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Existence and Global Exponential Stability of almost Periodic Solutions for BAM Neural Networks with variable Coefficients on Time Scales

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Abstract

In this paper, based on the theory of calculus on time scales and some basic results about almost periodic differential equations on almost periodic time scales, a class of BAM neural networks with variable coefficients are studied on almost periodic time scales, some sufficient conditions are established for the existence and global exponential stability of the almost periodic solution. Finally, two examples and numerical simulations are presented to illustrate the feasibility and effectiveness of the results.

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1 Introduction

In recent years, BAM neural networks have been extensively studied and applied in many different fields such as signal processing, pattern recognition, solving optimization problems and automatic control engineering. They have been widely studied both in theory and applications. In [6,10,17,24,28,30], some sufficient conditions have been obtained for global stability of delayed BAM networks, in [21,26,27]; the exponential stabilities of stochastic BAM neural networks have been studied; in [22,25]; the problems of passivity analysis for BAM neural networks have been investigated. Moreover,

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authors in [2, 9, 19, 23, 29, 31] investigated the periodic oscillatory solution of BAM neural networks. It is well known that studies on neural dynamical systems not only involve a discussion of stability properties, but also involve many dynamic behaviors such as periodic oscillatory behavior, almost periodic oscillatory properties, chaos, and bifurcation. In applications, almost periodic oscillatory is more accordant with fact.

In [3], the authors considered the BAM networks with variable coefficients of the following form:

$$\begin{cases} x_i'(t) = -a_i(t)x_i(t) + \sum_{j=1}^{p} p_{ji}f_j(y_j(t-\tau_{ji})) + I_i(t), \\ y_j'(t) = -b_j(t)y_j(t) + \sum_{i=1}^{n} q_{ij}f_i(x_i(t-\sigma_{ij})) + J_j(t), \end{cases} \tag{1.1}$$

where τ_{ji} and σ_{ij} are nonnegative constants, $i = 1, 2, \cdots, n,\ j = 1, 2, \cdots, p$. By using a Banach fixed point theorem and constructing suitable Lyapunov functions, some sufficient conditions are obtained ensuring existence, uniqueness and global stability of almost periodic solution of (1.1).

The discrete-time BAM networks of the following form:

$$\begin{cases} x_i(n+1) = -a_i x_i(n) + \sum_{j=1}^{m} c_{ij}f_j(y_j(n-k_n)) + I_i,\ i = 1, 2, \ldots, m, \\ y_j(n+1) = -b_j y_j(n) + \sum_{i=1}^{m} d_{ji}g_i(x_i(n-l_n)) + J_j,\ j = 1, 2, \ldots, m \end{cases} \tag{1.2}$$

has also been studied by many researchers (see, [5,18,20]). In these papers, the authors by using Lyapunov functionals or linear matrix inequality techniques (LMI), some sufficient conditions of exponential stability criterion are established. But they did not consider the almost periodic solutions of (1.2).

In fact, both continuous and discrete systems are very important in implementation and applications. But it is troublesome to study the existence of almost periodic solutions for continuous and discrete systems respectively. Therefore, it is meaningful to study that on time scale which can unify the continuous and discrete situations (see [8, 11, 12, 16]).

However, to the best of our knowledge, there is no paper published on the existence of almost periodic solutions for BAM neural networks with variable coefficients on time scales.

Motivated by the above, in this paper, we are concerned with the following BAM neural network on time scales:

$$\begin{cases} x_i^{\Delta}(t) &= -a_i(t)x_i(t) + \sum_{j=1}^{m} p_{ji}(t)f_j(y_j(t-\tau_{ji}(t))) \\ & \quad +I_i(t),\ t\in\mathbb{T},\ t>0,\ i=1,\cdots,n, \\ y_j^{\Delta}(t) &= -b_j(t)y_j(t) + \sum_{i=1}^{n} q_{ij}(t)g_i(x_i(t-\vartheta_{ij}(t))) \\ & \quad +L_j(t),\ t\in\mathbb{T},\ t>0,\ j=1,\cdots,m, \end{cases} \tag{1.3}$$

where $\mathbb{T}$ is an almost time scale which will be defined in the next section, $x_i(t)$ and $y_j(t)$ are the activations of the ith neuron and the jth neuron, respectively, p_{ji}, q_{ij} are the connection weights at time t, $I_i(t)$ and $L_j(t)$ denote the external inputs at time t, g_i, f_j are the input-output functions (the activation functions), time delays $\tau_{ji}(t), \vartheta_{ij}(t)$ correspond to finite speed of axonal transmission, $a_i(t), b_j(t)$ represent the rate with which the ith neuron and jth neuron will reset their potential to the resting state in isolation when they are disconnected from the network and the external inputs at time t, m, n correspond to the number of neurons in layers.

The system (1.3) is supplemented with initial values given by

$$\begin{cases} x_i(s)=\phi_i(s),\ s\in[-\vartheta,0]\cap\mathbb{T},\ \ \vartheta=\max\limits_{1\le i\le n,1\le j\le m}\sup\limits_{t\in\mathbb{T}}\big\{\vartheta_{ij}(t)\big\}, \quad i=1,2,\cdots,n, \\ y_j(s)=\varphi_j(s),\ s\in[-\hat{\tau},0]\cap\mathbb{T},\ \ \hat{\tau}=\max\limits_{1\le i\le n,1\le j\le m}\sup\limits_{t\in\mathbb{T}}\big\{\tau_{ji}(t)\big\}, \quad j=1,2,\cdots,m, \end{cases}$$

where $\phi_i(\cdot)$ and $\varphi_j(\cdot)$ denote real-valued continuous functions defined on $[-\hat{\tau},0]\cap\mathbb{T}$ and $[-\vartheta,0]\cap\mathbb{T}$, respectively.

Remark 1.1. When $\mathbb{T}=\mathbb{R}$, system (1.1) is a special case of system (1.3); when $\mathbb{T}=\mathbb{Z}$, system (1.2) is also a special case of system (1.3).

Our main purpose of this paper is first by using the exponential dichotomy of linear dynamic equations on time scales, the time scale calculus theory and contraction fixed point theorem to study the existence of almost periodic solutions to (1.3). Then we construct a suitable Lyapunov function to investigate the exponential stability of the almost periodic solutions to (1.3).

Remark 1.2. In order to describe various real-world problems in physical and engineering sciences subject to abrupt changes at certain instants during the evolution process, impulsive fractional differential equations has been used to the system model. But, the problem that how to establish the existence and stability of almost periodic solutions to BAM neural networks with impulses on time scales is still open.

