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Existence of Three Solutions for a Boundary Value Problem in the One-Dimensional Case

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Abstract

In this paper, we establish the existence of at least three solutions to a Navier boundary problem involving the biharmonic equation. The technical approach is mainly base on a three critical points theorem of B. Ricceri.

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1 Introduction and Main Results

Consider the Navier boundary value problem involving the biharmonic equation

$$\begin{cases} (|u''|u'')'' = \lambda f(x,u) + \mu g(x,u), & \text{in }]0,1[, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{$\mathcal{P}$}$$

where λ, $\mu \in [0, +\infty)$, $f, g\colon [0,1] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions.

Here in the sequel, X will be denoted the Sobolev space $W^{2,2}([0,1]) \cap W_0^{1,2}([0,1])$ and will be endowed with the norm

$$\|u\| = \left(\int_0^1 |u''|^3 \, dx \right)^{1/3}.$$

As usual, a weak solution of problem $(\mathcal{P})$ is any $u \in X$ such that

$$\int_0^1 |u''|u''\xi'' dx = \lambda \int_0^1 f(x,u)\xi dx + \mu \int_0^1 g(x,u)\xi dx \tag{1.1}$$

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for every $\xi \in X$.

The fourth-order equation of nonlinearity furnishes a model to study traveling waves in suspension bridges, so it's important to Physics. Many authors consider this type equation, we refer to [1–3] and there reference therein.

To the best of our knowledge, there are few results about multiple solutions to biharmonic equation. In this paper, we prove the existence of at least three solutions of problem $(\mathcal{P})$. The technical approach is based on the three critical points theorem of Ricceri [5]. Let $F(x,s) = \int_0^s f(x,\xi)d\xi$. Our main result is the following theorems.

Theorem 1.1. *Assume that there exist three positive constants* c, d *and* γ *with* $\gamma < 3$, $c\sqrt[3]{6} < 16d$ *and a function* $a(x) \in L^1([0,1])$, *such that*

- *(j_1)* $f(x,s) \geq 0$ *for every* $(x,s) \in [0,1/4] \cup [3/4,1] \times [0,d]$*;*
- *(j_2)* $\dfrac{1}{6c^3} \max\limits_{(x,s)\in[0,1]\times[-c,c]} F(x,s) < \dfrac{1}{4096d^3} \displaystyle\int_{1/4}^{3/4} F(x,d)dx$*;*
- *(j_3)* $F(x,s) \leq a(x)(1+|s|^\gamma)$ *for all* $s \in \mathbb{R}$.

Then there exist an open interval $\Lambda \subseteq [0,+\infty)$ *and a positive real number* ρ *with the following property: for each* $\lambda \in \Lambda$ *and for each Carathéodory function* $g\colon [0,1]\times\mathbb{R} \to \mathbb{R}$*, satisfying*

- *(j_4)* $\sup\limits_{\{|s|\leq\zeta\}} |g(\cdot,s)| \in L^1([0,1])$*, for all* $\zeta > 0$*,*

there exists $\delta > 0$ *such that, for each* $\mu \in [0,\delta]$*, problem* $(\mathcal{P})$ *has at least three solutions whose norms in* X *are less than* ρ.

We now want to point out a consequence of Theorem 1.1.

Theorem 1.2. *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a continuous function. Put* $F(s) = \int_0^s f(\eta)d\eta$ *for each* $s \in \mathbb{R}$ *and assume that there exist three positive constants* c, d *and* γ *with* $\gamma < 3$, $c\sqrt[3]{6} < 16d$ *and a positive constant* a*, such that*

- *(k_1)* $f(s) \geq 0$ *for every* $s \in [0,d]$*;*
- *(k_2)* $\dfrac{1}{6c^3} \max\limits_{s\in[-c,c]} F(s) < \dfrac{1}{4096d^3} F(d)$*;*
- *(k_3)* $F(s) \leq a(1+|s|^\gamma)$ *for all* $s \in \mathbb{R}$.

Then there exist an open interval $\Lambda \subseteq [0,+\infty)$ *and a positive real number* ρ *with the following property: for each* $\lambda \in \Lambda$ *and for each Carathéodory function* $g\colon [0,1]\times\mathbb{R} \to \mathbb{R}$*, satisfying*

(k_4) $\sup_{\{|s|\leq\zeta\}} |g(\cdot, s)| \in L^1([0,1])$, *for all* $\zeta > 0$,

there exists $\delta > 0$ *such that, for each* $\mu \in [0, \delta]$, *problem*

$$\begin{cases} (|u''|u'')'' = \lambda f(u) + \mu g(x,u), & \textit{in}\,]0,1[, \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{$\mathcal{P}'$}$$

has at least three solutions whose norms in X *are less than* ρ.

Remark 1.3. In Theorem 1.2, if $f(s) \geq 0$ for every $s \in [-c, d]$. Then, instead of condition (k_1) and (k_2), we put $\frac{F(c)}{6c^3} < \frac{F(d)}{4096d^3}$ and the result holds.

2 Proof of the Main Result

For the reader's convenience, we recall the revised form of Ricceri's three critical points theorem.

