytic continuations from each other in general. This fact is the *Stokes phenomenon* of System (1.1). It is characterized by the collection, for all anti-Stokes directions $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of System (1.1), of the automorphisms

$$St_{\theta^\star} : Y_{\theta^+} \longmapsto Y_{\theta^-}$$

that one calls *Stokes-Ramis automorphisms* relative to $\widetilde{Y}$.

The *Stokes-Ramis matrices* of System (1.1) are defined as matrix representations in $GL_n(\mathbb{C})$ of the $St_{\theta^\star}$'s.

Definition 2.3 (Stokes–Ramis matrices)**.** One calls the matrix of $St_{\theta^\star}$ in the basis Y_{θ^+} [2] the *Stokes–Ramis matrix* associated with $\widetilde{Y}$ in the direction θ. We denote it by $I_n + C_{\theta^\star}$.

Notice that the matrix $I_n + C_{\theta^\star}$ is uniquely determined by the relation

$$\boxed{Y_{\theta^-}(x) = Y_{\theta^+}(x)(I_n + C_{\theta^\star}) \qquad \text{for } \arg(x) \simeq \theta^\star.}$$

Split the matrix $C_{\theta^\star} = [C_{\theta^\star}^{j;\ell}]$ into blocks fitting to the Jordan structure of L ($C_{\theta^\star}^{j;\ell}$ is a $n_j \times n_\ell$-matrix). The block $C_{\theta^\star}^{j;\ell}$ is zero as soon as $e^{(q_j - q_\ell)(1/x)}$ is not flat in the direction θ. When $e^{(q_j - q_\ell)(1/x)}$ is flat in the direction θ and $r_{j,\ell} (= \deg(q_j - q_\ell)) = r_k$, the entries of the block $C_{\theta^\star}^{j;\ell}$ are called k^{th} *level's Stokes multipliers* of $\widetilde{F}^{\bullet;\ell}(x)$ *in the direction* θ.

Recall that the aim of this article is to display formulæ making explicit the first level's Stokes multipliers in terms of connection constants in the Borel plane. Our approach is based on the factorization theorem of $\widetilde{F}(x)$ which we recall in Section 2.3 below.

2.3 Factorization Theorem and Stokes–Ramis Matrices

The factorization theorem (Theorem 2.4 below) states that $\widetilde{F}(x)$ can be written essentially uniquely as a product of r_k-summable formal series $\widetilde{F}_k(x)$ for the different levels

[2] In the literature, a Stokes matrix has a more general meaning where one allows to compare any two asymptotic solutions whose domains of definition overlap. According to the custom initiated by J.-P. Ramis [15] in the spirit of Stokes' work, we exclude this case here. We consider only matrices providing the transition between the sums on each side of a same anti-Stokes direction.

r_k of System (1.1). It was first proved by J.-P. Ramis in [14, 15] by using a technical way based on Gevrey estimates. A quite different proof based on Stokes cocycles and mainly algebraic was given later by M. Loday–Richaud in [7]. Both proofs are nonconstructive. However, as we shall see in Section 3, the factorization theorem provides sufficient information about the first level to allow to make explicit the *first level's connection-to-Stokes formulæ* in full generality.

Theorem 2.4 (Factorization theorem, [7,14,15]). *Let* $R = \{r_1 < r_2 < \ldots < r_p\}$ *denote the set of levels of System* (1.1) [3]. *Then* $\widetilde{F}(x)$ *can be factored in*

$$\widetilde{F}(x) = \widetilde{F}_p(x) \cdots \widetilde{F}_2(x)\widetilde{F}_1(x),$$

where, for all $k = 1, \ldots, p$, $\widetilde{F}_k(x) \in M_n(\mathbb{C}[[x]])$ *is a* r_k*-summable formal series with singular directions the* k^{th} *level's anti-Stokes directions of System* (1.1)*. This factorization is essentially unique: Let*

$$\widetilde{F}(x) = \widetilde{G}_p(x) \cdots \widetilde{G}_2(x)\widetilde{G}_1(x)$$

be another decomposition of $\widetilde{F}(x)$*. Then, there exist* $p-1$ *invertible matrices*

$$P_1(x), \ldots, P_{p-1}(x) \in GL_n(\mathbb{C}\{x\}[x^{-1}])$$

with meromorphic entries at 0 *such that* $\widetilde{G}_1 = P_1\widetilde{F}_1$, $\widetilde{G}_k = P_k\widetilde{F}_kP_{k-1}^{-1}$ *for* $k = 2, \ldots, p-1$ *and* $\widetilde{G}_p = \widetilde{F}_pP_{p-1}^{-1}$*. In particular, we can always choose* $\widetilde{F}_k$ *so that* $\widetilde{F}_k(x) = I_n + O(x^{r_1})$ *for all* $k = 1, \ldots, p$ [4].

Denote $\widetilde{G}(x) := \widetilde{F}_p(x) \cdots \widetilde{F}_2(x)$. Denote also by

$$A_1(x) := \widetilde{G}^{-1}A(x)\widetilde{G} - x^{r+1}\widetilde{G}^{-1}\frac{d\widetilde{G}}{dx}$$

the matrix of the system obtained from System (1.1) by the formal gauge transformation $Y = \widetilde{G}(x)Y_1$. Then [7], $A_1(x)$ is analytic at 0 and the matrix $\widetilde{Y}_1(x) := \widetilde{F}_1(x)x^Le^{Q(1/x)}$ is a formal fundamental solution of the system

$$x^{r+1}\frac{dY}{dx} = A_1(x)Y. \tag{2.1}$$

Notice that System (2.1) has, like System (1.1), the levels $r_1 < r_2 < \ldots < r_p$. Notice also that $\widetilde{Y}_1(x)$ has same normalizations as $\widetilde{Y}(x)$.

[3] Recall that we suppose $p \geqslant 2$ in this paper.

[4] Actually, such conditions, like the initial condition $\widetilde{F}(x) = I_n + O(x^{r_1})$, allow us to have "good" normalizations for the r_1-reduced series and thus to simplify calculations below (see Sections 3.3 to 3.6).

The structure of $A_1(x)$ will be made precise in Theorem 3.3 below. In particular, we shall show that the matrix $A_1(x)$ (and, consequently, the matrix $\widetilde{F}_1(x)$) can always be chosen with a convenient "block-diagonal form".

Consider now $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ a first level's anti-Stokes direction of System (1.1). Recall that θ may also be a k^{th} level's anti-Stokes direction for some $k \in \{2, \ldots, p\}$.

By construction, θ is also a first level's anti-Stokes direction of System (2.1). Denote then by $I_n + C_{1,\theta^\star}$ the Stokes–Ramis matrix associated with $\widetilde{Y}_1$ in the direction θ and split as before $C_{1,\theta^\star} = [C_{1,\theta^\star}^{j;\ell}]$ into blocks $C_{1,\theta^\star}^{j;\ell}$ of size $n_j \times n_\ell$ fitting to the Jordan structure of L. Recall that $C_{1,\theta^\star}^{j;\ell} = 0$ as soon as $e^{(q_j - q_\ell)(1/x)}$ is not flat in the direction θ. Proposition 2.5 below determines the Stokes multipliers of $\widetilde{F}_1(x)$ in the direction θ.

Proposition 2.5 (See [7, 13, 15])**.** *Let $j, \ell \in \{1, \ldots, J\}$ be such that $e^{(q_j - q_\ell)(1/x)}$ is flat in the direction θ. Let $r_{j,\ell}$ denote the degree of $(q_j - q_\ell)(1/x)$ (see Section 2.1). Then,*

$$\boxed{C_{1,\theta^\star}^{j;\ell} = \begin{cases} C_{\theta^\star}^{j;\ell} & \textit{if } r_{j,\ell} = r_1 \\ 0_{n_j \times n_\ell} & \textit{if } r_{j,\ell} \in \{r_2, \ldots, r_p\}. \end{cases}}$$

In other words, Proposition 2.5 states that

1. the nontrivial Stokes multipliers of the ℓ^{th} column-block $\widetilde{F}_1^{\bullet;\ell}(x)$ are those of the first level,
2. the first level's Stokes multipliers of $\widetilde{F}_1^{\bullet;\ell}(x)$ and $\widetilde{F}^{\bullet;\ell}(x)$ coincide.

3 Main Results

Any of the J column-blocks $\widetilde{F}^{\bullet;\ell}(x)$ ($\ell = 1, \ldots, J$) of $\widetilde{F}(x)$ associated with the Jordan structure of L (matrix of exponents of formal monodromy) can be positioned at the first place by means of a permutation P on the columns of $\widetilde{Y}(x)$. Observe that the same permutation P acting on the rows of $\widetilde{Y}(x)$ allows to keep initial normalizations of $\widetilde{Y}(x)$. More precisely, the new formal fundamental solution $P\widetilde{Y}(x)P$ reads $P\widetilde{Y}(x)P = P\widetilde{F}(x)Px^{P^{-1}LP}e^{P^{-1}Q(1/x)P}$ with $P\widetilde{F}(x)P = I_n + O(x^{r_1})$.

Thereby, we can restrict our study to the first column-block $\widetilde{F}^{\bullet;1}(x)$ denoted below $\widetilde{f}(x)$ (the size of $\widetilde{f}(x)$ is $n \times n_1$). Note that $\widetilde{f}(x) = I_{n,n_1} + O(x^{r_1})$, where I_{n,n_1} denotes the first n_1 columns of the identity matrix I_n.

Remark 3.1. It is worth to notice here that, by means of a convenient permutation on the columns and the rows with indices $\geqslant n_1 + 1$ of $\widetilde{Y}(x)$, we can always order the polynomials q_j, $j = 2, \ldots, J$, as we want, while maintaining the initial normalizations of $\widetilde{Y}(x)$ and the first place of $\widetilde{f}(x)$.

3.1 Setting the Problem

In addition to normalizations of $\widetilde{Y}(x)$, we suppose that

$$\lambda_1 = 0 \quad \text{and} \quad q_1 \equiv 0, \tag{3.1}$$

conditions that can be always fulfilled by means of the change of unknown vector $Y = x^{\lambda_1} e^{q_1(1/x)} Z$.

According to (3.1), the anti-Stokes directions of System (1.1) associated with $\widetilde{f}(x)$ are the directions of maximal decay of the exponentials $e^{q_j(1/x)}$ with $q_j \not\equiv 0$ (cf. Def. 2.2, 2.). Denote then by

$$R' := \{r'_1 < \ldots < r'_{p'}\}, \quad p' \geqslant 1,$$

the set of degrees in $1/x$ of polynomials $q_j \not\equiv 0$. Obviously, $R' \subseteq R$ (the degrees r'_j's are levels of System (1.1)), $r'_{p'} = r_p$ the highest level of System (1.1) and $r_1 \leqslant r'_1 \leqslant r_p$. Notice that, when $r'_1 > r_1$, there exists no first level's anti-Stokes direction (hence, no first level's Stokes multipliers) for $\widetilde{f}(x)$. Henceforward, we suppose $p' \geqslant 2$ and $r'_1 = r_1$.

The aim of Section 3 is to display formulæ making explicit the first level's Stokes multipliers of $\widetilde{f}(x)$ in terms of the connection constants of the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ of the r_1-reduced series $\widetilde{\boldsymbol{f}}^{[u]}(t)$ of $\widetilde{f}(x)$ (Theorem 3.12), generalizing thus formulæ given in [9, 16] for single-leveled systems.

Recall that the r_1-reduced series of $\widetilde{f}(x) \in M_{n,n_1}(\mathbb{C}[[x]])$ are the formal series $\widetilde{\boldsymbol{f}}^{[u]}(t) \in M_{n,n_1}(\mathbb{C}[[t]])$, $u = 0, \ldots, r_1 - 1$, defined by the relation

$$\widetilde{f}(x) = \widetilde{\boldsymbol{f}}^{[0]}(x^{r_1}) + x\widetilde{\boldsymbol{f}}^{[1]}(x^{r_1}) + \ldots + x^{r_1-1}\widetilde{\boldsymbol{f}}^{[r_1-1]}(x^{r_1}). \tag{3.2}$$

Notice that the normalization $\widetilde{f}(x) = I_{n,n_1} + O(x^{r_1})$ implies $\widetilde{\boldsymbol{f}}^{[0]}(t) = I_{n,n_1} + O(t)$ and $\widetilde{\boldsymbol{f}}^{[u]}(t) = O(t)$ for $u = 1, \ldots, r_1 - 1$.

Our approach is based on the relation between $\widetilde{F}(x)$ and $\widetilde{F}_1(x)$ (Factorization Theorem 2.4 and Proposition 2.5) and on Block-Diagonalisation Theorem 3.3 below allowing to "reduce" System (2.1) into a convenient single-leveled system.