The organization of this paper is as follows: In Section 2, we introduce some notations, definitions and state some preliminary results needed in the later sections. In Section 3, we study the existence of almost periodic solutions of system (1.3) by using a fixed point theorem. In Section 4, we shall derive sufficient conditions to ensure that the almost periodic solution of (1.3) is globally exponentially stable. In Section 5, two examples are given to illustrate that our results are feasible and more general.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]\big| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$. Let y be right-dense continuous. If $Y^\Delta(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R})$. We define the set

$$\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}.$$

If $r \in \mathcal{R}$, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{ \int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau \right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \dfrac{\mathrm{Log}(1 + \mathrm{hz})}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Definition 2.1 (See [1]). Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions. We define

$$p \oplus q = p + q + \mu pq, \quad \ominus p = -\frac{p}{1 + \mu p}, \quad p \ominus q = p \oplus (\ominus q).$$

Lemma 2.2 (See [1]). *Assume that* $p : \mathbb{T} \to R$ *is regressive. Then*

(i) $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$*;*

(ii) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$*;*

(iii) $e_p(t,s) = \dfrac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$*;*

(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$*;*

(v) $(e_{\ominus p}(t,s))^{\Delta} = (\ominus p)(t)e_{\ominus p}(t,s)$*;*

(vi) If $a,b,c \in \mathbb{T}$*, then* $\displaystyle\int_a^b p(t)e_p(c,\sigma(t))\Delta t = e_p(c,a) - e_p(c,b)$.

Definition 2.3 (See [13]). A time scale $\mathbb{T}$ is called an almost periodic time scale if

$$\Pi := \big\{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\big\} \neq \{0\}.$$

Throughout this paper, we always restrict our discussion to almost periodic time scales. In this section, $|\cdot|$ denotes a norm of $\mathbb{R}^n$.

Definition 2.4 (See [13, 14]). Let $\mathbb{T}$ be an almost periodic time scale. A function $f \in C(\mathbb{T},\mathbb{R}^n)$ is called an almost periodic function if the ε-translation set of f

$$E\{\varepsilon, f\} = \{\tau \in \Pi : |f(t+\tau) - f(t)| < \varepsilon, \text{ for all } t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$ such that each interval of length $l(\varepsilon)$ contains a $\tau(\varepsilon) \in E\{\varepsilon, f\}$ such that

$$|f(t+\tau) - f(t)| < \varepsilon, \quad \text{for all } t \in \mathbb{T}.$$

τ is called the ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

Lemma 2.5 (See [13, 14]). *If* $f \in C(\mathbb{T},\mathbb{R}^n)$ *be an almost periodic function, then* $f(t)$ *is bounded on* $\mathbb{T}$*.*

Definition 2.6 (See [13, 14]). Let $x \in \mathbb{R}^n$ and $A(t)$ be an $n \times n$ rd-continuous matrix on $\mathbb{T}$. The linear system

$$x^{\Delta}(t) = A(t)x(t),\ t \in \mathbb{T} \tag{2.1}$$

is said to admit an exponential dichotomy on $\mathbb{T}$ if there exist positive constant k, α, projection P and the fundamental solution matrix $X(t)$ of (2.1), satisfying

$$|X(t)PX^{-1}(\sigma(s))|_0 \le ke_{\ominus\alpha}(t,\sigma(s)),\ s,t \in \mathbb{T},\ t \ge \sigma(s),$$
$$|X(t)(I-P)X^{-1}(\sigma(s))|_0 \le ke_{\ominus\alpha}(\sigma(s),t),\ s,t \in \mathbb{T},\ t \le \sigma(s),$$

where $|\cdot|_0$ is a matrix norm (say, for example, if $A = (a_{ij})_{n\times m}$, then we can take $|A|_0 = \big(\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} |a_{ij}|^2\big)^{\frac{1}{2}}$).

Lemma 2.7 (See [14]). *Let $c_i(t)$ be an almost periodic function on $\mathbb{T}$, where $c_i(t) > 0$, $-c_i(t) \in \mathcal{R}^+$, $\forall\, t \in \mathbb{T}$ and*

$$\min_{1\le i\le n}\big\{\inf_{t\in\mathbb{T}} c_i(t)\big\} = \widetilde{m} > 0.$$

Then the linear system

$$x^{\Delta}(t) = \mathrm{diag}\big(-c_1(t), -c_2(t), \cdots, -c_n(t)\big)x(t) \tag{2.2}$$

admits an exponential dichotomy on $\mathbb{T}$.

Consider the almost periodic system

$$x^{\Delta}(t) = A(t)x(t) + f(t),\ t \in \mathbb{T}, \tag{2.3}$$

where $A(t)$ is an almost periodic matrix function, $f(t)$ is an almost periodic vector function.

Lemma 2.8 (See [13, 14]). *If the linear system* (2.1) *admits an exponential dichotomy, then system* (2.3) *has a unique almost periodic solution as follows:*

$$x(t) = \int_{-\infty}^{t} X(t)PX^{-1}(\sigma(s))f(s)\Delta s - \int_{t}^{+\infty} X(t)(I-P)X^{-1}(\sigma(s))f(s)\Delta s, \tag{2.4}$$

where $X(t)$ is the fundamental solution matrix of (2.1).

Lemma 2.9 (See [1]). *Let A be a regressive $n \times n$-matrix-valued function on $\mathbb{T}$. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}^n$. Then the initial value problem*

$$y^{\Delta}(t) = A(t)y(t),\ \ y(t_0) = y_0$$

has a unique solution $y : \mathbb{T} \to \mathbb{R}^n$. Moreover, the solution is given by

$$y(t) = e_A(t, t_0)y_0.$$

Definition 2.10 (See [8]). For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then we defined the generalized derivation (of Dini derivative), $D^+u^{\Delta}(t)$ to mean that, given $\epsilon > 0$, there exists a right neighborhood $N(\epsilon) \subset N$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\sigma(t) - s} < D^+u^{\Delta}(t) + \epsilon$$

for each $s \in N(\epsilon), s > t$. In case t is right-scattered and $u(t)$ is continuous at t, this reduces to

$$D^+u^{\Delta}(t) = \frac{u(\sigma(t)) - u(t)}{\sigma(t) - t}.$$

Definition 2.11. The almost periodic solution $x^* = (x_1^*, x_2^*, \dots, x_n^*, y_1^*, y_2^*, \dots, y_m^*)^T$ of system (1.3) is said to be globally exponentially stable, if there exist constants λ and $M = M(\lambda) \geq 1$, for any solution

$$x(t) = (x_1(t), x_2(t), \dots, x_n(t), y_1(t), y_2(t), \dots, y_m(t))^T$$

of (1.3) with initial value $\varphi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t), \varphi_1(t), \varphi_2(t), \dots, \varphi_m(t))^T$, where

$$(\phi_1, \phi_2, \dots, \phi_n) \in C([-\vartheta, 0]_{\mathbb{T}}, \mathbb{R}^n), \quad (\varphi_1, \varphi_2, \dots, \varphi_m) \in C([-\hat{\tau}, 0]_{\mathbb{T}}, \mathbb{R}^m)$$

such that

$$\sum_{i=1}^{n} |x_i(t) - x_i^*(t)| + \sum_{j=1}^{m} |y_j(t) - y_j^*(t)| \leq M(\lambda) e_{\ominus\lambda}(t, s) \Bigg(\sum_{i=1}^{n} \|x_i - x^*\| + \sum_{j=1}^{m} \|y_i - y^*\| \Bigg),$$

where

$$\|x_i - x^*\| = \sum_{i=1}^{n} \max_{\delta \in [-\vartheta, 0]_{\mathbb{T}}} |\phi_i(\delta) - x_i^*(\delta)|, \;\; \delta \in [-\vartheta, 0]_{\mathbb{T}},$$

$$||y_j - y^*|| = \sum_{j=1}^{m} \max_{\delta \in [-\hat{\tau}, 0]_{\mathbb{T}}} |\varphi_j(\delta) - y_j^*(\delta)|, \;\; \delta \in [-\hat{\tau}, 0]_{\mathbb{T}}.$$