Theorem 2.1 (See [5, Theorem 1])**.** *Let X be a reflexive real Banach space. $\Phi\colon X \mapsto \mathbb{R}$ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^* and Φ is bounded on each bounded subset of X; $\Psi\colon X \mapsto \mathbb{R}$ is a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact; $I \subseteq \mathbb{R}$ an interval. Assume that*

$$\lim_{\|x\|\to+\infty} (\Phi(x) + \lambda\Psi(x)) = +\infty$$

for all $\lambda \in I$, and that there exists $h \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\Phi(x) + \lambda(\Psi(x) + h)) < \inf_{x\in X} \sup_{\lambda\in I} (\Phi(x) + \lambda(\Psi(x) + h)). \tag{2.1}$$

Then, there exists an open interval $\Lambda \subseteq I$ and a positive real number ρ with the following property: for every $\lambda \in \Lambda$ and every C^1 functional $J\colon X \mapsto \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$ the equation

$$\Phi'(x) + \lambda\Psi'(x) + \mu J'(x) = 0$$

has a least three solutions in X whose norms are less than ρ.

Now we can give the proof of our main result.

Proof of Theorem 1.1. For each $u \in X$, let

$$\Phi(u) = \frac{\|u\|^3}{3}, \qquad \Psi(u) = -\int_0^1 F(x,u)dx, \qquad -J(u) = \int_0^1 \int_0^{u(x)} g(x,\xi)d\xi dx.$$

Under the condition of Theorem 1.1, Φ is a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional. Moreover, Φ admits a continuous inverse on X^*. Ψ and J are continuously Gâteaux differential functional whose Gâteaux derivative is compact. Obviously, Φ is bounded on each bounded subset of X.

Thanks to (j_3), for each $\lambda > 0$, one has that

$$\lim_{\|u\|\to+\infty}(\Phi(u)+\lambda\Psi(u)) = +\infty, \tag{2.2}$$

and so the first assumption of Theorem 2.1 holds.

Let

$$u^*(x) = \begin{cases} d - 16d(1/4 - |x-1/2|)^2, & x \in \left[0, \dfrac{1}{4}\right] \cup \left[\dfrac{3}{4}, 1\right], \\ d, & x \in \left]\dfrac{1}{4}, \dfrac{3}{4}\right[, \end{cases}$$

It is easy to verify that $u^* \in W^{2,3}([0,1]) \cap W_0^{1,3}([0,1])$, and in particular, one has

$$\|u^*\|^3 = \frac{(32d)^3}{2}. \tag{2.3}$$

Now, let $r = (2c)^3$ and by the assumption of $c\sqrt[3]{6} < 16d$ we have that

$$\frac{\|u^*\|^3}{3} > r > 0.$$

Moreover, it follows from (j_1) that

$$\int_{\frac{1}{4}}^{\frac{3}{4}} F(x, u^*(x))dx \geq 0.$$

One has

$$\max_{(x,s)\in[0,1]\times[-c,c]} F(x,s) < \frac{6c^3}{4096d^3}\int_{1/4}^{3/4} F(x,d)dx \leq 3r\frac{\int_0^1 F(x,u^*(x))dx}{\|u^*\|}. \tag{2.4}$$

Namely

$$\max_{(x,s)\in[0,1]\times[-c,c]} F(x,s) < 3r\frac{\int_0^1 F(x,u^*(x))dx}{\|u^*\|}.$$

For each $r > 0$, if $u \in X$ satisfying $\|u\| \leq \sqrt[3]{3r}$, due to the inequality $\max_{x\in[0,1]}|u(x)| \leq \frac{1}{2}\sqrt[3]{3}\|u\|$, one has $\max_{x\in[0,1]}|u(x)| \leq \frac{\sqrt[3]{r}}{2} = c$.

So, we have that

$$\begin{aligned}\sup_{\{\Phi(u)\leq r\}} -\Psi(u) &= \sup_{\{u \| \|u\|^p \leq pr\}} \int_0^1 F(x,u)dx \\ &\leq \max_{(x,s)\in[0,1]\times[-c,c]} F(x,s) \\ &< \frac{\int_0^1 F(u^*(x))dx}{\|u^*\|^p}.\end{aligned}$$

Therefore, using [4, Proposition 3.1], with $u_0 = 0$ and $u_1 = u^*$, we obtain

$$\sup_{\lambda\geq 0}\inf_{x\in X}(\Phi(x) + \lambda(h + \Psi(x))) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) + \lambda(h + \Psi(x))), \tag{2.5}$$

and so the assumption (2.1) of Theorem 2.1 holds.

Now, set $I = [0, +\infty)$, by (2.2), (2.5), all the assumptions of Theorem 2.1 are satisfied. Hence, our conclusion follows from Theorem 2.1. □

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References

[1] G. Bonanno, B. Di Bella, A boundary value problem for fourth-order elastic beam equations, J. Math. Anal. Appl. 343 (2) (2008) 1166–1176.

[2] J. Chabrowski, J. Marcos do Ó, On some fourth-order semilinear elliptic problems in $\mathbb{R}^N$, Nonlinear Anal. 49 (6) (2002) 861–884.

[3] A. M. Micheletti, A. Pistoia, Multiplicity results for a fourth-order semilinear elliptic problem, Nonlinear Anal. 31 (7) (1998) 895–908.

[4] B. Ricceri, Existence of three solutions for a class of elliptic eigenvalue problems, Math. Comput. Modelling 32 (11-13) (2000) 1485–1494.

[5] B. Ricceri, A three critical points theorem revisited, Nonlinear Anal. 70 (9) (2009) 3084–3089.