3.2 A Block-Diagonalisation Theorem

According to Remark 2.1, we suppose from now on that the polynomials q_j for $j = 2, \ldots, J$ are ordered so that the matrix Q read in the form

$$Q = Q_1 \oplus Q_2 \oplus \ldots \oplus Q_{p'} \tag{3.3}$$

where

- Q_1 is formed by all the polynomials $q_j \equiv 0$ and all the polynomials q_j of degree r_1, i.e., by all the polynomials q_j of degrees $\leqslant r_1$,

- for $k = 2, \ldots, p'$, Q_k is formed by all the polynomials q_j of degree r'_k and its leading term $\mathcal{Q}_k := x^{r'_k} Q_k \mid_{x=0}$ has a block-decomposition of the form $\bigoplus_{\ell=1}^{s_k} Q_{k,\ell} I_{m_{k,\ell}}$ with $Q_{k,\ell} \in \mathbb{C}^*$ and $Q_{k,\ell} \neq Q_{k,\ell'}$ if $\ell \neq \ell'$.

We denote by N_k, $k = 1, \ldots, p'$, the size of the square matrix Q_k and we split the matrix L of exponents of formal monodromy like Q:

$$L = L_1 \oplus L_2 \oplus \ldots \oplus L_{p'} \text{ with } L_k \in M_{N_k}(\mathbb{C}).$$

Observe that each sub-matrix L_k has a Jordan structure induced by the one of L.

Block-Diagonalisation Theorem 3.3 below states that, up to analytic gauge transformation, System (2.1) can be split into p' sub-systems fitting to the block-decomposition (3.3), i.e., the matrix $A_1(x)$ can be reduced into a block-diagonal form like Q.

Recall that a (formal, meromorphic) gauge transformation $Z = T(x)W$ transforms any system of the form

$$x^{r+1} \frac{dW}{dx} = \mathcal{A}(x) W$$

into the system

$$x^{r+1} \frac{dZ}{dx} = {}^{T}\mathcal{A}(x) Z, \quad \text{where} \quad {}^{T}\mathcal{A}(x) = T\mathcal{A}(x)T^{-1} + x^{r+1} \frac{dT}{dx} T^{-1}.$$

Let us start with a technical lemma based on the results of [10].

Lemma 3.2. *Let $d \in \{2, \ldots, p'\}$. Denote*

- $N_{<d} = N_1 + \ldots + N_{d-1}$ *and* $N_{\leqslant d} = N_{<d} + N_d$,
- $L_{<d} = L_1 \oplus \ldots \oplus L_{d-1}$ *and* $L_{\leqslant d} = L_{<d} \oplus L_d$,
- $Q_{<d} = Q_1 \oplus \ldots \oplus Q_{d-1}$ *and* $Q_{\leqslant d} = Q_{<d} \oplus Q_d$.

Consider a system

$$x^{r'_d+1} \frac{dW}{dx} = \mathcal{A}(x) W, \quad \mathcal{A}(x) \in M_{N_{\leqslant d}}(\mathbb{C}\{x\}) \tag{3.4}$$

together with a formal fundamental solution at 0 of the form

$$\widetilde{W}(x) = \widetilde{H}(x) x^{L_{\leqslant d}} e^{Q_{\leqslant d}(1/x)}$$

where $\widetilde{H}(x) \in M_{N_{\leqslant d}}(\mathbb{C}[[x]])$ verifies $\widetilde{H}(x) = I_{N_{\leqslant d}} + O(x^{r_1})$. Suppose that $\widetilde{H}(x)$ is r_1-summable. Then, there exists an invertible matrix $T_d(x) \in GL_{N_{\leqslant d}}(\mathbb{C}\{x\})$ with analytic entries at 0 such that

1. $T_d(x) = I_{N_{\leqslant d}} + O(x^{r_1})$,

2. *the gauge transformation $Z = T_d(x)W$ transforms System* (3.4) *into a system*

$$x^{r'_d+1}\frac{dZ}{dx} = \begin{bmatrix} \mathcal{A}_{<d}(x) & 0 \\ 0 & \mathcal{A}_d(x) \end{bmatrix} Z \tag{3.5}$$

with $\mathcal{A}_{<d}(x) \in M_{N_{<d}}(\mathbb{C}\{x\})$ and $\mathcal{A}_d(x) \in M_{N_d}(\mathbb{C}\{x\})$,

3. *the formal fundamental solution $\widetilde{Z}(x) = T_d(x)\widetilde{W}(x)$ of System* (3.5) *has a block-diagonal decomposition*

$$\widetilde{Z}(x) = \widetilde{H}_{<d}(x)x^{L_{<d}}e^{Q_{<d}(1/x)} \oplus \widetilde{H}_d(x)x^{L_d}e^{Q_d(1/x)}$$

where

   (a) *the formal series $\widetilde{H}_{<d}(x) \in M_{N_{<d}}(\mathbb{C}[[x]])$ and $\widetilde{H}_d(x) \in M_{N_d}(\mathbb{C}[[x]])$ verify $\widetilde{H}_{<d}(x) = \widetilde{H}_d(x) = I_* + O(x^{r_1})$,*

   (b) *the matrix $\widetilde{Z}_{<d}(x) = \widetilde{H}_{<d}(x)x^{L_{<d}}e^{Q_{<d}(1/x)}$ is a formal fundamental solution of the system*

$$x^{r'_{d-1}+1}\frac{dZ_{<d}}{dx} = \mathcal{A}_{<d}(x)Z_{<d}, \tag{3.6}$$

   (c) *the matrix $\widetilde{Z}_d(x) = \widetilde{H}_d(x)x^{L_d}e^{Q_d(1/x)}$ is a formal fundamental solution of the system*

$$x^{r'_d+1}\frac{dZ_d}{dx} = \mathcal{A}_d(x)Z_d.$$

Moreover, both formal series $\widetilde{H}_{<d}(x)$ and $\widetilde{H}_d(x)$ are r_1-summable.

Proof. Since $\widetilde{H}(0) = I_{N_{\leqslant d}}$, the matrix $\mathcal{A}(x)$ of System (3.4) reads

$$\mathcal{A}(x) = x^{r'_d+1}\frac{dQ_{\leqslant d}}{dx} + x^{r'_d}\mathcal{B}(x)$$

with $\mathcal{B}(x)$ analytic at 0. Hence, according to the block-decomposition (3.3) of the matrix Q, the heading term $\mathcal{A}(0) = 0_{N_{<d}} \oplus (-r'_d\mathcal{Q}_d)$ of $\mathcal{A}(x)$ has the block-decomposition

$$\mathcal{A}(0) = 0_{N_{<d}} \oplus \left(\bigoplus_{\ell=1}^{s_d} -r'_d Q_{d,\ell} I_{m_{d,\ell}}\right)$$

with $Q_{k,\ell} \neq 0$ and $Q_{k,\ell} \neq Q_{k,\ell'}$ if $\ell \neq \ell'$. Thus, by applying [10, Thm. 1.5], there exists an invertible matrix $T_{d,1}(x) \in GL_{N_{\leqslant d}}(\mathbb{C}[[x]]_{1/r'_d}[x^{-1}])$ with meromorphic $1/r'_d$-Gevrey entries at 0 [5] such that the matrix ${}^{T_{d,1}}\mathcal{A}(x)$ has a block-decomposition like $\mathcal{A}(0)$.

[5] Recall that a series $\sum a_m x^m \in \mathbb{C}[[x]]$ is said to be $1/k$-*Gevrey* and denoted $\sum a_m x^m \in \mathbb{C}[[x]]_{1/k}$ when the series $\sum \frac{a_m}{(m!)^{1/k}} x^m$ is convergent.

Observe that the entries of ${}^{T_{d,1}}\mathcal{A}(x)$ are in general meromorphic $1/r'_d$-Gevrey and not convergent. Denote then by $\mathcal{A}^{(\ell)}(x)$, $\ell = 0, \ldots, s_d$, the blocks of ${}^{T_{d,1}}\mathcal{A}(x)$. By construction, the sub-systems

$$x^{r'_d+1}\frac{dW}{dx} = \mathcal{A}^{(\ell)}(x)W, \quad \ell = 0, \ldots, s_d$$

have levels $< r'_d$. Therefore, [10, Thm. 1.4] applies: for all $\ell = 0, \ldots, s_d$, there exists an invertible matrix $T^{(\ell)}_{d,2}(x)$ with meromorphic $1/r'_d$-Gevrey entries at 0 such that the matrix ${}^{T^{(\ell)}_{d,2}}\mathcal{A}^{(\ell)}(x)$ has meromorphic entries at 0. Finally, by normalizing if neccessary the formal fundamental solutions of these last systems by means of convenient polynomial gauge transformations in x and $1/x$, we deduce from calculations above that there exists a matrix $T_d(x) \in GL_{N_{\leqslant d}}(\mathbb{C}[[x]]_{1/r'_d}[x^{-1}])$ satisfying Points 2. and 3. of Lemma 3.2. Notice that Point 1. results from equalities

$$T_d(x)\widetilde{H}(x) = \widetilde{H}_{<d}(x) \oplus \widetilde{H}_d(x) = I_{N_{\leqslant d}} + O(x^{r_1}) \tag{3.7}$$

and from the assumption $\widetilde{H}(x) = I_{N_{\leqslant d}} + O(x^{r_1})$. Notice also that, by construction, the formal series $\widetilde{H}_{<d}(x)$ and $\widetilde{H}_d(x)$ are both summable of levels $< r'_d$. In particular, the first equality of (3.7) and the hypothesis "$\widetilde{H}(x)$ is r_1-summable" show that $T_d(x)$ is both $1/r'_d$-Gevrey and summable of levels $< r'_d$ (indeed, $r_1 < r'_d$ for all $d = 2, \ldots, p'$). Thus, due to [12, Prop. 7, p. 349], $T_d(x)$ is analytic at 0. Therefore, $T_d(x)\widetilde{H}(x)$ keeps being r_1-summable and, consequently, $\widetilde{H}_{<d}(x)$ and $\widetilde{H}_d(x)$ are also both r_1-summable. This ends the proof of Lemma 3.2. □

Note that the hypothesis "$\widetilde{H}(x)$ is r_1-summable" plays a fundamental role in the proof of Lemma 3.2. Note also that Lemma 3.2 can be again applied to sub-system (3.6) when $d \geqslant 3$... and so on as long as $d \neq 2$.

In the case of System (2.1), an iterative application of Lemma 3.2 starting with $d = p'$ allows us to state the following result:

Theorem 3.3 (Block-diagonalisation theorem)**.** *There exists an invertible matrix* $T(x) \in GL_n(\mathbb{C}\{x\})$ *with analytic entries at* 0 *such that*

1. $T(x) = I_n + O(x^{r_1})$,
2. *the gauge transformation* $Z_1 = T(x)Y_1$ *transforms System* (2.1) *into a system*

$$x^{r+1}\frac{dZ}{dx} = {}^{T}A_1(x)Z, \tag{3.8}$$

where the matrix ${}^{T}A_1(x) \in M_n(\mathbb{C}\{x\})$ *has a block-diagonal decomposition like* Q*:*

$${}^{T}A_1(x) = \bigoplus_{k=1}^{p'} A_{1,k}(x) \textit{ with } A_{1,k}(x) \in M_{N_k}(\mathbb{C}\{x\}),$$

3. *the formal fundamental solution* $\widetilde{Z}_1(x) = T(x)\widetilde{Y}_1(x)$ *of System* (3.8) *has a block-diagonal decomposition*

$$\widetilde{Z}_1(x) = \bigoplus_{k=1}^{p'} \widetilde{F}_{1,k}(x) x^{L_k} e^{Q_k(1/x)}$$

where, for all $k = 1, \dots, p'$,

   (a) $\widetilde{F}_{1,k}(x) \in M_{N_k}(\mathbb{C}[[x]])$ *verifies* $\widetilde{F}_{1,k}(x) = I_{N_k} + O(x^{r_1})$,

   (b) the matrix $\widetilde{Z}_{1,k}(x) = \widetilde{F}_{1,k}(x) x^{L_k} e^{Q_k(1/x)}$ *is a formal fundamental solution of the system*

$$x^{r'_k+1} \frac{dZ_{1,k}}{dx} = A_{1,k}(x) Z_{1,k} \tag{3.9}$$

   (recall that r'_k *is the degree of* Q_k, $r'_1 = r_1$ *and* $r'_{p'} = r_p = r$*).*

In particular, the matrix $T(x)\widetilde{F}_1(x)$ *has the block-decomposition*

$$T(x)\widetilde{F}_1(x) = \bigoplus_{k=1}^{p'} \widetilde{F}_{1,k}(x)$$

and all the formal series $\widetilde{F}_{1,k}(x)$ *are* r_1*-summable.*

Notice that, by construction, System (3.9) has (multi)-levels $\leqslant r'_k$ when $k = 2, \dots, p'$ and has the unique level r_1 when $k = 1$ (indeed, r_1 is the smallest level of System (1.1), hence, of Systems (3.9) for all k).

Let us now make two remarks about the interest of Block-Diagonalisation Theorem 3.3:

1. Since $T(x)$ is analytic at 0, the "unicity" of Factorization Theorem 2.4 implies that we can respectively choose for $\widetilde{F}_1(x)$ and $A_1(x)$ the two matrices $\bigoplus_{k=1}^{p'} \widetilde{F}_{1,k}(x)$ and ${}^{T}A_1(x)$.