Lemma 2.12 (See [4, 7])**.** *Let N be a positive integer and $\mathbb{B}$ be a Banach space. If the mapping $\phi^N : \mathbb{B} \to \mathbb{B}$ is a contraction mapping, then $\phi : \mathbb{B} \to \mathbb{B}$ has exactly one fixed point in $\mathbb{B}$, where $\phi^N = \phi(\phi^{N-1})$.*

3 Existence of almost Periodic Solutions

We denote the radius of the spectrum of matrix F by $\rho(F)$. Hereafter of this paper, we use the following norm:

$$\|z\| = \max \big\{ \max_{1 \leq i \leq n} \sup_{t \in \mathbb{T}} |x_i(t)|, \max_{1 \leq j \leq m} \sup_{t \in \mathbb{T}} |y_j(t)| \big\}.$$

For convenience, we denote

$$p_{ji}^+ = \sup_{t \in \mathbb{T}} |p_{ji}(t)|, \; q_{ij}^+ = \sup_{t \in \mathbb{T}} |p_{ij}(t)|, \; \overline{a_i} = \sup_{t \in \mathbb{T}} |a_i(t)|,$$
$$\underline{a_i} = \inf_{t \in \mathbb{T}} |a_i(t)|, \; \overline{b_j} = \sup_{t \in \mathbb{T}} |b_j(t)|, \; \underline{b_j} = \inf_{t \in \mathbb{T}} |b_j(t)|.$$

We make the following assumptions:

(H$_1$) $a_i(t) > 0, b_j(t) > 0, p_{ji}(t), q_{ij}(t), I_i(t), L_j(t), 0 < \vartheta_{ij}(t) < \vartheta, 0 < \tau_{ji}(t) < \hat{\tau}$ are all almost periodic functions on $\mathbb{T}$, for $t \in \mathbb{T}, t - \vartheta_{ij}(t), t - \tau_{ji}(t) \in \mathbb{T}, i = 1, 2, \cdots, n, j = 1, 2, \cdots, m.$

(H$_2$) $f_j, g_i \in C(\mathbb{R}, \mathbb{R})(i = 1, 2, \cdots, n, j = 1, 2, \cdots, m)$ are Lipschitzian with Lipschitz constants $\eta_j, \lambda_i > 0$, that is,

$$|f_j(x) - f_j(y)| \le \eta_j |x - y|, \ |g_i(x) - g_i(y)| \le \lambda_i |x - y|, \ \forall \, x, y \in \mathbb{R}.$$

(H$_3$) $\min\big\{ \min\limits_{1 \le i \le n} \underline{a}_i, \ \min\limits_{1 \le j \le m} \underline{b}_j \big\} > 0$, and $-a_i(t) \in \mathcal{R}^+, \ -b_j(t) \in \mathcal{R}^+, \ \forall \, t \in \mathbb{T}, \ i = 1, \cdots, n, \ j = 1, \cdots, m.$

Theorem 3.1. *Assume that* (*H*$_1$)–(*H*$_3$) *hold and*

(*H*$_4$) $\rho(F) < 1$, *where*

$$F := \begin{bmatrix} A^{-1}PL & 0 \\ 0 & B^{-1}Q\Lambda \end{bmatrix}_{(n+m)\times(n+m)}$$

with $A^{-1} = \operatorname{diag}(\underline{a}_1^{-1}, \underline{a}_2^{-1}, \cdots, \underline{a}_n^{-1})_{n\times n}$, $B^{-1} = \operatorname{diag}(\underline{b}_1^{-1}, \underline{b}_2^{-1}, \cdots, \underline{b}_m^{-1})_{m\times m}$, $P = (p_{ji}^+)_{m\times n}$,
$Q = (q_{ij}^+)_{n\times m}$, $L = \operatorname{diag}(\eta_1, \eta_2, \cdots, \eta_m)$, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n)$.

Then system (1.3) *has exactly one almost periodic solution.*

Proof. Let $\mathbb{B} = \big\{ z | z = \big(\psi_1, \psi_2, \cdots, \psi_n, \Psi_1, \Psi_2, \cdots, \Psi_m\big)^T \big\}$, where ψ_i and Ψ_j are almost periodic functions on $\mathbb{T}$ with the norm

$$\|z\| = \max\big\{ \max_{1\le i\le n} \sup_{t\in\mathbb{T}} |\psi_i(t)|, \max_{1\le j\le m} \sup_{t\in\mathbb{T}} |\Psi_j(t)| \big\}.$$

Then, $\mathbb{B}$ is a Banach space. For any

$$z = z_{(\psi,\Psi)^T} = \big(\psi_1, \psi_2, \cdots, \psi_n, \Psi_1, \Psi_2, \cdots, \Psi_m\big)^T \in \mathbb{B},$$

we consider the almost solution $z_{(\psi,\Psi)^T}$ of the nonlinear almost periodic system

$$\begin{cases} x_i^{\Delta}(t) &= -a_i(t)x_i(t) + \sum\limits_{j=1}^{m} p_{ji}(t) f_j(\Psi_j(t - \tau_{ji}(t))) \\ & \quad + I_i(t), \ t \in \mathbb{T}, \ i = 1, 2, \cdots, n, \\ y_j^{\Delta}(t) &= -b_j(t)y_j(t) + \sum\limits_{i=1}^{n} q_{ij}(t) g_i(\psi_i(t - \vartheta_{ij}(t))) \\ & \quad + L_j(t), \ t \in \mathbb{T}, \ j = 1, 2, \cdots, m. \end{cases} \tag{3.1}$$

Since $\underline{a}_i > 0, \underline{b}_j > 0$, then by Lemma 2.7, the linear system

$$\begin{cases} x_i^{\Delta}(t) = -a_i(t)x_i(t), \\ y_j^{\Delta}(t) = -b_j(t)y_j(t) \end{cases} \tag{3.2}$$

admits an exponential dichotomy. By Lemma 2.8, the solution of system (3.1) can be expressed as

$$
\begin{aligned}
z_{(\psi,\Psi)^T}(t) = \Bigg\{ &\int_{-\infty}^{t} e_{-a_1}(t,\sigma(s))\Bigg\{ \sum_{j=1}^{m} p_{j1}(s) f_j(\Psi_j(s-\tau_{j1}(s))) + I_1(s)\Bigg\}\Delta s, \cdots, \\
&\int_{-\infty}^{t} e_{-a_n}(t,\sigma(s))\Bigg\{ \sum_{j=1}^{m} p_{jn}(s) f_j(\Psi_j(s-\tau_{jn}(s))) + I_n(s)\Bigg\}\Delta s, \\
&\int_{-\infty}^{t} e_{-b_1}(t,\sigma(s))\Bigg\{ \sum_{i=1}^{n} q_{i1}(s) g_i(\psi_i(s-\vartheta_{i1}(s))) + L_1(s)\Bigg\}\Delta s, \cdots, \\
&\int_{-\infty}^{t} e_{-b_m}(t,\sigma(s))\Bigg\{ \sum_{i=1}^{n} q_{im}(s) g_i(\psi_i(s-\vartheta_{im}(s))) + L_m(s)\Bigg\}\Delta s\Bigg\}.
\end{aligned}
\tag{3.3}
$$

Define a mapping $\Phi : \mathbb{B} \to \mathbb{B}$ by setting

$$\Phi z(t) = z_{(\psi,\Psi)^T}(t),\ \forall\, z \in \mathbb{B}.$$

In view of (H_1)–(H_3), for any $z, \bar{z} \in \mathbb{B}$, where

$$z = \big(\psi_1, \psi_2, \cdots, \psi_n, \Psi_1, \Psi_2, \cdots, \Psi_m\big)^T,$$

$$\bar{z} = \big(\bar{\psi}_1, \bar{\psi}_2, \cdots, \bar{\psi}_n, \bar{\Psi}_1, \bar{\Psi}_2, \cdots, \bar{\Psi}_m\big)^T,$$

we have

$$
\begin{aligned}
|\Phi(z(t)) - \Phi(\bar{z}(t))| \le \Bigg\{ &\int_{-\infty}^{t} e_{-a_1}(t,\sigma(s))\Bigg\{ \sum_{j=1}^{m} |p_{j1}(s)(f_j(\Psi_j(s-\tau_{j1}(s))) \\
&-f_j(\bar{\Psi}_j(s-\tau_{j1}(s))))|\Bigg\}\Delta s, \cdots, \\
&\int_{-\infty}^{t} e_{-a_n}(t,\sigma(s))\Bigg\{ \sum_{j=1}^{m} |p_{jn}(s)(f_j(\Psi_j(s-\tau_{jn}(s))) \\
&-f_j(\bar{\Psi}_j(s-\tau_{jn}(s))))|\Bigg\}\Delta s, \\
&\int_{-\infty}^{t} e_{-b_1}(t,\sigma(s))\Bigg\{ \sum_{i=1}^{n} |q_{i1}(s)(g_i(\psi_i(s-\vartheta_{i1}(s))) \\
&-g_i(\bar{\psi}_i(s-\vartheta_{i1}(s))))|\Bigg\}\Delta s, \cdots, \\
&\int_{-\infty}^{t} e_{-b_m}(t,\sigma(s))\Bigg\{ \sum_{i=1}^{n} |q_{im}(s)(g_i(\psi_i(s-\vartheta_{im}(s)))
\end{aligned}
$$

$$
\begin{aligned}
&\quad -g_i(\bar{\psi}_i(s-\vartheta_{im}(s))))|\Big\}\Delta s\Big\} \\
\leq\ & \bigg(\frac{1}{\underline{a}_1}\Big(\sum_{j=1}^{m}\eta_j p_{j1}^{+}\sup_{t\geq -\hat{\tau}}|\Psi_j(t)-\bar{\Psi}_j(t)|\Big),\cdots,\frac{1}{\underline{a}_n}\Big(\sum_{j=1}^{m}\eta_j p_{jn}^{+}\sup_{t\geq -\hat{\tau}}|\Psi_j(t)-\bar{\Psi}_j(t)|\Big), \\
& \frac{1}{\underline{b}_1}\Big(\sum_{i=1}^{n}\lambda_i q_{1i}^{+}\sup_{t\geq -\vartheta}|\psi_i(t)-\bar{\psi}_i(t)|\Big),\cdots,\frac{1}{\underline{b}_m}\Big(\sum_{i=1}^{n}\lambda_i q_{mi}^{+}\sup_{t\geq \vartheta}|\psi_i(t)-\bar{\psi}_i(t)|\Big)\bigg)^T \\
=\ & \begin{bmatrix} A^{-1}PL & 0 \\ 0 & B^{-1}Q\Lambda \end{bmatrix}_{(n+m)\times(n+m)} \\
& \bigg(\sup_{t\geq -\hat{\tau}}|\Psi_1(t)-\bar{\Psi}_1(t)|,\cdots,\sup_{t\geq -\hat{\tau}}|\Psi_m(t)-\bar{\Psi}_m(t)|, \\
& \sup_{t\geq -\vartheta}|\psi_1(t)-\bar{\psi}_1(t)|,\cdots,\sup_{t\geq -\vartheta}|\psi_n(t)-\bar{\psi}_n(t)|\bigg)^T \\
=\ & F\bigg(\sup_{t\geq -\hat{\tau}}|\Psi_1(t)-\bar{\Psi}_1(t)|,\cdots,\sup_{t\geq -\hat{\tau}}|\Psi_m(t)-\bar{\Psi}_m(t)|, \\
& \sup_{t\geq -\vartheta}|\psi_1(t)-\bar{\psi}_1(t)|,\cdots,\sup_{t\geq -\vartheta}|\psi_n(t)-\bar{\psi}_n(t)|\bigg)^T \\
=\ & F\bigg(\sup_{t\geq -\hat{\tau}}|(z(t)-\bar{z}(t))_{n+1}|,\cdots,\sup_{t\geq -\hat{\tau}}|(z(t)-\bar{z}(t))_{n+m}|, \\
& \sup_{t\geq -\vartheta}|(z(t)-\bar{z}(t))_1|,\cdots,\sup_{t\geq -\vartheta}|(z(t)-\bar{z}(t))_n|\bigg)^T \\
\leq\ & F\bigg(\max_{1\leq p\leq n+m}\sup_{t\geq -\hat{\tau}}|(z(t)-\bar{z}(t))_p|,\cdots,\max_{1\leq p\leq n+m}\sup_{t\geq -\hat{\tau}}|(z(t)-\bar{z}(t))_p|, \\
& \max_{1\leq p\leq n+m}\sup_{t\geq -\vartheta}|(z(t)-\bar{z}(t))_p|,\cdots,\max_{1\leq p\leq n+m}\sup_{t\geq -\vartheta}|(z(t)-\bar{z}(t))_p|\bigg)^T, \qquad (3.4)
\end{aligned}
$$

where F is defined in Theorem 3.1. Let l be a positive integer. Then from (3.4), we get

$$
\begin{aligned}
|\Phi^l(z(t))-\Phi^l(\bar{z}(t))| \leq\ & F\bigg(\max_{1\leq p\leq n+m}\sup_{t\geq -\hat{\tau}}|(\Phi^{l-1}(z(t))-\Phi^{l-1}(\bar{z}(t)))_p|,\cdots, \\
& \max_{1\leq p\leq n+m}\sup_{t\geq -\hat{\tau}}|(\Phi^{l-1}(z(t))-\Phi^{l-1}(\bar{z}(t)))_p|, \\
& \max_{1\leq p\leq n+m}\sup_{t\geq -\vartheta}|(\Phi^{l-1}(z(t))-\Phi^{l-1}(\bar{z}(t)))_p|,\cdots, \\
& \max_{1\leq p\leq n+m}\sup_{t\geq -\vartheta}|(\Phi^{l-1}(z(t))-\Phi^{l-1}(\bar{z}(t)))_p|\bigg)^T, \\
=\ & F^l\bigg(\max_{1\leq p\leq n+m}\sup_{t\geq -\hat{\tau}}|(z(t)-\bar{z}(t))_p|,\cdots,\max_{1\leq p\leq n+m}\sup_{t\geq -\hat{\tau}}|(z(t)-\bar{z}(t))_p|,
\end{aligned}
$$

$$\max_{1\le p\le n+m}\sup_{t\ge-\vartheta}|(z(t)-\bar{z}(t))_p|,\cdots,\max_{1\le p\le n+m}\sup_{t\ge-\vartheta}|(z(t)-\bar{z}(t))_p|\Big)^T. \quad (3.5)$$