2. With these choices, Proposition 2.5 implies that the first level's Stokes multipliers of $\widetilde{f}(x)$ are actually the Stokes multipliers of the system with the unique level r_1

$$x^{r_1+1} \frac{dZ_{1,1}}{dx} = A_{1,1}(x) Z_{1,1} \tag{3.10}$$

   associated with the first n_1 columns $\widetilde{f}'(x)$ of $\widetilde{F}_{1,1}(x)$.

Denote as before by $\widetilde{\boldsymbol{f}}'^{[u]}(t)$, $u = 0, \dots, r_1 - 1$, the r_1-reduced series of $\widetilde{f}'(x)$ and by $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ their Borel transforms. According to Point 2. above and normalizations of the formal fundamental solution $\widetilde{Z}_{1,1}(x) = \widetilde{F}_{1,1}(x)x^{L_1}e^{Q_1(1/x)}$ of System (3.10) (cf. Thm. 3.3, 3.), [9, Thm. 4.3] and [16, Thm. 4.4] tell us that the first level's Stokes multipliers of $\widetilde{f}(x)$ are expressed in terms of the connection constants of the $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$'s.

Hence, to state the first level's connection-to-Stokes formulæ, we are left to prove that the connection constants of the $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$'s are also connection constants of the $\widehat{\boldsymbol{f}}^{[u]}(\tau)$'s. To this end, we shall compare the structure of the singularities of the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ and $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ for all $u = 0, \ldots, r_1 - 1$.

Lemma 3.4 below allows us to connect $\widehat{\boldsymbol{f}}^{[u]}$ and $\widehat{\boldsymbol{f}}'^{[u]}$.

3.3 A Fundamental Identity

According to Factorization Theorem 2.4, the first n_1 columns $\widetilde{f}(x)$ of $\widetilde{F}(x)$ are related to the first n_1 columns $\widetilde{f}'(x)$ of $\widetilde{F}_{1,1}(x)$ by the relation

$$\widetilde{f}(x) = \widetilde{F}_p(x) \cdots \widetilde{F}_2(x)\widetilde{f}_1(x), \quad \widetilde{f}_1(x) := \begin{bmatrix} \widetilde{f}'(x) \\ 0_{(N_2+\ldots+N_{p'})\times n_1} \end{bmatrix}$$

where

- $\widetilde{F}_k(x)$ is r_k-summable and $\widetilde{F}_k(x) = I_n + O(x^{r_1})$ for all $k = 2, \ldots, p$,
- $0_{(N_2+\ldots+N_{p'})\times n_1}$ denotes the null-matrix of size $(N_2 + \ldots + N_{p'}) \times n_1$.

Denote by

- $\widetilde{\boldsymbol{f}}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{[0]}(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}^{[r_1-1]}(t) \end{bmatrix} \in M_{r_1n,n_1}(\mathbb{C}[[t]])$ the matrix of size $r_1n \times n_1$ formed by the r_1-reduced series of $\widetilde{f}(x)$,
- $\widetilde{\boldsymbol{f}}_1^{[u]}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}'^{[u]}(t) \\ 0_{(N_2+\ldots+N_{p'})\times n_1} \end{bmatrix}$ for all $u = 0, \ldots, r_1 - 1$ and

$$\widetilde{\boldsymbol{f}}_1(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}_1^{[0]}(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}_1^{[r_1-1]}(t) \end{bmatrix} \in M_{r_1n,n_1}(\mathbb{C}[[t]]).$$

Denote also by $\widetilde{\boldsymbol{F}}_k^{[u]}(t)$, $u = 0, \ldots, r_1 - 1$, the r_1-reduced series of $\widetilde{F}_k(x)$.

Then, the r_1-reduced series $\widetilde{\boldsymbol{f}}^{[u]}(t)$ of $\widetilde{f}(x)$ are related to the r_1-reduced series $\widetilde{\boldsymbol{f}}'^{[u]}(t)$ of $\widetilde{f}'(x)$ by the relation

$$\widetilde{\boldsymbol{f}}(t) = \widetilde{\boldsymbol{F}}_p(t) \cdots \widetilde{\boldsymbol{F}}_2(t)\widetilde{\boldsymbol{f}}_1(t) \tag{3.11}$$

where

$$\widetilde{\boldsymbol{F}}_k(t) := \begin{bmatrix} \widetilde{\boldsymbol{F}}_k^{[0]}(t) & t\widetilde{\boldsymbol{F}}_k^{[r_1-1]}(t) & \cdots & \cdots & t\widetilde{\boldsymbol{F}}_k^{[1]}(t) \\ \widetilde{\boldsymbol{F}}_k^{[1]}(t) & \widetilde{\boldsymbol{F}}_k^{[0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \widetilde{\boldsymbol{F}}_k^{[0]}(t) & t\widetilde{\boldsymbol{F}}_k^{[r_1-1]}(t) \\ \widetilde{\boldsymbol{F}}_k^{[r_1-1]}(t) & \cdots & \cdots & \widetilde{\boldsymbol{F}}_k^{[1]}(t) & \widetilde{\boldsymbol{F}}_k^{[0]}(t) \end{bmatrix} \text{ for all } k.$$

Notice that $\widetilde{\boldsymbol{F}}_k(t) = I_{r_1 n} + O(t)$ and $\widetilde{\boldsymbol{F}}_k(t)$ is $\dfrac{r_k}{r_1}$-summable with $\dfrac{r_k}{r_1} > 1$ for all $k = 2, \ldots, p$. In particular, the Borel transform $\widehat{\boldsymbol{F}}_k(\tau)$ of $\widetilde{\boldsymbol{F}}_k(t)$ reads for all k in the form $\widehat{\boldsymbol{F}}_k = \delta I_{r_1 n} + \widehat{\boldsymbol{G}}_k$ with $\widehat{\boldsymbol{G}}_k$ an entire function on all $\mathbb{C}$ with exponential growth of order $\leqslant r_k/(r_k - r_1)$ at infinity [1, p. 81]. Denoting $r_{1,k} := r_k/(r_k - r_1)$, we have $r_{1,p} < \ldots < r_{1,2}$. Hence, since the Borel transformed identity of (3.11) reads

$$\widehat{\boldsymbol{f}} = \widehat{\boldsymbol{F}}_p * \ldots * \widehat{\boldsymbol{F}}_2 * \widehat{\boldsymbol{f}}_1,$$

the following lemma:

Lemma 3.4. *The Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ of $\widetilde{\boldsymbol{f}}^{[u]}(t)$ and the Borel transforms $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ of $\widetilde{\boldsymbol{f}}'^{[u]}(t)$ are related, for all $u = 0, \ldots, r_1 - 1$, by the relations*

$$\widehat{\boldsymbol{f}}^{[u]} = \begin{bmatrix} \widehat{\boldsymbol{f}}'^{[u]} \\ 0_{(N_2+\ldots+N_{p'})\times n_1} \end{bmatrix} + \boldsymbol{E}_u * \begin{bmatrix} \widehat{\boldsymbol{f}}'^{[u]} \\ 0_{(N_2+\ldots+N_{p'})\times n_1} \end{bmatrix}$$

where $\boldsymbol{E}_u$ is an entire function on all $\mathbb{C}$ with exponential growth of order $\leqslant r_{1,2}$ at infinity. Recall that $r_{1,2} = r_2/(r_2 - r_1)$.

We are now able to compare the structure of the singularities of the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}$ and $\widehat{\boldsymbol{f}}'^{[u]}$ for all $u = 0, \ldots, r_1 - 1$.

Let us first start by a resurgence theorem to locate their possible singular points.

We denote below

$$Q_1\left(\frac{1}{x}\right) = \bigoplus_{j=1}^{J_1} q_j\left(\frac{1}{x}\right) I_{n_j}$$

where $q_j(1/x)$ is a polynomial in $1/x$ of the form

$$q_j\left(\frac{1}{x}\right) = -\frac{a_{j,r_1}}{x^{r_1}} - \frac{a_{j,r_1-1}}{x^{r_1-1}} - \ldots - \frac{a_{j,1}}{x} \in \frac{1}{x}\mathbb{C}\left[\frac{1}{x}\right].$$

Recall that n_j denotes the size of the j^{th} Jordan block of the matrix L of exponents of formal monodromy of System (1.1) (cf. page 248). In particular, the sub-matrix L_1 of L corresponding to Q_1 has the Jordan structure

$$L_1 = \bigoplus_{j=1}^{J_1} (\lambda_j I_{n_j} + J_{n_j}).$$

Recall also that, by definition of Q_1 (cf. Section 3.2), the polynomials q_j for $j = 1, \ldots, J_1$ are zero or of degree r_1. In particular,

$$q_j \equiv 0 \Leftrightarrow a_{j,r_1} = 0.$$

We denote also by

- $\mathcal{S}_1(Q) := \{q_j \; ; \; j = 1, \ldots, J_1\}$ the set of polynomials q_j of degree $\leqslant r_1$ of Q, i.e., the set of all the polynomials of Q_1,
- $\Omega_1 := \{a_{j,r_1} \; ; \; j = 1, \ldots, J_1\}$ the set of first level's Stokes values of System (1.1) associated with $\widetilde{f}(x)$ (cf. Def. 2.2, 2.)

Notice that, following Section 3.1, $a_{1,r_1} = 0$ (since $q_1 \equiv 0$) and there exists $j \in \{1, \ldots, J_1\}$ such that $a_{j,r_1} \neq 0$. Notice also that Ω_1 is also the set of Stokes values of System (3.10) associated with $\widetilde{f}'(x)$.

3.4 Resurgence Theorem

Recall that *a resurgent function* is an analytic function at $0 \in \mathbb{C}$ which can be analytically continued to an adequate Riemann surface $\mathcal{R}_\Omega$ associated with a so-called *singular support* $\Omega \subset \mathbb{C}$. For a more precise definition, we refer to [17] and [9, Def. 2.1 and 2.2]. Recall that the difference between $\mathcal{R}_\Omega$ and the universal cover of $\mathbb{C}\backslash\Omega$ lies in the fact that $\mathcal{R}_\Omega$ has no branch point at 0 in the first sheet.

In the linear case, the singular support Ω is a finite set containing 0. In a more general framework, convolutions of singularities may occur what requires to consider for Ω a lattice, possibly dense in $\mathbb{C}$ (cf. [5, 11, 17] for instance).

To state Resurgence Theorem 3.7 below, we need to extend the classical definition of sectorial regions of $\mathbb{C}$ used in summation theory into the one of sectorial regions of $\mathcal{R}_\Omega$. These regions are called *ν-sectorial regions* (cf. [9, Def. 2.3]) and are defined for all $\nu > 0$ small enough by the data of

- an open disc D_ν centered at $0 \in \mathbb{C}$,
- an open sector Σ_ν with bounded opening at infinity,
- a tubular neighborhood $\mathcal{N}_\nu$ of a piecewise-$\mathcal{C}^1$ path γ connecting D_ν to Σ_ν after a finite number of turns around points of Ω,

such that the distance of D_ν to $\Omega^* = \Omega\backslash\{0\}$ and the distance of $\mathcal{N}_\nu \cup \Sigma_\nu$ to Ω have to be greater than ν.

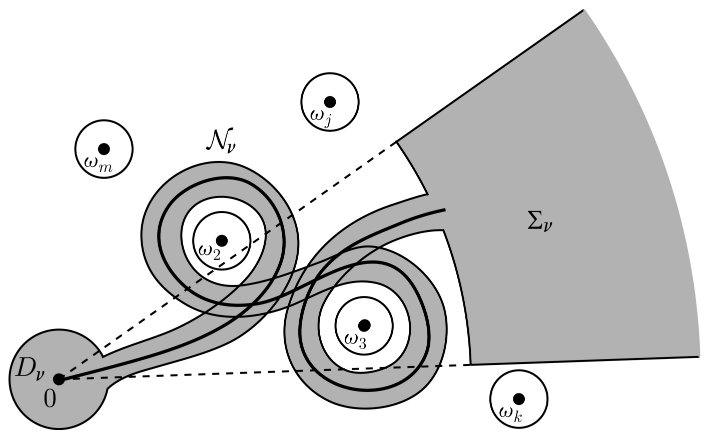

Figure 3.1 - A ν-sectorial region

Definition 3.5 (Resurgent function with exponential growth of order $\leqslant \rho$)**.** Given $\rho > 0$, a resurgent function defined on $\mathcal{R}_\Omega$ is said to be *with exponential growth of order* $\leqslant \rho$ *and with singular support* Ω when it grows at most exponentially at infinity with an order $\leqslant \rho$ on any ν-sectorial region Δ_ν of $\mathcal{R}_\Omega$. We denote by $\widehat{\mathcal{R}es}_\Omega^{\leqslant\rho}$ the set of resurgent functions with exponential growth of order $\leqslant \rho$ and with singular support Ω.

When $\rho = 1$, any function of $\widehat{\mathcal{R}es}_\Omega^{\leqslant 1}$ is said to be *summable-resurgent with singular support* Ω. Following notations of [9], we denote $\widehat{\mathcal{R}es}_\Omega^{sum}$ for $\widehat{\mathcal{R}es}_\Omega^{\leqslant 1}$ the set of summable-resurgent functions with singular support Ω.