From the assumption $\rho(F) < 1$, we obtain

$$\lim_{l\to\infty} F^l = 0,$$

which implies that there exist a positive integer N and positive constant $\Theta < 1$ such that

$$\begin{aligned} F^N &= \begin{bmatrix} A^{-1}PL & 0 \\ 0 & B^{-1}Q\Lambda \end{bmatrix}^N \\ &= (h_{kp})_{(n+m)\times(n+m)}, \ \sum_{p=1}^{n+m} h_{kp} \le \Theta, k = 1,\cdots,n+m. \end{aligned} \quad (3.6)$$

In view of (3.5) and (3.6), one has

$$\begin{aligned} \|\Phi^N z - \Phi^N \bar{z}\| &= \max_{1\le p\le n+m}\sup_{t\ge-\max\{\hat{\tau},\vartheta\}} |\Phi^N(z(t)) - \Phi^N(\bar{z}(t))| \\ &\le \max_{1\le k\le n+m}\left\{\sum_{p=1}^{n+m} h_{kp}\right\}\max_{1\le p\le n+m}\sup_{t\ge-\max\{\hat{\tau},\vartheta\}}|(z(t)-\bar{z}(t))_p| \\ &\le \Theta\|z-\bar{z}\|. \end{aligned}$$

This implies that the mapping $\Phi^N\mathbb{B} \to \mathbb{B}$ is a contraction mapping.

By Lemma 2.12, Φ has exactly a fixed point z^* in $\mathbb{B}$ such that $\Phi(z^*) = z^*$. By (3.3) and (3.1), z^* satisfies (1.3). So, system (1.3) has an unique almost periodic solution. This completes the proof of Theorem 3.1. □

4 Global Exponential Stability

Now, in this section, we denote that $p_{ji}^+ = \sup_{t\in\mathbb{T}}|p_{ji}(t)|$, $q_{ij}^+ = \sup_{t\in\mathbb{T}}|p_{ij}(t)|$, $\underline{a_i} = a_i^m = \inf_{t\in\mathbb{T}}|a_i(t)|$, $\underline{b_j} = b_j^m = \inf_{t\in\mathbb{T}}|b_j(t)|$, $\overline{a_i} = a_i^M = \sup_{t\in\mathbb{T}}|a_i(t)|$, $\overline{b_j} = b_j^M = \sup_{t\in\mathbb{T}}|b_j(t)|$.

Suppose that $z^* = (x_1^*, x_2^*, \cdots, x_n^*, y_1^*, y_2^*, \cdots, y_m^*)^T = (z_1^*, z_2^*, \cdots, z_{n+m}^*)^T$ is an almost periodic solution of system (1.3). In this section, we will construct some suitable differential inequality to study the global exponential stability of the almost periodic solution.

Lemma 4.1 (See [15]). *Let $f \in C(\mathbb{T},\mathbb{R})$ be Δ-differentiable at t. Then*

$$D^+|f(t)|^\Delta \le \mathrm{sign}(f^\sigma(t))f^\Delta(t), \quad \textit{where } f^\sigma(t) = f(\sigma(t)).$$

Theorem 4.2. *Assume* (H_1)–(H_4) *and let $\tau_{ji}(t) \equiv \tau_{ji}$, $\vartheta_{ij}(t) \equiv \vartheta_{ij}$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, be nonnegative constants. Suppose further that*

(H_5) *There exists a positive constant* α *such that* $\sup_{t\in\mathbb{T}} w_i(\alpha,t) < 0,\ \sup_{t\in\mathbb{T}} w_j^*(\alpha,t) < 0$, *where*

$$\begin{aligned} w_i(\alpha,t) &= \alpha+(1+\alpha\mu(t))(2\mu(t)(a_i^M)^2-a_i^m)+\sum_{j=1}^{m}(1+\alpha\mu(t+\vartheta_{ij})) \\ &\quad\times(1+2\mu(t+\vartheta_{ij})b_j^M)e_\alpha(t+\vartheta_{ij},t)q_{ij}^+\lambda_i,\ i=1,2,\dots,n, \\ w_j^*(\alpha,t) &= \alpha+(1+\alpha\mu(t))(2\mu(t)(b_j^M)^2-b_j^m)+\sum_{i=1}^{n}(1+\alpha\mu(t+\tau_{ji})) \\ &\quad\times(1+2\mu(t+\tau_{ji})a_i^M)e_\alpha(t+\tau_{ji},t)p_{ji}^+\eta_j,\ j=1,2,\dots,m. \end{aligned}$$

Then the almost periodic solution of system (1.3) *is globally exponentially stable.*

Proof. According to Theorem 3.1, we know that system (1.3) has an almost periodic solution $z^*=(x_1^*,x_2^*,\cdots,x_n^*,y_1^*,y_2^*,\cdots,y_m^*)^T=(z_1^*,z_2^*,\cdots,z_{n+m}^*)^T$ is an almost periodic solution of system (1.3), suppose that $x(t)=(x_1(t),x_2(t),\dots,x_n(t),\ y_1(t),y_2(t),\dots,y_n(t))^T$ is an arbitrary solution of system (1.3). Set $x_i(t)-x_i^*(t)=\alpha_i(t), i=1,2,\dots,n,\ y_j(t)-y_j^*(t)=\beta_j(t),\ j=1,2,\dots,m$. Then it follows from system (1.3) that

$$\begin{cases} \alpha_i^\Delta(t)=-a_i(t)\alpha_i(t)+\displaystyle\sum_{j=1}^{m}p_{ji}(t)(f_j(y_j(t-\tau_{ji}(t)))-f_j(y_j^*(t-\tau_{ji}(t)))), \\ \beta_j^\Delta(t)=-b_j(t)\beta_j(t)+\displaystyle\sum_{i=1}^{n}q_{ij}(t)(g_i(x_i(t-\vartheta_{ij}(t)))-g_i(x_i^*(t-\vartheta_{ij}(t)))), \end{cases} \tag{4.1}$$

where $t\in\mathbb{T},\ t>0,\ i=1,\cdots,n,\ j=1,2\dots,m$. In view of system (4.1) and Lemma 4.1, for $t\in\mathbb{T}, t>0,\ i=1,2,\dots,n$, we have

$$\begin{aligned} D^+|\alpha_i(t)|^\Delta &\le -a_i(t)\mathrm{sign}(\alpha_i^\sigma(t))\alpha_i(t)+\sum_{j=1}^{m}p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})| \\ &= -a_i(t)\mathrm{sign}(\alpha_i^\sigma(t))[\alpha_i^\sigma(t)-\mu(t)\alpha_i^\Delta(t)]+\sum_{j=1}^{m}p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})| \\ &\le -a_i^m|\alpha_i^\sigma(t)|+\mu(t)a_i^M|\alpha_i^\Delta(t)|+\sum_{j=1}^{m}p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})| \\ &= -a_i^m|\alpha_i(t)+\mu(t)\alpha_i^\Delta(t)|+\mu(t)a_i^M|\alpha_i^\Delta(t)|+\sum_{j=1}^{m}p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})| \\ &\le -a_i^m|\alpha_i(t)|+2\mu(t)a_i^M|\alpha_i^\Delta(t)|+\sum_{j=1}^{m}p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})| \end{aligned}$$

$$\leq \quad (2\mu(t)(a_i^M)^2 - a_i^m)|\alpha_i(t)| + (1 + 2\mu(t)a_i^M)\sum_{j=1}^{m} p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})|.$$