Definition 3.6 (Resurgent series with exponential growth of order $\leqslant \rho$)**.** Given $\rho > 0$, a formal series is said to be *a resurgent series with exponential growth of order* $\leqslant \rho$ *and with singular support* Ω when its formal Borel transform belongs to $\widehat{\mathcal{R}es}_\Omega^{\leqslant\rho}$. The set of resurgent series with exponential growth of order $\leqslant \rho$ and with singular support Ω is denoted $\widetilde{\mathcal{R}es}_\Omega^{\leqslant\rho}$. As above, we denote $\widetilde{\mathcal{R}es}_\Omega^{sum}$ for $\widetilde{\mathcal{R}es}_\Omega^{\leqslant 1}$ the set of *summable-resurgent series with singular support* Ω.

We are now able to state the result in view in this section.

Theorem 3.7 (Resurgence Theorem)**.** *With notations as above:*

1. *For all* $u = 0, \ldots, r_1 - 1$,

$$\boxed{\widetilde{\boldsymbol{f}}'^{[u]}(t) \in \widetilde{\mathcal{R}es}_{\Omega_1}^{sum}.}$$

2. *For all* $u = 0, \dots, r_1 - 1$,

$$\boxed{\widetilde{\boldsymbol{f}}^{[u]}(t) \in \widetilde{\mathcal{R}es}_{\Omega_1}^{\leqslant r_{1,2}}, \text{ where } r_{1,2} = \frac{r_2}{r_2 - r_1}.}$$

Proof. Point 1. is proved by applying [9, Thm. 2.7] (case $r_1 = 1$) and [16, Thm. 1.2] (case $r_1 \geqslant 2$) to the single-leveled system (3.10). Point 2. is straightforward from Point 1. and Lemma 3.4. □

In particular, Theorem 3.7 tells us that, for all $u = 0, \dots, r_1 - 1$, the Borel transforms $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ and $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ are all analytic on the same Riemann surface $\mathcal{R}_{\Omega_1}$, their possible singular points being the first level's Stokes values of Ω_1, including 0 out of the first sheet. Section 3.5 below is devoted to the analysis of these singularities.

3.5 Singularities in the Borel Plane

For the convenience of the reader, we first recall some vocabulary used in resurgence theory (see [5, 11, 17] for instance).

Denote by $\mathcal{O}$ the space of holomorphic germs at 0 on $\mathbb{C}$ and $\widetilde{\mathcal{O}}$ the space of holomorphic germs at 0 on the Riemann surface $\widetilde{\mathbb{C}}$ of the logarithm. One calls any element of the quotient space $\mathcal{C} := \widetilde{\mathcal{O}}/\mathcal{O}$ [6] a *singularity at* 0.

A singularity is usually denoted with a nabla. A representative of the singularity $\overset{\triangledown}{\varphi}$ in $\widetilde{\mathcal{O}}$ is called *a major* of $\overset{\triangledown}{\varphi}$ and is often denoted by $\check{\varphi}$.

Given $\omega \neq 0$ in $\mathbb{C}$, the space of the singularities at ω is the space $\mathcal{C}$ translated from 0 to ω. Then, a function $\check{\varphi}_\omega$ is a major of a singularity at ω if $\check{\varphi}_\omega(\omega + \tau)$ is a major of a singularity at 0.

3.5.1 Front of a Singularity

For any $\omega \in \Omega_1$, we call *first level's front of* ω (or simply *front of* ω when we refer to the single-leveled system (3.10)) the set

$$Fr_1(\omega) := \{q_j \in \mathcal{S}_1(Q)\ ;\ a_{j,r_1} = \omega\}$$

of polynomials $q_j(1/x)$'s of degree r_1, the leading term of which is $-\omega/x^{r_1}$.

Since r_1 is the smallest level of Systems (1.1) and (3.10), $Fr_1(\omega)$ is a singleton:

$$Fr_1(\omega) = \left\{ -\frac{\omega}{x^{r_1}} + \dot{q}_{1,\omega}\left(\frac{1}{x}\right) \right\}$$

[6] The elements of $\mathcal{C}$ are also called *micro-functions* by B. Malgrange [11] by analogy with hyper- and micro-functions defined by Sato, Kawai and Kashiwara in higher dimensions.

where $\dot{q}_{1,\omega} \equiv 0$ or $\dot{q}_{1,\omega}(1/x)$ is a polynomial in $1/x$ of degree $\leqslant r_1 - 1$ and with no constant term.

When $\dot{q}_{1,\omega} \equiv 0$, ω is said to be *with monomial front*; the corresponding singularities of $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ and $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$, $u = 0, \ldots, r_1 - 1$, at ω are then called *singularities with monomial front*. As in the case of single-leveled systems, the study of these singularities is sufficient to state the first level's connection-to-Stokes formulæ in full generality (see Section 3.6.2 below).

3.5.2 Structure of Singularities with Monomial Front

For all $u = 0, \ldots, r_1 - 1$, the behavior of the functions $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ and $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ at any point $\omega \in \Omega_1$ depends on the sheet of the Riemann surface $\mathcal{R}_{\Omega_1}$ where we are, i.e., it depends on the "homotopic class" of the path γ of analytic continuation followed from 0 (first sheet) to a neighborhood of ω. We denote by $\overset{\triangledown}{\boldsymbol{f}}{}^{[u]}_{\omega,\gamma}$ (resp. $\overset{\triangledown}{\boldsymbol{f}}{}'^{[u]}_{\omega,\gamma}$) the singularity defined by the analytic continuation of $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ (resp. $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$) along the path γ.

Besides, given a matrix M split into blocks fitting to the Jordan structure of L (matrix of exponents of formal monodromy of System (1.1), cf. p. 248) or L_1 (matrix of exponents of formal monodromy of System (3.10), cf. p. 261), we denote by $M^{j;\bullet}$ the j^{th} row-block of M. So, $M^{j;\bullet}$ is a $n_j \times p$-matrix for all $j = 1, \ldots, J$ (resp. $j = 1, \ldots, J_1$) when M is a $n \times p$-matrix (resp. $N_1 \times p$-matrix). Recall that n_j is the size of the j^{th} Jordan block of L and L_1.

Since System (3.10) has the unique level r_1, the structure of the singularities $\overset{\triangledown}{\boldsymbol{f}}{}'^{[u]}_{\omega,\gamma}$ at any point $\omega \in \Omega_1 \backslash \{0\}$ with monomial front was displayed in [9, Thm. 3.7] (case $r_1 = 1$) and [16, Thm. 3.5] (case $r_1 \geqslant 2$). More precisely, we have the following.

Proposition 3.8 (Singularities with monomial front of $\widehat{\boldsymbol{f}}'^{[u]}$). *Fix $u \in \{0, \ldots, r_1 - 1\}$ and $\omega \in \Omega_1 \backslash \{0\}$ a singular point of $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ with monomial front. For any path γ on $\mathbb{C} \backslash \Omega_1$ from 0 to a neighborhood of ω, the singularity $\overset{\triangledown}{\boldsymbol{f}}{}'^{[u]}_{\omega,\gamma}$ admits a major $\check{\boldsymbol{f}}'^{[u]}_{\omega,\gamma}$ of the form*

$$\boxed{\check{\boldsymbol{f}}'^{[u]j;\bullet}_{\omega,\gamma}(\omega + \tau) = \tau^{\frac{\lambda_j - u}{r_1} - 1} \tau^{\frac{J_{n_j}}{r_1}} \boldsymbol{K}'^{[u]j;\bullet}_{\omega,\gamma} \tau^{-\frac{J_{n_1}}{r_1}} + rem'^{[u]j;\bullet}_{\omega,\gamma}(\tau)}$$

for all $j = 1, \ldots, J_1$ with a remainder

$$rem'^{[u]j;\bullet}_{\omega,\gamma}(\tau) = \sum_{\lambda_\ell; a_{\ell,r_1} = \omega} \sum_{v=0}^{r_1 - 1} \tau^{\frac{\lambda_\ell - v}{r_1}} \boldsymbol{R}'^{[u]j;\bullet}_{\lambda_\ell, v; \omega,\gamma}(\ln \tau)$$

where

- $\boldsymbol{K}'^{[u]j;\bullet}_{\omega,\gamma}$ *denotes a constant $n_j \times n_1$-matrix such that $\boldsymbol{K}'^{[u]j;\bullet}_{\omega,\gamma} = 0$ as soon as $a_{j,r_1} \neq \omega$,*

- $\boldsymbol{R}'^{[u]j;\bullet}_{\lambda_\ell,v;\omega,\gamma}(X)$ *denotes a polynomial matrix with coefficients in* $\widehat{\mathcal{R}es}^{\,sum}_{\Omega_1-\omega}$ *whose the columns are of log-degree*

$$N[\ell] = \begin{cases} \big[(n_\ell - 1) \quad (n_\ell - 1) + 1 \quad \cdots \quad (n_\ell - 1) + (n_1 - 1)\big] & \textit{if } \lambda_\ell \neq 0 \\ \big[n_\ell \quad n_\ell + 1 \quad \cdots \quad n_\ell + (n_1 - 1)\big] & \textit{if } \lambda_\ell = 0. \end{cases}$$

The constants $\boldsymbol{K}'^{[u]j;\bullet}_{\omega,\gamma}$ and the remainders $rem'^{[u]j;\bullet}_{\omega,\gamma}$ depend on the path of analytic continuation γ and on the chosen determination of the argument around ω. Recall (cf. [9, Def. 3.10] and [16, Def. 4.3]) that the *connection constants* of $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ at ω are the entries of the nontrivial matrices $\boldsymbol{K}'^{[u]j;\bullet}_{\omega^\star,+} := \boldsymbol{K}'^{[u]j;\bullet}_{\omega,\gamma^+}$ obtained with the following choices:

- γ^+ is a path going along the straight line $[0,\omega]$ from 0 to a point τ close to ω and avoiding all singular points of $\Omega_1 \cap]0,\omega]$ to the right (see Figure 3.2 below),
- we choose the principal determination of the variable τ around ω, say $\arg(\tau) \in]-2\pi, 0]$ as in Section 2.2 (cf. Note 1).

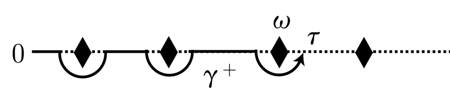

Figure 3.2

By using Lemma 3.4 and [9, Lem. 3.2], we deduce from Proposition 3.8 above the following theorem:

Theorem 3.9 (Singularities with monomial front of $\widehat{\boldsymbol{f}}^{[u]}$)**.** *Fix* $u \in \{0, \ldots, r_1 - 1\}$ *and* $\omega \in \Omega_1 \backslash \{0\}$ *a singular point of* $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ *with monomial front. For any path* γ *on* $\mathbb{C} \backslash \Omega_1$ *from* 0 *to a neighborhood of* ω*, the singularity* $\overset{\nabla}{\boldsymbol{f}}{}^{[u]}_{\omega,\gamma}$ *admits a major* $\check{\boldsymbol{f}}^{[u]}_{\omega,\gamma}$ *of the form*

$$\boxed{\check{\boldsymbol{f}}^{[u]j;\bullet}_{\omega,\gamma}(\omega + \tau) = \tau^{\frac{\lambda_j - u}{r_1} - 1} \tau^{\frac{J_{n_j}}{r_1}} \boldsymbol{K}^{[u]j;\bullet}_{\omega,\gamma} \tau^{-\frac{J_{n_1}}{r_1}} + rem^{[u]j;\bullet}_{\omega,\gamma}(\tau)}$$

for all $j = 1, \ldots, J$ *with a remainder*

$$rem^{[u]j;\bullet}_{\omega,\gamma}(\tau) = \sum_{\lambda_\ell; a_{\ell,r_1} = \omega} \sum_{v=0}^{r_1 - 1} \tau^{\frac{\lambda_\ell - v}{r_1}} \boldsymbol{R}^{[u]j;\bullet}_{\lambda_\ell,v;\omega,\gamma}(\ln \tau)$$

where

- $\boldsymbol{K}_{\omega,\gamma}^{[u]j;\bullet}$ *denotes a constant $n_j \times n_1$-matrix such that*

$$\boldsymbol{K}_{\omega,\gamma}^{[u]j;\bullet} = \begin{cases} 0_{n_j \times n_1} & \text{if } j \notin \{1,\ldots,J_1\} \text{ or } a_{j,r_1} \neq \omega \\ \\ \boldsymbol{K'}_{\omega,\gamma}^{[u]j;\bullet} & \text{otherwise,} \end{cases}$$

- $\boldsymbol{R}_{\lambda_\ell,v;\omega,\gamma}^{[u]j;\bullet}(X)$ *denotes a polynomial matrix with coefficients in $\widehat{\mathcal{R}es}_{\Omega_1-\omega}^{\leqslant r_{1,2}}$ whose the columns are of log-degree $N[\ell]$ (cf. notation just above).*

Observe that the nontrivial constant matrices $\boldsymbol{K'}_{\omega,\gamma}^{[u]j;\bullet}$ and $\boldsymbol{K}_{\omega,\gamma}^{[u]j;\bullet}$ obtained in Proposition 3.8 and Theorem 3.9 coincide. In particular, the connection constants of $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$ at ω can be directly calculate by considering the singularity $\overset{\nabla}{\boldsymbol{f}}{}_{\omega^\star,+}^{[u]} := \overset{\nabla}{\boldsymbol{f}}{}_{\omega,\gamma^+}^{[u]}$.

Definition 3.10 (Connection constants of $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ at ω)**.** Given $u \in \{0,\ldots,r_1-1\}$, we call *connection constants* of $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ at ω the entries of the nontrivial constant matrices $\boldsymbol{K}_{\omega^\star,+}^{[u]j;\bullet} := \boldsymbol{K'}_{\omega^\star,+}^{[u]j;\bullet}$ for $j = 1,\ldots,J_1$ and $a_{j,r_1} = \omega$.

Notice that, in practice, the matrix $\boldsymbol{K}_{\omega^\star,+}^{[u]j;\bullet}$ for $j = 1,\ldots,J_1$ and $a_{j,r_1} = \omega$ can be determined as the coefficient of the monomial $\tau^{(\lambda_j-u)/r_1-1}$ in the major $\check{\boldsymbol{f}}_{\omega^\star,+}^{[u]j;\bullet}(\omega+\tau)$.

We are now able to state the first level's connection-to-Stokes formulæ.

3.6 First Level's Connection-to-Stokes Formulæ

Recall (cf. Def. 2.2, 2.) that the first level's anti-Stokes directions of System (1.1) associated with $\widetilde{f}(x)$ are the directions of maximal decay of the exponentials $e^{q_j(1/x)}$ with $q_j \in \mathcal{S}_1(Q)$ and $q_j \not\equiv 0$ (we refer to page 261 for the notations). Therefore, each nonzero first level's Stokes value $a_{j,r_1} \in \Omega_1^* := \Omega_1 \backslash \{0\}$ generates a collection of r_1 first level's anti-Stokes directions $\theta_0, \theta_1, \ldots, \theta_{r_1-1} \in \mathbb{R}/2\pi\mathbb{Z}$ respectively given by the r_1^{th} roots of a_{j,r_1}. Of course, when $r_1 = 1$, such a collection just reduces to the direction $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$ given by a_{j,r_1}. Note besides that, when $r_1 \geqslant 2$, the directions θ_k's are regularly distributed around the origin $x = 0$.

Such a collection (θ_k) being chosen, we assume, to fix ideas, that their principal determinations $\theta_k^\star \in]-2\pi, 0]$ verify

$$-2\pi < \theta_{r_1-1}^\star < \ldots < \theta_1^\star < \theta_0^\star \leqslant 0$$

Notice that a first level's Stokes value $\omega \in \Omega_1^*$ generates the collection $(\theta_k)_{k=0,\ldots,r_1-1}$ if and only if $\omega \in \Omega_{1,r_1\theta_0}$ the set of nonzero first level's Stokes values of System (1.1) associated with $\widetilde{f}(x)$ and with argument $r_1\theta_0$.

For all $k = 0,\ldots,r_1-1$, we denote by $I_n + C_{\theta_k^\star}$ the Stokes–Ramis matrix of $\widetilde{Y}$ in the direction θ_k. Let $c_{\theta_k^\star}$ be the first n_1 columns of $C_{\theta_k^\star}$. As previously, we split $c_{\theta_k^\star}$ into row-blocks $c_{\theta_k^\star}^{j;\bullet}$ fitting to the Jordan structure of L.

The first level's Stokes multipliers of $\widetilde{f}(x)$ in the direction θ_k are the entries of $c^{j;\bullet}_{\theta^\star_k}$ for $j = 1, \ldots, J_1$ and $a_{j,r_1} \in \Omega_{1,r_1\theta_0}$. We shall make explicit here below formulæ to express these entries in terms of the connection constants of the $\widehat{\boldsymbol{f}}^{[u]}$'s, $u = 0, \ldots, r_1 - 1$. To this end, we need the following more precise definition:

Definition 3.11. When $j = 1, \ldots, J_1$ and $a_{j,r_1} = \omega \in \Omega_{1,r_1\theta_0}$, the entries of the matrix $c^{j;\bullet}_{\theta^\star_k}$ are called *first level's Stokes multipliers of* $\widetilde{f}(x)$ *associated with* ω *in the direction* θ_k.

3.6.1 Case of Singularities with Monomial Front

We denote by

- $\rho_1 := e^{-2i\pi/r_1}$,
- $\Lambda_j := \lambda_j I_{n_j} + J_{n_j}$ the j^{th} Jordan block of the matrix L of exponents of formal monodromy of System (1.1).

Let $\omega \in \Omega_{1,r_1\theta_0}$ be a nonzero first level's Stokes value of System (1.1) associated with $\widetilde{f}(x)$ generating the collection $(\theta_k)_{k=0,\ldots,r_1-1}$. We assume besides, in this section, that the front of ω is monomial.

As we said at the end of Section 3.2, [9, Thm. 4.3] and [16, Thm. 4.4] tell us that the first level's Stokes multipliers of $\widetilde{f}(x)$ associated with ω in the directions θ_k, $k = 0, \ldots, r_1 - 1$, are expressed in terms of the connection constants at ω of the Borel transforms $\widehat{\boldsymbol{f}}'^{[u]}(\tau)$'s, $u = 0, \ldots, r_1 - 1$. On the other hand, we showed in Section 3.5 above that these connection constants are also the connection constants at ω of the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$'s. Consequently, the connection-to-Stokes formulæ relative to $\widetilde{f}'(x)$ displayed in [9,16] coincide with the first level's connection-to-Stokes formulæ relative to $\widetilde{f}(x)$. Hence, the following theorem holds.

Theorem 3.12 (First level's connection-to-Stokes formulæ)**.** *Let $j = 1, \ldots, J_1$ be such that $a_{j,r_1} = \omega$. Then, the data of $(c^{j;\bullet}_{\theta^\star_k})_{k=0,\ldots,r_1-1}$ and of $(\boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+})_{u=0,\ldots,r_1-1}$ are equivalent and are related, for all $k = 0, \ldots, r_1 - 1$, by the relations*

$$\boxed{c^{j;\bullet}_{\theta^\star_k} = \sum_{u=0}^{r_1-1} \rho_1^{k(uI_{n_j} - \Lambda_j)} \boldsymbol{I}^{[u]j;\bullet}_{\omega^\star} \rho_1^{kJ_{n_1}}} \tag{3.12}$$

where

$$\boxed{\boldsymbol{I}^{[u]j;\bullet}_{\omega^\star} := \int_{\gamma_0} \tau^{\frac{\lambda_j - u}{r_1} - 1} \tau^{\frac{J_{n_j}}{r_1}} \boldsymbol{K}^{[u]j;\bullet}_{\omega^\star,+} \tau^{-\frac{J_{n_1}}{r_1}} e^{-\tau} d\tau} \tag{3.13}$$

and where γ_0 is a Hankel type path around the nonnegative real axis $\mathbb{R}^+$ with argument from -2π to 0.

An expanded form providing each entry of First Level's Connection-to-Stokes Formulæ(3.12) is given in [16, Cor. 4.6]. This can be useful for effective numerical calculations. We recall this expanded form below in the particular case where the matrix L of exponents of formal monodromy is diagonal: $L = \bigoplus_{j=1}^{n} \lambda_j$ (we keep denoting by $j = 1, \ldots, J_1$ the indices of polynomials $q_j \in \mathcal{S}_1(Q)$). In this case, the matrices $c_{\theta_k^\star}^{j;\bullet}$ and $\boldsymbol{K}_{\omega^\star,+}^{[u]j;\bullet}$ are reduced to just one entry which we respectively denote $c_{\theta_k^\star}^{j}$ and $\boldsymbol{K}_{\omega^\star,+}^{[u]j}$.

Since the Jordan blocks J_{n_j} are zero for all j, Identity (3.13) becomes

$$\int_{\gamma_0} \tau^{\frac{\lambda_j-u}{r_1}-1} \boldsymbol{K}_{\omega^\star,+}^{[u]j} e^{-\tau} d\tau = 2i\pi \frac{e^{-i\pi\frac{\lambda_j-u}{r_1}}}{\Gamma\left(1-\frac{\lambda_j-u}{r_1}\right)} \boldsymbol{K}_{\omega^\star,+}^{[u]j}.$$

Therefore, for all $j = 1, \ldots, J_1$ such that $a_{j,r_1} = \omega$, the first level's Stokes multipliers $c_{\theta_k^\star}^{j}$ are related to the connection constants $\boldsymbol{K}_{\omega^\star,+}^{[u]j}$ by the formulæ

$$\boxed{c_{\theta_k^\star}^{j} = 2i\pi \sum_{u=0}^{r_1-1} \rho_1^{k(u-\lambda_j)} \frac{e^{-i\pi\frac{\lambda_j-u}{r_1}}}{\Gamma\left(1-\frac{\lambda_j-u}{r_1}\right)} \boldsymbol{K}_{\omega^\star,+}^{[u]j} \text{ for all } k = 0, \ldots, r_1 - 1.} \tag{3.14}$$

3.6.2 General Case

Let us now consider a nonzero first level's Stokes value $\omega \in \Omega_{1,r\theta_0}$ of System (1.1) associated with $\widetilde{f}(x)$ generating the collection $(\theta_k)_{k=0,\ldots,r_1-1}$. Recall that the first level's front of ω reads

$$Fr_1(\omega) = \left\{ q_{1,\omega}\left(\frac{1}{x}\right) := -\frac{\omega}{x^{r_1}} + \dot{q}_{1,\omega}\left(\frac{1}{x}\right) \right\}$$

where $\dot{q}_{1,\omega} \equiv 0$ or $\dot{q}_{1,\omega}(1/x)$ is a polynomial in $1/x$ of degree $\leqslant r_1 - 1$ and with no constant term (cf. Section 3.5.1).

When ω is with monomial front (i.e., $\dot{q}_{1,\omega} \equiv 0$), Theorem 3.12 above allows us to express the first level's Stokes multipliers of $\widetilde{f}(x)$ associated with ω in terms of connection constants in the Borel plane. In particular, in the special case where $r_1 = 1$, Theorem 3.12 allows us to calculate *all* the first level's Stokes multipliers since *all* the singularities of $\widehat{f}$ are with monomial front.

In the case when $r_1 \geqslant 2$ and ω is not with monomial front (i.e., $\dot{q}_{1,\omega} \not\equiv 0$), a result of the same type exists, but requires to reduce ω into a first level's Stokes value with monomial front by means of a convenient change of the variable x in System (1.1) (see Lemma 3.13 below). Recall that such a method was already used in [16] to state the connection-to-Stokes formulæ in the case of systems with a single level $\geqslant 2$.

Lemma 3.13 (M. Loday–Richaud, [6])**.**

1. *There exists, in the x-plane (also called Laplace plane), a change of the variable x of the form*

$$x = \frac{y}{1 + \alpha_1 y + \ldots + \alpha_{r-1} y^{r-1}} \qquad , \alpha_1, \ldots, \alpha_{r-1} \in \mathbb{C} \tag{3.15}$$

such that the polar part $p_{1,\omega}(1/y)$ of $q_{1,\omega}(1/x(y))$ reads

$$p_{1,\omega}\left(\frac{1}{y}\right) = -\frac{\omega}{y^r}.$$

2. *The Stokes–Ramis matrices of System* (1.1) *are preserved by the change of variable* (3.15).

Observe that, although Lemma 3.13 be proved in [6] in the case of systems of dimension 2 (hence, with a single level), it can be extended to any system of dimension $n \geqslant 3$. Indeed, the change of variable (3.15) being tangent to identity, it "preserves" levels, Stokes values and summation operators.

Lemma 3.13 allows us to construct a *new* system, denoted below (S), verifying the following properties:

- (S) has levels $r_1 < r_2 < \ldots < r_p$ and satisfies normalizations as System (1.1) (cf. page 247),
- (S) has the same first level's Stokes values as System (1.1),
- ω is a first level's Stokes value of (S) with monomial front,
- (S) has the same Stokes–Ramis matrices as System (1.1).

Hence, applying Theorem 3.12 to System (S), we can again express the first level's Stokes multipliers of $\widetilde{f}(x)$ associated with ω in terms of connection constants in the Borel plane. Note however that these constants are calculated from System (S) and not from System (1.1).

3.6.3 Effective Calculation of the First Level's Stokes Multipliers

According to Theorem 3.12, the effective calculation of the first level's Stokes multipliers of $\widetilde{f}(x)$ is reduced, after possibly applying Lemma 3.13, to the effective calculation of the connection constants of the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$'s of the r_1-reduced series $\widetilde{\boldsymbol{f}}^{[u]}(t)$'s of $\widetilde{f}(x)$.

For the convenience of the reader, we briefly recall here below how to characterize the series $\widetilde{\boldsymbol{f}}^{[u]}(t)$'s and their Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$'s.