Similarly, for $t \in \mathbb{T}, t > 0,\ j = 1, 2, \ldots, m$, we can also get

$$D^+|\beta_i(t)|^\Delta \quad \leq \quad (2\mu(t)(b_j^M)^2 - b_j^m)|\beta_j(t)| + (1 + 2\mu(t)b_i^M)\sum_{i=1}^{n} q_{ij}^+\lambda_i|\alpha_i(t-\vartheta_{ij})|.$$

Now, we construct the Lyapunov function

$$\begin{aligned} V(t) &= V_1(t) + V_2(t) + V_3(t) + V_4(t), \\ &\quad V_1(t) = \sum_{i=1}^{n} e_\alpha(t,s)|\alpha_i(t)|, \\ V_2(t) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{t-\tau_{ji}}^{t}(1+\alpha\mu(v+\tau_{ji}))(1+2\mu(v+\tau_{ji})a_i^M) \\ &\quad \times e_\alpha(v+\tau_{ji},s)p_{ji}^+\eta_j|\beta_j(v)|\Delta v, \\ V_3(t) &= \sum_{j=1}^{m} e_\alpha(t,s)|\beta_j(t)|, \\ V_4(t) &= \sum_{i=1}^{n}\sum_{j=1}^{m}\int_{t-\vartheta_{ij}}^{t}(1+\alpha\mu(v+\vartheta_{ij}))(1+2\mu(v+\vartheta_{ij})b_j^M) \\ &\quad \times e_\alpha(v+\vartheta_{ij},s)q_{ij}^+\lambda_i|\alpha_i(v)|\Delta v. \end{aligned}$$

For $t \in \mathbb{T}, t > 0$, calculating the delta derivative $D^+V(t)^\Delta$ of $V(t)$ along system (4.1), we can get

$$\begin{aligned} D^+V_1^\Delta(t) &= \sum_{i=1}^{n}\Big[\alpha e_\alpha(t,s)|\alpha_i(t)| + e_\alpha(\sigma(t),s)D^+|\alpha_i(t)|^\Delta\Big] \\ &\leq \sum_{i=1}^{n}\Big\{\alpha e_\alpha(t,s)|\alpha_i(t)| + (1+\alpha\mu(t))e_\alpha(t,s)\Big[(2\mu(a_i^M)^2 - a_i^m)|\alpha_i(t)| \\ &\quad +(1+2\mu a_i^M)\sum_{j=1}^{m}p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})|\Big]\Big\} \\ &= \sum_{i=1}^{n}\Big[\alpha + (1+\alpha\mu(t))(2\mu(t)(a_i^M)^2 - a_i^m)\Big]e_\alpha(t,s)|\alpha_i(t)| \\ &\quad +(1+\alpha\mu(t))e_\alpha(t,s)\sum_{i=1}^{n}\sum_{j=1}^{m}(1+2\mu a_i^M)p_{ji}^+\eta_j|\beta_j(t-\tau_{ji})| \end{aligned}$$

and

$$\begin{aligned} D^+V_2^\Delta(t) &\le \sum_{i=1}^{n}\sum_{j=1}^{m}(1+\alpha\mu(t+\tau_{ji}))(1+2\mu(t+\tau_{ji})a_i^M)e_\alpha(t+\tau_{ji},s)p_{ji}^+\eta_j|\beta_j(t)| \\ &\quad -(1+\alpha\mu(t))e_\alpha(t,s)\sum_{i=1}^{n}\sum_{j=1}^{m}(1+2\mu(t)a_i^M)p_{ji}^+\eta_j\big|\beta_j(t-\tau_{ji})\big|. \end{aligned}$$

Similarly, for $t \in \mathbb{T}, t>0,\ j=1,2,\ldots,m$, we can also obtain

$$\begin{aligned} D^+V_3^\Delta(t) &\le \sum_{j=1}^{m}\Big[\alpha+(1+\alpha\mu(t))(2\mu(t)(b_j^M)^2-b_j^m)\Big]e_\alpha(t,s)|\beta_j(t)| \\ &\quad +(1+\alpha\mu(t))e_\alpha(t,s)\sum_{j=1}^{m}\sum_{i=1}^{n}(1+2\mu b_j^M)q_{ij}^+\lambda_i|\alpha_i(t-\vartheta_{ij})| \end{aligned}$$

and

$$\begin{aligned} D^+V_4^\Delta(t) &\le \sum_{j=1}^{m}\sum_{i=1}^{n}(1+\alpha\mu(t+\vartheta_{ij}))(1+2\mu(t+\vartheta_{ij})b_j^M)e_\alpha(t+\vartheta_{ij},s)q_{ij}^+\lambda_i|\alpha_i(t)| \\ &\quad -(1+\alpha\mu(t))e_\alpha(t,s)\sum_{j=1}^{m}\sum_{i=1}^{n}(1+2\mu(t)b_j^M)q_{ij}^+\lambda_i\big|\alpha_i(t-\vartheta_{ij})\big|. \end{aligned}$$

By assumption (H_5), it follows that

$$\begin{aligned} & D^+(V(t))^\Delta = D^+(V_1(t)+V_2(t)+V_3(t)+V_4(t))^\Delta \\ &\le \sum_{i=1}^{n}\Big[\alpha+(1+\alpha\mu(t))(2\mu(t)(a_i^M)^2-a_i^m)\Big]e_\alpha(t,s)|\alpha_i(t)| \\ &\quad +\sum_{i=1}^{n}\sum_{j=1}^{m}(1+\alpha\mu(t+\tau_{ji}))(1+2\mu(t+\tau_{ji})a_i^M)e_\alpha(t+\tau_{ji},s)p_{ji}^+\eta_j|\beta_j(t)| \\ &\quad +\sum_{j=1}^{m}\Big[\alpha+(1+\alpha\mu(t))(2\mu(t)(b_j^M)^2-b_j^m)\Big]e_\alpha(t,s)|\beta_j(t)| \\ &\quad +\sum_{j=1}^{m}\sum_{i=1}^{n}(1+\alpha\mu(t+\vartheta_{ij}))(1+2\mu(t+\vartheta_{ij})b_j^M)e_\alpha(t+\vartheta_{ij},s)q_{ij}^+\lambda_i|\alpha_i(t)| \\ &= \sum_{i=1}^{n}\Big[\alpha+(1+\alpha\mu(t))(2\mu(t)(a_i^M)^2-a_i^m) \\ &\quad +\sum_{j=1}^{m}(1+\alpha\mu(t+\vartheta_{ij}))(1+2\mu(t+\vartheta_{ij})b_j^M)e_\alpha(t+\vartheta_{ij},t)q_{ij}^+\lambda_i\Big]e_\alpha(t,s)|\alpha_i(t)| \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m}\Big[\alpha+(1+\alpha\mu(t))(2\mu(t)(b_j^M)^2-b_j^m)\\
&+\sum_{i=1}^{n}(1+\alpha\mu(t+\tau_{ji}))(1+2\mu(t+\tau_{ji})a_i^M)e_\alpha(t+\tau_{ji},t)p_{ji}^{+}\eta_j\Big]e_\alpha(t,s)|\beta_j(t)|\\
\leq\ & 0,\ t\in\mathbb{T}^+.
\end{aligned}
$$