• *Case* $r_1 = 1$:
The series $\widetilde{\boldsymbol{f}}^{[u]}(t)$'s are reduced to just one series $\widetilde{\boldsymbol{f}}^{[0]}(t) = \widetilde{f}(x)$; we keep denoting

the variable x for t. According to normalizations of the formal fundamental solution $\widetilde{Y}(x)$ of System (1.1) (cf. p. 247), the formal series $\widetilde{F}(x)$ is uniquely determined by the homological system

$$x^{r+1}\frac{dF}{dx} = A(x)F - FA_0(x), \quad A_0(x) := x^{r+1}\frac{dQ}{dx} + x^r L$$

of System (1.1) jointly with the initial condition $\widetilde{F}(0) = I_n$ [2]. Hence, by considering its first n_1 columns, we deduce that $\widetilde{f}(x)$ is uniquely determined by the system

$$\boxed{x^2\frac{df}{dx} = x^{1-r}A(x)f - xfJ_{n_1}} \tag{3.16}$$

jointly with the initial condition $\widetilde{f}(0) = I_{n,n_1}$ (first n_1 columns of the identity matrix of size n). Recall that $q_1 \equiv 0$ and $\lambda_1 = 0$ (cf. Assumption (3.1)).

• *Case* $r_1 \geqslant 2$:

In this case, a system characterizing the formal series $\widetilde{\boldsymbol{f}}^{[u]}(t)$'s, $u = 0, \ldots, r_1 - 1$, is provided by the classical method of rank reduction [8] by considering the homological system of the r_1-reduced system associated with System (1.1). More precisely, writing System (1.1) in the form

$$x^{r_1+1}\frac{dY}{dx} = \mathcal{A}(x)Y, \quad \mathcal{A}(x) := x^{r_1-r}A(x) \in M_n(\mathbb{C}\{x\}[x^{-1}])$$

one can prove, similarly as in the case $r_1 = 1$, that the formal series

$$\widetilde{\boldsymbol{f}}(t) = \begin{bmatrix} \widetilde{\boldsymbol{f}}^{[0]}(t) \\ \vdots \\ \widetilde{\boldsymbol{f}}^{[r_1-1]}(t) \end{bmatrix} \in M_{r_1n,n_1}(\mathbb{C}[[t]])$$

is uniquely determined by the system

$$\boxed{r_1 t^2\frac{d\boldsymbol{f}}{dt} = \boldsymbol{A}(t)\boldsymbol{f} - t\boldsymbol{f}J_{n_1}} \tag{3.17}$$

jointly with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{r_1n,n_1}$ (first n_1 columns of the identity matrix of size r_1n); the matrix $\boldsymbol{A}(t) \in M_{r_1n}(\mathbb{C}\{t\}[t^{-1}])$ is defined by

$$\boldsymbol{A}(t) = \begin{bmatrix} \mathcal{A}^{[0]}(t) & t\mathcal{A}^{[r_1-1]}(t) & \cdots & \cdots & t\mathcal{A}^{[1]}(t) \\ \mathcal{A}^{[1]}(t) & \mathcal{A}^{[0]}(t) & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \mathcal{A}^{[0]}(t) & t\mathcal{A}^{[r_1-1]}(t) \\ \mathcal{A}^{[r_1-1]}(t) & \cdots & \cdots & \mathcal{A}^{[1]}(t) & \mathcal{A}^{[0]}(t) \end{bmatrix} - \bigoplus_{u=0}^{r_1-1} utI_n$$

where $\mathcal{A}^{[u]}(t)$, $u = 0, \dots, r_1 - 1$, denote the r_1-reduced series of $\mathcal{A}(x)$.

• Then, by applying the formal Borel transformation to Systems (3.16) and (3.17), we obtain convolution equations satisfied by the Borel transforms $\widehat{\boldsymbol{f}}^{[u]}(\tau)$'s, $u = 0, \dots, r_1 - 1$. In the special case where $r_1 = 1$, we simply denote $\widehat{f}(\xi)$ for $\widehat{\boldsymbol{f}}^{[0]}(\tau)$.

Recall that the formal Borel transformation is an isomorphism from the $\mathbb{C}$-differential algebra $\left(\mathbb{C}[[t]], +, \cdot, t^2\dfrac{d}{dt}\right)$ to the $\mathbb{C}$-differential algebra $(\delta\mathbb{C} \oplus \mathbb{C}[[\tau]], +, *, \tau\cdot)$ that changes ordinary product $\cdot$ into convolution product $*$ and changes derivation $t^2\dfrac{d}{dt}$ into multiplication by τ. It also changes multiplication by $\dfrac{1}{t}$ into derivation $\dfrac{d}{d\tau}$ allowing thus to extend the isomorphism from the meromorphic series $\mathbb{C}[[t]][t^{-1}]$ to $\mathbb{C}[\delta^{(k)}, k \in \mathbb{N}] \oplus \mathbb{C}[[\tau]]$.

4 Examples

To end this article, we develop three examples. Although the given systems may seem a little bit involved, they are simple enough to allow the *exact* calculation of the connection constants and so of the first level's Stokes multipliers. This "simplicity" is due to the fact that the matrices of these systems are triangular. Of course, for more general systems, such exact calculations no longer hold in general.

4.1 An Example with a Three-Leveled System

We consider the system

$$x^4\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2x^4 & x^2 + \dfrac{x^3}{3} & 0 & 0 & 0 \\ -3x^3 & 2x^3 & 2x^2 & 0 & 0 \\ x^2 & 0 & 0 & 2x + x^2 & 0 \\ x^4 + x^5 & 0 & 0 & 0 & 1 \end{bmatrix} Y \tag{4.1}$$

and its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ where

- $Q\left(\dfrac{1}{x}\right) = \operatorname{diag}\left(0, -\dfrac{1}{x}, -\dfrac{2}{x}, -\dfrac{1}{x^2} - \dfrac{1}{x}, -\dfrac{1}{3x^3}\right)$,
- $L = \operatorname{diag}\left(0, \dfrac{1}{3}, 0, 0, 0\right)$,

- $\widetilde{F}(x)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ \widetilde{f}_2(x) & 1 & 0 & 0 & 0\\ \widetilde{f}_3(x) & * & 1 & 0 & 0\\ \widetilde{f}_4(x) & 0 & 0 & 1 & 0\\ \widetilde{f}_5(x) & 0 & 0 & 0 & 1\end{bmatrix}$ verifies $\widetilde{F}(x)=I_5+O(x)$. More precisely,

$$\widetilde{f}_2(x)=O(x^2),\ \widetilde{f}_3(x)=\frac{3x}{2}+O(x^2),\ \widetilde{f}_4(x)=-\frac{x}{2}+O(x^2),\ \widetilde{f}_5(x)=O(x^4).$$

We denote as before by $\widetilde{f}(x)$ the first column of $\widetilde{F}(x)$.

System (4.1) has levels $(1,2,3)$ and the set Ω_1 of first level's Stokes values associated with $\widetilde{f}(x)$ is $\Omega_1=\{0,1,2\}$. In particular, System (4.1) admits the direction $\theta=0$ (direction of maximal decay of the exponentials $e^{-1/x}$ and $e^{-2/x}$) as unique first level's anti-Stokes directions associated with $\widetilde{f}(x)$. Note that this direction is also a second and a third level's anti-Stokes direction associated with $\widetilde{f}(x)$.

Obviously, the Stokes–Ramis matrix I_5+C_0 is of the form

$$C_0=\begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ c_0^2 & 0 & 0 & 0 & 0\\ c_0^3 & * & 0 & 0 & 0\\ c_0^4 & 0 & 0 & 0 & 0\\ c_0^5 & 0 & 0 & 0 & 0\end{bmatrix}.$$

The Stokes multipliers c_0^2 and c_0^3 are respectively the first level's Stokes multipliers of $\widetilde{f}(x)$ associated with the first level's Stokes values $\xi=1$ and $\xi=2$. The Stokes multipliers c_0^4 and c_0^5 are respectively a second level's and a third level's Stokes multiplier.

Our aim is the calculation of c_0^2 and c_0^3. Observe that, due to Theorem 3.12, c_0^2 (resp. c_0^3) is expressed in terms of the connection constants of $\widehat{f}(\xi)$ at $\xi=1$ (resp. $\xi=2$). Indeed, the two first level's Stokes values 1 and 2 are both with monomial front.

According to (3.16), $\widetilde{f}(x)$ is uniquely determined by the system

$$x^2\frac{df}{dx}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0\\ 2x^2 & 1+\dfrac{x}{3} & 0 & 0 & 0\\ -3x & 2x & 2 & 0 & 0\\ 1 & 0 & 0 & \dfrac{2}{x}+1 & 0\\ x^2+x^3 & 0 & 0 & 0 & \dfrac{1}{x^2}\end{bmatrix}f$$

jointly with the initial condition $\widetilde{f}(0) = I_{5,1}$ (first column of the identity matrix of size 5). Therefore, the $\widetilde{f}_j$'s are the unique formal series solutions of the equations

$$\begin{cases} x^2 \dfrac{d\widetilde{f}_2}{dx} - \left(1 + \dfrac{x}{3}\right) \widetilde{f}_2 = 2x^2 \\[2ex] x^2 \dfrac{d\widetilde{f}_4}{dx} - \dfrac{2}{x}\widetilde{f}_4 + \widetilde{f}_4 = 1 \\[2ex] x^2 \dfrac{d\widetilde{f}_3}{dx} - 2\widetilde{f}_3 = -3x + 2x\widetilde{f}_2 \\[2ex] x^2 \dfrac{d\widetilde{f}_5}{dx} - \dfrac{1}{x^2}\widetilde{f}_5 = x^2 + x^3 \end{cases}$$

satisfying the condition $\widetilde{f}_j(x) = O(x)$. As a result, their Borel transforms $\widehat{f}_j$'s verify the equations

$$\begin{cases} (\xi - 1)\dfrac{d\widehat{f}_2}{d\xi} + \dfrac{2}{3}\widehat{f}_2 = 2 & , \ \widehat{f}_2(0) = 0 \\[2ex] (\xi - 2)\widehat{f}_3 = -3 + 2 * \widehat{f}_2 & \\[2ex] -2\dfrac{d\widehat{f}_4}{d\xi} + (\xi + 1)\widehat{f}_4 = 0 & , \ \widehat{f}_4(0) = -\dfrac{1}{2} \\[2ex] -\dfrac{d^2\widehat{f}_5}{d\xi} + \xi\widehat{f}_5 = \xi + \dfrac{\xi^2}{2} & , \ \widehat{f}_5(0) = \dfrac{d\widehat{f}_5}{d\xi}(0) = 0. \end{cases}$$

Hence, for all $|\xi| < 1$,

$$\begin{cases} \widehat{f}_2(\xi) = -3(1 - \xi)^{-2/3} + 3 \\[2ex] \widehat{f}_3(\xi) = \dfrac{-21 + 6\xi + 18(1 - \xi)^{1/3}}{\xi - 2} \\[2ex] \widehat{f}_4(\xi) = -\dfrac{1}{2} \exp\left(\dfrac{\xi^2}{4} + \dfrac{\xi}{2}\right) \\[2ex] \widehat{f}_5(\xi) = 1 + \dfrac{\xi}{2} - {}_0F_1\left(\cdot, \dfrac{2}{3}; \dfrac{\xi^3}{9}\right) - \dfrac{\xi}{2} {}_0F_1\left(\cdot, \dfrac{4}{3}; \dfrac{\xi^3}{9}\right) \end{cases}$$

where ${}_0F_1(\cdot, b; \xi)$ denotes the confluent hypergeometric function with parameters $(\cdot, b)$. In particular, $\widehat{f}_4$ and $\widehat{f}_5$ are entire on all $\mathbb{C}$ and, for $j = 2, 3$, the analytic continuations

$\widehat{f}^{+}_{j,\omega^\star}$'s of the $\widehat{f}_j$'s to the right of points $\omega \in \{1,2\}$ verify

$$\boxed{\begin{aligned}
&\widehat{f}^{+}_{2,1}(1+\xi) = \frac{3+3i\sqrt{3}}{2}\xi^{-2/3} + 3\\
&\widehat{f}^{+}_{2,2}(2+\xi) \in \mathbb{C}\{\xi\}\\
&\widehat{f}^{+}_{3,1}(1+\xi) \in \mathbb{C}\{\xi\} + \xi^{1/3}\mathbb{C}\{\xi\}\\
&\widehat{f}^{+}_{3,2}(2+\xi) = \frac{-9+6\xi+(9+9i\sqrt{3})(1+\xi)^{1/3}}{\xi}
\end{aligned}}.$$

Consequently, the connection matrices $K_{1,+}$ and $K_{2,+}$ of $\widehat{f}(\xi)$ at the points $\xi = 1$ and $\xi = 2$ are given by

$$\boxed{K_{1,+} = \begin{bmatrix} 0 \\ k^2_{1,+} = \dfrac{3+3i\sqrt{3}}{2} \\ 0 \\ 0 \end{bmatrix} \qquad K_{2,+} = \begin{bmatrix} 0 \\ 0 \\ k^3_{2,+} = 9i\sqrt{3} \\ 0 \end{bmatrix}}.$$

Since the matrix L of exponents of formal monodromy is diagonal, it results from (3.14) that the Stokes multipliers c^2_0 and c^3_0 are related to the connection constants $k^2_{1,+}$ and $k^3_{2,+}$ above by the relations

$$\boxed{c^2_0 = 2i\pi\frac{e^{-i\pi/3}}{\Gamma(2/3)}k^2_{1,+} \qquad c^3_0 = 2i\pi k^3_{2,+}}$$