It follows that $V(t)\leq V(0)$ for all $t\in\mathbb{T}^+$. On the other hand, we have

$$
\begin{aligned}
V(0) &= V_1(0)+V_2(0)+V_3(0)+V_4(0)\\
&= \sum_{i=1}^{n}e_\alpha(0,s)|\alpha_i(0)|+\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{-\tau_{ji}}^{0}(1+\alpha\mu(v+\tau_{ji}))(1+2\mu(v+\tau_{ji})a_i^M)\\
&\quad \times e_\alpha(v+\tau_{ji},s)p_{ji}^{+}\eta_j|\beta_j(v)|\Delta v\\
&\quad +\sum_{j=1}^{m}e_\alpha(0,s)|\beta_j(0)|+\sum_{i=1}^{n}\sum_{j=1}^{m}\int_{-\vartheta_{ij}}^{0}(1+\alpha\mu(v+\vartheta_{ij}))(1+2\mu(v+\vartheta_{ij})b_j^M)\\
&\quad \times e_\alpha(v+\vartheta_{ij},s)q_{ij}^{+}\lambda_i|\alpha_i(v)|\Delta v\\
&\leq \sum_{i=1}^{n}\Big\{e_\alpha(0,s)+\sum_{j=1}^{m}\int_{-\vartheta_{ij}}^{0}(1+\alpha\mu(v+\vartheta_{ij}))(1+2\mu(v+\vartheta_{ij})b_j^M)\\
&\quad \times e_\alpha(v+\vartheta_{ij},s)q_{ij}^{+}\lambda_i\Delta v\Big\}\max_{v\in[-\vartheta,0]_{\mathbb{T}}}|\alpha_i(v)|\\
&\quad +\sum_{j=1}^{m}\Big\{e_\alpha(0,s)+\sum_{i=1}^{n}\int_{-\tau_{ji}}^{0}(1+\alpha\mu(v+\tau_{ji}))(1+2\mu(v+\tau_{ji})a_i^M)\\
&\quad \times e_\alpha(v+\tau_{ji},s)p_{ji}^{+}\eta_j\Delta v\Big\}\max_{v\in[-\hat{\tau},0]_{\mathbb{T}}}|\beta_j(v)|\\
&\leq \Gamma(\alpha)\bigg(\sum_{i=1}^{n}\max_{v\in[-\vartheta,0]_{\mathbb{T}}}|\phi_i(v)-x_i^*(v)|+\sum_{j=1}^{m}\max_{v\in[-\hat{\tau},0]_{\mathbb{T}}}|\varphi_i(v)-y_j^*(v)|\bigg),
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_1(\alpha) &= \max_{1\leq i\leq n}\bigg\{e_\alpha(0,s)+\sum_{j=1}^{m}\int_{-\vartheta_{ij}}^{0}(1+\alpha\mu(v+\vartheta_{ij}))(1+2\mu(v+\vartheta_{ij})b_j^M)\\
&\quad \times e_\alpha(v+\vartheta_{ij},s)q_{ij}^{+}\lambda_i\Delta v\bigg\},\\
\Gamma_2(\alpha) &= \max_{1\leq j\leq m}\bigg\{e_\alpha(0,s)+\sum_{i=1}^{n}\int_{-\tau_{ji}}^{0}(1+\alpha\mu(v+\tau_{ji}))(1+2\mu(v+\tau_{ji})a_i^M)\\
&\quad \times e_\alpha(v+\tau_{ji},s)p_{ji}^{+}\eta_j\Delta v\bigg\}
\end{aligned}
$$

and $\Gamma(\alpha) = \max\{\Gamma_1(\alpha), \Gamma_2(\alpha)\}$. It is obvious that

$$\bigg(\sum_{i=1}^{n}|x_i(t)-x_i^*(t)|+\sum_{j=1}^{m}|y_j(t)-y_j^*(t)|\bigg)e_\alpha(t,s)\leq V(t)\leq V(0)$$
$$\leq \Gamma(\alpha)\bigg(\sum_{i=1}^{n}\max_{v\in[-\vartheta,0]_{\mathbb{T}}}|\phi_i(v)-x_i^*(v)|+\sum_{j=1}^{m}\max_{v\in[-\hat{\tau},0]_{\mathbb{T}}}|\varphi_i(v)-y_j^*(v)|\bigg).$$

Hence,

$$\sum_{i=1}^{n}|x_i(t)-x_i^*(t)|+\sum_{j=1}^{m}|y_j(t)-y_j^*(t)|\leq\Gamma(\alpha)e_{\ominus\alpha}(t,s)\bigg(\sum_{i=1}^{n}\|x_i-x^*\|+\sum_{j=1}^{m}\|y_i-y^*\|\bigg).$$

Since $\Gamma(\alpha)\geq 1$, from Definition 2.11, the positive periodic solution of system (1.3) is globally exponentially stable. This completes the proof. □

5 Examples and Numerical Simulations

Consider the BAM neural network with delays

$$\begin{cases} x_i^{\Delta}(t) = -a_i(t)x_i(t)+\sum_{j=1}^{m}p_{ji}(t)f_j(y_j(t-\tau_{ji}(t))) \\ \qquad +I_i(t),\ t\in\mathbb{T},\ i=1,2,\cdots,m, \\ y_j^{\Delta}(t) = -b_j(t)y_j(t)+\sum_{i=1}^{n}q_{ij}(t)g_i(x_i(t-\vartheta_{ij}(t))) \\ \qquad +L_j(t),\ t\in\mathbb{T},\ j=1,2,\cdots,n, \end{cases} \tag{5.1}$$

where $I_i(t)=\sin t$, $L_j(t)=\cos\sqrt{3}t$, $g_i(x_i(t-\vartheta_{ij}))=\frac{1}{2}\sin(x_i(t-\vartheta_{ij}))$, $f_j(y_j(t-\tau_{ji}))=\cos(y_j(t-\tau_{ji}))$, $t\in\mathbb{T}$, $\lambda_i=\frac{1}{2}$, $\eta_j=1$, $i=j=1,2$.