(recall that $\rho_1 = e^{-2i\pi}$ and $k = 0$ since $r_1 = 1$). Hence,

$$\boxed{c^2_0 = \frac{6i\pi}{\Gamma(2/3)} \qquad c^3_0 = -18\pi\sqrt{3}}.$$

4.2 An Example with Rank Reduction

We consider now the system

$$x^4\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 \\ x^4 - 2x^5 & 2x & 0 \\ -x^3 & 0 & 3+x^2 \end{bmatrix} Y \tag{4.2}$$

and its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)e^{Q(1/x)}$ where

- $Q\left(\frac{1}{x}\right) = \operatorname{diag}\left(0, -\frac{1}{x^2}, -\frac{1}{x^3} - \frac{1}{x}\right)$,

- $\widetilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2(x) & 1 & 0 \\ \widetilde{f}_3(x) & 0 & 1 \end{bmatrix}$ verifies $\widetilde{F}(x) = I_3 + O(x^3)$. More precisely,

$$\widetilde{f}_2(x) = -\frac{x^3}{2} + x^4 - \frac{3x^5}{4} + O(x^6) \quad \text{and} \quad \widetilde{f}_3(x) = \frac{x^3}{3} - \frac{x^5}{9} + O(x^6). \tag{4.3}$$

System (4.2) has levels $(2, 3)$ and $\Omega_1 = \{0, 1\}$. In particular, the first level's anti-Stokes directions of System (4.2) associated with the first column $\widetilde{f}(x)$ of $\widetilde{F}(x)$ are given by the unique collection $(\theta_0 = 0, \theta_1 = -\pi)$ generated by $\tau = 1$. Note that $\theta_0 = 0$ is also a second level's anti-Stokes direction associated with $\widetilde{f}(x)$. Obviously, the Stokes–Ramis matrices $I_3 + C_0$ and $I_3 + C_{-\pi}$ are of the form

$$C_0 = \begin{bmatrix} 0 & 0 & 0 \\ c_0^2 & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \quad \text{and} \quad C_{-\pi} = \begin{bmatrix} 0 & 0 & 0 \\ c_{-\pi}^2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Indeed, $\widetilde{f}(x)$ is the unique column of $\widetilde{F}(x)$ which is divergent.

As in the previous example, the first level's Stokes value $\tau = 1$ is with monomial front. Hence, Theorem 3.12 implies that the two first level's Stokes multipliers c_0^2 and $c_{-\pi}^2$ are expressed in terms of the connection constants of $\widehat{\boldsymbol{f}}^{[0]}(\tau)$ and $\widehat{\boldsymbol{f}}^{[1]}(\tau)$ at $\tau = 1$.

According to Relation (3.2), the 2-reduced series of $\widetilde{f}(x)$ are of the form

$$\widetilde{\boldsymbol{f}}^{[0]}(t) = \begin{bmatrix} 1 \\ \widetilde{\boldsymbol{f}}_2(t) \\ \widetilde{\boldsymbol{f}}_3(t) \end{bmatrix} \quad \text{and} \quad \widetilde{\boldsymbol{f}}^{[1]}(t) = \begin{bmatrix} 0 \\ \widetilde{\boldsymbol{f}}_5(t) \\ \widetilde{\boldsymbol{f}}_6(t) \end{bmatrix}$$

where the $\widetilde{\boldsymbol{f}}_j(t)$'s are power series in t satisfying $\widetilde{\boldsymbol{f}}_j(t) = O(t)$. More precisely, it results from (4.3) that

$$\widetilde{\boldsymbol{f}}_2(t) = t^2 + O(t^3) \qquad \widetilde{\boldsymbol{f}}_5(t) = -\frac{t}{2} - \frac{3t^2}{4} + O(t^3)$$

$$\widetilde{\boldsymbol{f}}_3(t) = O(t^3) \qquad \widetilde{\boldsymbol{f}}_6(t) = \frac{t}{3} - \frac{t^2}{9} + O(t^3).$$

Following (3.17), the matrix $\widetilde{\boldsymbol{f}}(t) := \begin{bmatrix} \widetilde{\boldsymbol{f}}^{[0]}(t) \\ \widetilde{\boldsymbol{f}}^{[1]}(t) \end{bmatrix} \in M_{6,1}(\mathbb{C}[[t]])$ is uniquely determined by the system

$$2t^2\frac{d\boldsymbol{f}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -2t^2 & 2 & 0 & t^2 & 0 & 0 \\ -t & 0 & 0 & 0 & 0 & 3+t \\ 0 & 0 & 0 & -t & 0 & 0 \\ t & 0 & 0 & -2t^2 & 2-t & 0 \\ 0 & 0 & \dfrac{3}{t}+1 & -t & 0 & -t \end{bmatrix} \boldsymbol{f}$$

jointly with the initial condition $\widetilde{\boldsymbol{f}}(0) = I_{6,1}$. Then, the $\widetilde{\boldsymbol{f}}_j$'s are the unique formal series solutions of the equations

$$2t^2\frac{d\widetilde{\boldsymbol{f}}_2}{dt} - 2\widetilde{\boldsymbol{f}}_2 = -2t^2 \qquad 2t^2\frac{d\widetilde{\boldsymbol{f}}_5}{dt} - (2-t)\,\widetilde{\boldsymbol{f}}_5 = t$$

$$2t^2\frac{d\widetilde{\boldsymbol{f}}_3}{dt} = -t + (3+t)\widetilde{\boldsymbol{f}}_6 \qquad 2t^2\frac{d\widetilde{\boldsymbol{f}}_6}{dt} + t\widetilde{\boldsymbol{f}}_6 = \left(\frac{3}{t}+1\right)\widetilde{\boldsymbol{f}}_3$$

satisfying the conditions $\widetilde{\boldsymbol{f}}_j(t) = O(t)$. Hence,

- the Borel transforms $\widehat{\boldsymbol{f}}_2$ and $\widehat{\boldsymbol{f}}_5$ verify the equations

$$\begin{cases} (\tau-1)\widehat{\boldsymbol{f}}_2 = -\tau \\ (\tau-1)\dfrac{d\widehat{\boldsymbol{f}}_5}{d\tau} + \dfrac{3}{2}\widehat{\boldsymbol{f}}_5 = 0 \qquad , \ \widehat{\boldsymbol{f}}_5(0) = -\dfrac{1}{2}, \end{cases}$$

- denoting $\boldsymbol{\varphi} := \begin{bmatrix} \widehat{\boldsymbol{f}}_3 \\ \widehat{\boldsymbol{f}}_6 \end{bmatrix}$, the Borel transforms $\widehat{\boldsymbol{f}}_3$ and $\widehat{\boldsymbol{f}}_6$ verify the system

$$\begin{cases} \begin{bmatrix} 3 & 0 \\ -2\tau & 3 \end{bmatrix} \dfrac{d^2\boldsymbol{\varphi}}{d\tau} + \begin{bmatrix} 1 & -2\tau \\ -4 & 1 \end{bmatrix} \dfrac{d\boldsymbol{\varphi}}{d\tau} + \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix} \boldsymbol{\varphi} = 0 \\ \boldsymbol{\varphi}(0) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \ \dfrac{d\boldsymbol{\varphi}}{d\tau}(0) = \begin{bmatrix} 0 \\ -\dfrac{1}{9} \end{bmatrix}. \end{cases}$$

As a result, $\widehat{\boldsymbol{f}}_3$ and $\widehat{\boldsymbol{f}}_6$ are entire on all $\mathbb{C}$ and $\widehat{\boldsymbol{f}}_2$ and $\widehat{\boldsymbol{f}}_5$ are defined by

$$\widehat{\boldsymbol{f}}_2(\tau) = \frac{\tau}{1-\tau} \quad \text{and} \quad \widehat{\boldsymbol{f}}_5(\tau) = -\frac{1}{2}(1-\tau)^{-3/2}$$

for all $|\tau| < 1$. In particular, the analytic continuations $\widehat{\boldsymbol{f}}^{+}_{j,1}$'s of the $\widehat{\boldsymbol{f}}_j$'s to the right of 1 verify

$$\boxed{\begin{array}{ll} \widehat{\boldsymbol{f}}^{+}_{2,1}(1+\tau) = -\dfrac{\tau+1}{\tau} & \widehat{\boldsymbol{f}}^{+}_{5,1}(1+\tau) = -\dfrac{i}{2}\tau^{-3/2} \\[2ex] \widehat{\boldsymbol{f}}^{+}_{3,1}(1+\tau) \in \mathbb{C}\{\tau\} & \widehat{\boldsymbol{f}}^{+}_{6,1}(1+\tau) \in \mathbb{C}\{\tau\} \end{array}}\,.$$

Consequently, the connection matrices $\boldsymbol{K}^{[u]}_{1,+}$ of $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ at the point $\tau = 1$ are given by

$$\boxed{\boldsymbol{K}^{[0]}_{1,+} = \begin{bmatrix} 0 \\ k^{[0]2}_{1,+} = -1 \\ 0 \end{bmatrix} \qquad \boldsymbol{K}^{[1]}_{1,+} = \begin{bmatrix} 0 \\ k^{[1]2}_{1,+} = -\dfrac{i}{2} \\ 0 \end{bmatrix}}\,.$$

From Theorem 3.12 and more precisely Formula (3.14) (recall that $L = 0$), we deduce that the two first level's Stokes multipliers c^2_0 and $c^2_{-\pi}$ are related to the connection constants $k^{[0]2}_{1,+}$ and $k^{[1]2}_{1,+}$ above by the relations

$$\boxed{c^2_0 = 2i\pi k^{[0]2}_{1,+} + 2i\pi\frac{e^{i\pi/2}}{\Gamma(3/2)}k^{[1]2}_{1,+} \qquad c^2_{-\pi} = 2i\pi k^{[0]2}_{1,+} + 2i\pi e^{-i\pi}\frac{e^{i\pi/2}}{\Gamma(3/2)}k^{[1]2}_{1,+}}$$

(recall that $\rho_1 = e^{-i\pi}$ since $r_1 = 2$). Hence,

$$\boxed{c^2_0 = -2i(\pi - \sqrt{\pi}) \qquad c^2_{-\pi} = -2i(\pi + \sqrt{\pi})}\,.$$

4.3 An Example with a Singularity with Non-Monomial Front

Let us now consider the system

$$x^5\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 \\ -x^7 & x^2 + x^3 & 0 \\ x^4 & 0 & 1 \end{bmatrix} Y \tag{4.4}$$

together with its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)e^{Q(1/x)}$, where

- $Q\left(\dfrac{1}{x}\right) = \operatorname{diag}\left(0, -\dfrac{1}{2x^2} - \dfrac{1}{x}, -\dfrac{1}{4x^4}\right)$,
- $\widetilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2(x) & 1 & 0 \\ \widetilde{f}_3(x) & 0 & 1 \end{bmatrix}$ verifies $\widetilde{F}(x) = I_3 + O(x^4)$.

System (4.4) has the levels $(2,4)$ and $\Omega_1 = \{0, 1/2\}$. In particular, the first level's anti-Stokes directions of System (4.4) associated with the first column of $\widetilde{F}(x)$ are given by the unique collection $(\theta_0 = 0, \theta_1 = -\pi)$ generated by $\tau = 1/2$. Note that these two directions are also second level's anti-Stokes directions.

Since just the first column of $\widetilde{F}(x)$ is divergent, the Stokes–Ramis matrices $I_3 + C_0$ and $I_3 + C_{-\pi}$ are of the form

$$C_0 = \begin{bmatrix} 0 & 0 & 0 \\ c_0^2 & 0 & 0 \\ * & 0 & 0 \end{bmatrix} \quad \text{and} \quad C_{-\pi} = \begin{bmatrix} 0 & 0 & 0 \\ c_{-\pi}^2 & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$$

where c_0^2 and $c_{-\pi}^2$ are the first level's Stokes multipliers associated with the first level's Stokes value $\tau = 1/2$. Our aim is the calculation of c_0^2 and $c_{-\pi}^2$. However, since $\tau = 1/2$ is not with monomial front, we can not directly apply Theorem 3.12 as in the previous examples.

Let us first reduce the Stokes value $\tau = 1/2$ into a first level's Stokes value with monomial front by considering the change of variable

$$x = \frac{y}{1-y}.$$

System (4.4) becomes

$$y^5 \frac{d\mathcal{Y}}{dy} = \begin{bmatrix} 0 & 0 & 0 \\ -\dfrac{y^7}{(1-y)^4} & y^2 & 0 \\ \dfrac{y^4}{1-y} & 0 & (1-y)^3 \end{bmatrix} \mathcal{Y}$$

and its formal fundamental solution $\widetilde{\mathcal{Y}}(y) := \widetilde{Y}(x(y))$ reads $\widetilde{\mathcal{Y}}(y) = \widetilde{G}(y) e^{P(1/y)}$ where

- $P\left(\dfrac{1}{y}\right) = \operatorname{diag}\left(0, -\dfrac{1}{2y^2}, -\dfrac{1}{4y^4} + \dfrac{1}{y^3} - \dfrac{3}{2y^2} + \dfrac{1}{y}\right)$,
- $\widetilde{G}(y) = \widetilde{F}(x(y)) \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{1/2} & 0 \\ 0 & 0 & e^{-1/4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{f}_2(x(y)) & e^{1/2} & 0 \\ \widetilde{f}_3(x(y)) & 0 & e^{-1/4} \end{bmatrix} \in M_3(\mathbb{C}[[y]])$.

To normalize $\widetilde{G}(y)$ to $I_3 + O(y^4)$, we consider the constant gauge transformation

$$Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e^{-1/2} & 0 \\ 0 & 0 & e^{1/4} \end{bmatrix} \mathcal{Y}.$$

Hence, the system

$$y^5 \frac{dZ}{dy} = \begin{bmatrix} 0 & 0 & 0 \\ -\dfrac{y^7 e^{-1/2}}{(1-y)^4} & y^2 & 0 \\ \dfrac{y^4 e^{1/4}}{1-y} & 0 & (1-y)^3 \end{bmatrix} Z \tag{4.5}$$

and its formal fundamental solution $\widetilde{Z}(y) = \widetilde{H}(y) e^{P(1/y)}$ where

$$\widetilde{H}(y) = \begin{bmatrix} 1 & 0 & 0 \\ \widetilde{h}_2(y) & 1 & 0 \\ \widetilde{h}_3(y) & 0 & 1 \end{bmatrix}$$

is a power series in y such that $\widetilde{H}(y) = I_3 + O(y^4)$. More precisely,

$$\widetilde{h}_2(y) = e^{-1/2} y^5 + O(y^6) \quad \text{and} \quad \widetilde{h}_3(y) = -e^{1/4} y^4 - 4e^{1/4} y^5 + O(y^6). \tag{4.6}$$

System (4.5) has, like System (4.4), the levels $(3,4)$ and the set of first level's Stokes values associated with the first column $\widetilde{h}(x)$ of $\widetilde{H}(x)$ is again $\Omega_1 = \{0, 1/2\}$. Due to Lemma 3.13, the Stokes–Ramis matrices $I_3 + C_0$ and $I_3 + C_{-\pi}$ of System (4.4) are also Stokes–Ramis matrices of System (4.5). Moreover, since the first level's Stokes value $\tau = 1/2$ of System (4.5) is now with monomial front, Theorem 3.12 applies allowing thus to make explicit the two first level's Stokes multipliers c_0^2 and $c_{-\pi}^2$ in terms of the connection constants of $\widehat{\boldsymbol{h}}^{[0]}(\tau)$ and $\widehat{\boldsymbol{h}}^{[1]}(\tau)$ at $\tau = 1/2$.

According to Relations (3.2) and (4.6), the 2-reduced series of $\widetilde{h}(x)$ are of the form

$$\widetilde{\boldsymbol{h}}^{[0]}(t) = \begin{bmatrix} 1 \\ \widetilde{\boldsymbol{h}}_2(t) \\ \widetilde{\boldsymbol{h}}_3(t) \end{bmatrix} \quad \text{and} \quad \widetilde{\boldsymbol{h}}^{[1]}(t) = \begin{bmatrix} 0 \\ \widetilde{\boldsymbol{h}}_5(t) \\ \widetilde{\boldsymbol{h}}_6(t) \end{bmatrix}$$

where the $\widetilde{\boldsymbol{h}}_j(t)$'s are power series in t verifying

$$\widetilde{\boldsymbol{h}}_2(t) = O(t^3) \qquad \widetilde{\boldsymbol{h}}_5(t) = e^{-1/2} t^2 + O(t^3)$$

$$\widetilde{\boldsymbol{h}}_3(t) = -e^{1/4} t^2 + O(t^3) \qquad \widetilde{\boldsymbol{h}}_6(t) = -4e^{1/4} t^2 + O(t^3).$$

Following (3.17), the matrix $\widetilde{\boldsymbol{h}}(t) := \begin{bmatrix} \widetilde{\boldsymbol{h}}^{[0]}(t) \\ \widetilde{\boldsymbol{h}}^{[1]}(t) \end{bmatrix} \in M_{6,1}(\mathbb{C}[[t]])$ is uniquely determined by the system

$$2t^2\frac{d\boldsymbol{h}}{dt} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ T_1^{[0]}(t) & 1 & 0 & tT_1^{[1]}(t) & 0 & 0 \\ T_2^{[0]}(t) & 0 & \dfrac{1}{t}+3 & tT_2^{[1]}(t) & 0 & -3-t \\ 0 & 0 & 0 & -t & 0 & 0 \\ T_1^{[1]}(t) & 0 & 0 & T_1^{[0]}(t) & 1-t & 0 \\ T_2^{[1]}(t) & 0 & -\dfrac{3}{t}-1 & T_2^{[0]}(t) & 0 & \dfrac{1}{t}+3-t \end{bmatrix}\boldsymbol{h}$$

jointly with the initial condition $\widetilde{\boldsymbol{h}}(0) = I_{6,1}$ (first column of the identity matrix of size 6) where

$$\begin{cases} T_1^{[0]}(t) = -\dfrac{4e^{-1/2}(1+t)t^3}{(1-t)^4} = -\dfrac{2e^{-1/2}}{3}\displaystyle\sum_{m\geqslant 3}(m-1)(m-2)(2m-3)t^m \\[2ex] T_1^{[1]}(t) = -\dfrac{e^{-1/2}(1+6t+t^2)t^2}{(1-t)^4} = -\dfrac{e^{-1/2}}{3}\displaystyle\sum_{m\geqslant 2}(m-1)(2m-1)(2m-3)t^m \\[2ex] T_2^{[0]}(t) = T_2^{[1]}(t) = \dfrac{e^{1/4}t}{1-t} = e^{1/4}\displaystyle\sum_{m\geqslant 1}t^m. \end{cases}$$

Therefore, the $\widetilde{\boldsymbol{h}}_j$'s are the unique formal series solutions of the equations

$$\begin{cases} 2t^2\dfrac{d\widetilde{\boldsymbol{h}}_2}{dt} - \widetilde{\boldsymbol{h}}_2 = T_1^{[0]}(t) \\[2ex] 2t^2\dfrac{d\widetilde{\boldsymbol{h}}_3}{dt} - \left(\dfrac{1}{t}+3\right)\widetilde{\boldsymbol{h}}_3 = T_2^{[0]}(t) - (3+t)\widetilde{\boldsymbol{h}}_6 \\[2ex] 2t^2\dfrac{d\widetilde{\boldsymbol{h}}_5}{dt} - (1-t)\,\widetilde{\boldsymbol{h}}_5 = T_1^{[1]}(t) \\[2ex] 2t^2\dfrac{d\widetilde{\boldsymbol{h}}_6}{dt} - \left(\dfrac{1}{t}+3-t\right)\widetilde{\boldsymbol{h}}_6 = T_2^{[1]}(t) - \left(\dfrac{3}{t}+3\right)\widetilde{\boldsymbol{h}}_3 \end{cases}$$

satisfying the conditions $\widetilde{\boldsymbol{h}}_j(t) = O(t^2)$. Hence,

- the Borel transforms $\widehat{\boldsymbol{h}}_2$ and $\widehat{\boldsymbol{h}}_5$ verify the equations

$$\begin{cases} (2\tau-1)\widehat{\boldsymbol{h}}_2 = \widehat{T}_1^{[0]}(\tau) \\ (2\tau-1)\dfrac{d\widehat{\boldsymbol{h}}_5}{d\tau} + 3\widehat{\boldsymbol{h}}_5 = \dfrac{d\widehat{T}_1^{[1]}}{d\tau}(\tau) \qquad , \ \widehat{\boldsymbol{h}}_5(0) = 0 \end{cases}$$

  where the Borel transforms $\widehat{T}_1^{[u]}(\tau)$ of $T_1^{[u]}(t)$ are defined by

$$\begin{cases} \widehat{T}_1^{[0]}(\tau) = -\dfrac{2e^{-1/2}}{3}\displaystyle\sum_{m\geqslant 2}\dfrac{(2m-1)}{(m-2)!}\tau^m = -\dfrac{2\tau^2(2\tau+3)}{3}e^{\tau-1/2} \\ \widehat{T}_1^{[1]}(\tau) = -\dfrac{e^{-1/2}}{3}\displaystyle\sum_{m\geqslant 1}\dfrac{4m^2-1}{(m-1)!}\tau^m = -\dfrac{\tau(4\tau^2+12\tau+3)}{3}e^{\tau-1/2}, \end{cases}$$

- denoting $\boldsymbol{\varphi} := \begin{bmatrix}\widehat{\boldsymbol{h}}_3\\ \widehat{\boldsymbol{h}}_6\end{bmatrix}$, the Borel transforms $\widehat{\boldsymbol{h}}_3$ and $\widehat{\boldsymbol{h}}_6$ verify the system

$$\begin{cases} \begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}\dfrac{d^2\boldsymbol{\varphi}}{d\tau} + \begin{bmatrix}3-2\tau & -3\\ 0 & 3-2\tau\end{bmatrix}\dfrac{d\boldsymbol{\varphi}}{d\tau} + \begin{bmatrix}-2 & 1\\ 0 & -3\end{bmatrix}\boldsymbol{\varphi} = -\dfrac{d}{d\tau}\begin{bmatrix}\widehat{T}_2^{[0]}\\ \widehat{T}_2^{[1]}\end{bmatrix} \\ \boldsymbol{\varphi}(0) = 0, \ \dfrac{d\boldsymbol{\varphi}}{d\tau}(0) = \begin{bmatrix}-e^{1/4}\\ -4e^{1/4}\end{bmatrix} \end{cases}$$

  where the Borel transforms $\widehat{T}_2^{[u]}(\tau)$ of $T_2^{[u]}(t)$ are defined by

$$\widehat{T}_2^{[0]}(\tau) = \widehat{T}_2^{[1]}(\tau) = e^{1/4}\sum_{m\geqslant 0}\frac{\tau^m}{m!} = e^{\tau+1/4}.$$

As a result, $\widehat{\boldsymbol{h}}_3$ and $\widehat{\boldsymbol{h}}_6$ are entire on all $\mathbb{C}$ and, for $j = 2, 5$, the analytic continuations $\widehat{\boldsymbol{h}}^{+}_{j,1/2}$'s of the $\widehat{\boldsymbol{h}}_j$'s to the right of $\tau = 1/2$ verify

$$\boxed{\begin{aligned} \widehat{\boldsymbol{h}}^{+}_{2,1/2}\left(\frac{1}{2}+\tau\right) &= -\frac{(1+2\tau)^2(2+\tau)}{6\tau}e^{\tau} \\ \widehat{\boldsymbol{h}}^{+}_{5,1/2}\left(\frac{1}{2}+\tau\right) &= -i\alpha\tau^{-3/2} + E(\tau) \end{aligned}}$$

with $E(\tau)$ an entire function on $\mathbb{C}$ and

$$\boxed{\alpha = \frac{1}{8}\sqrt{\frac{2}{e}} + \frac{\sqrt{2}}{6}\,{}_1F_1\left(\frac{1}{2}, \frac{3}{2}; -\frac{1}{2}\right)}$$

where ${}_1F_1\left(\frac{1}{2},\frac{3}{2};\tau\right)$ denotes the confluent hypergeometric function with parameters $\frac{1}{2}$ and $\frac{3}{2}$.

Consequently, the connection matrices $\boldsymbol{K}^{[u]}_{1/2,+}$ of $\widehat{\boldsymbol{f}}^{[u]}(\tau)$ at the point $\tau = 1/2$ are given by

$$\boxed{\boldsymbol{K}^{[0]}_{1/2,+} = \begin{bmatrix} 0 \\ k^{[0]2}_{1/2,+} = -\dfrac{1}{3} \\ 0 \end{bmatrix} \qquad \boldsymbol{K}^{[1]}_{1/2,+} = \begin{bmatrix} 0 \\ k^{[1]2}_{1/2,+} = i\alpha \\ 0 \end{bmatrix}}.$$

From Theorem 3.12 and more precisely Formula (3.14) (recall that $L = 0$), we deduce that the two first level's Stokes multipliers c_0^2 and $c_{-\pi}^2$ are related to the connection constants $k^{[0]2}_{1,+}$ and $k^{[1]2}_{1,+}$ above by the relations

$$\boxed{c_0^2 = 2i\pi k^{[0]2}_{1/2,+} + 2i\pi\frac{e^{i\pi/2}}{\Gamma(3/2)}k^{[1]2}_{1/2,+} \qquad c_{-\pi}^2 = 2i\pi k^{[0]2}_{1/2,+} + 2i\pi e^{-i\pi}\frac{e^{i\pi/2}}{\Gamma(3/2)}k^{[1]2}_{1/2,+}}$$

(recall that $\rho_1 = e^{-i\pi}$ since $r_1 = 2$). Hence,

$$\boxed{c_0^2 = -\frac{2i}{3}(\pi + 6\alpha\sqrt{\pi}) \qquad c_{-\pi}^2 = -\frac{2i}{3}(\pi - 6\alpha\sqrt{\pi})}.$$

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