Example 5.1. Consider $\mathbb{T}=\mathbb{R}$, $\vartheta_{ij}=\tau_{ji}\equiv 0.003$, $i=j=1,2$,

$$a_1(t)=b_2(t)=2-\sin t,\ a_2(t)=b_1(t)=2-\cos t.$$

Then

$$\underline{a}_1=\underline{a}_2=\underline{b}_1=\underline{b}_2=1.$$

Let

$$p_{11}(t)=0.05\sin\sqrt{2}t,\ p_{12}(t)=0.1\cos\sqrt{5}t,\ p_{21}(t)=0.15\cos t,\ p_{22}(t)=0.05\sin t.$$
$$q_{11}(t)=0.25\sin\sqrt{3}t,\ q_{12}(t)=0.05\cos t,\ q_{21}(t)=0.05\cos t,\ q_{22}(t)=0.5\sin t.$$

Then

$$F = \begin{bmatrix} 0.05 & 0.1 & 0 & 0 \\ 0.15 & 0.05 & 0 & 0 \\ 0 & 0 & 0.125 & 0.025 \\ 0 & 0 & 0.025 & 0.25 \end{bmatrix}.$$

By a direct calculation, we know that $\rho(F) = 0.2548$ and $w_1(0.001, t) = -0.8490 < 0$, $w_2(0.001, t) = -0.7240 < 0$, $w_1^*(0.001, t) = -0.8490 < 0$, $w_2^*(0.001, t) = -0.7990 < 0$. Thus, ($H_1$)–($H_5$) are satisfied. According to Theorems 3.1 and Theorem 4.2, system (5.1) has an unique almost periodic solution, which is globally exponentially stable (see Figure 5.1).

Example 5.2. Consider $\mathbb{T} = \mathbb{Z}$, $\vartheta_{ij} = \tau_{ji} = 0.001$, $i = j = 1, 2$,

$$a_1(t) = b_2(t) = 0.02 + 0.01 \sin t,\ a_2(t) = b_1(t) = 0.02 + 0.01 \cos t.$$

Then

$$\underline{a}_1 = \underline{a}_2 = \underline{b}_1 = \underline{b}_2 = 0.01,\ \overline{a}_1 = \overline{a}_2 = \overline{b}_1 = \overline{b}_2 = 0.03.$$

Let

$$p_{11}(t) = 0.005 \sin \sqrt{2}t,\ p_{12}(t) = 0.001 \cos \sqrt{5}t,$$

$$p_{21}(t) = 0.002 \cos t,\ p_{22}(t) = 0.005 \sin t.$$

$$q_{11}(t) = 0.001 \sin \sqrt{3}t,\ q_{12}(t) = 0.005 \cos t,$$

$$q_{21}(t) = 0.005 \cos t,\ q_{22}(t) = 0.001 \sin t.$$

Then

$$F = \begin{bmatrix} 0.05 & 0.01 & 0 & 0 \\ 0.02 & 0.05 & 0 & 0 \\ 0 & 0 & 0.005 & 0.025 \\ 0 & 0 & 0.025 & 0.005 \end{bmatrix}.$$

By a direct calculation, we know that

$$\rho(F) = 0.0641 < 1$$

and

$$w_1(0.001, t) = -0.0755 < 0,\ w_2(0.001, t) = -0.0755 < 0,$$

$$w_1^*(0.001, t) = -0.0719 < 0,\ w_2^*(0.001, t) = -0.0707 < 0.$$

Thus, (H_1)–(H_5) are satisfied. According to Theorems 3.1 and Theorem 4.2, system (5.1) has an unique almost periodic solution, which is globally exponentially stable (see Figure 5.2).

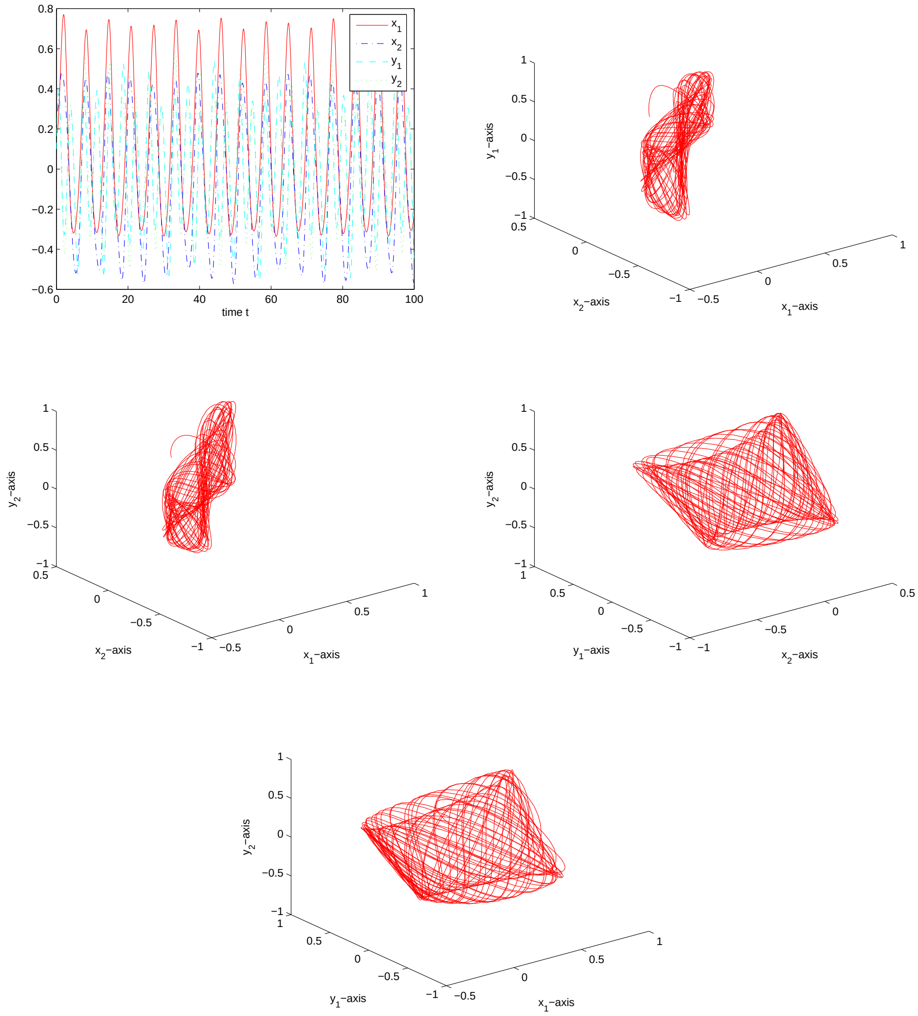

Figure 5.1: Transient responses of states x_1, x_2, y_1, y_2 in Example 5.1

6 Conclusion

Using the exponential dichotomy of linear dynamic equations on time scales and the time scale calculus theory, some sufficient conditions are derived to guarantee the existence and exponential stability of the almost periodic solution to a class of BAM neural networks with variable coefficients are studied on almost periodic time scales. To the best of our knowledge, the results presented here have been not appeared in the related literature. Besides, the results obtained in this paper possess feasibility. Moreover, the method in this paper may be applied to some other type neural networks with or without impulse on time scales.

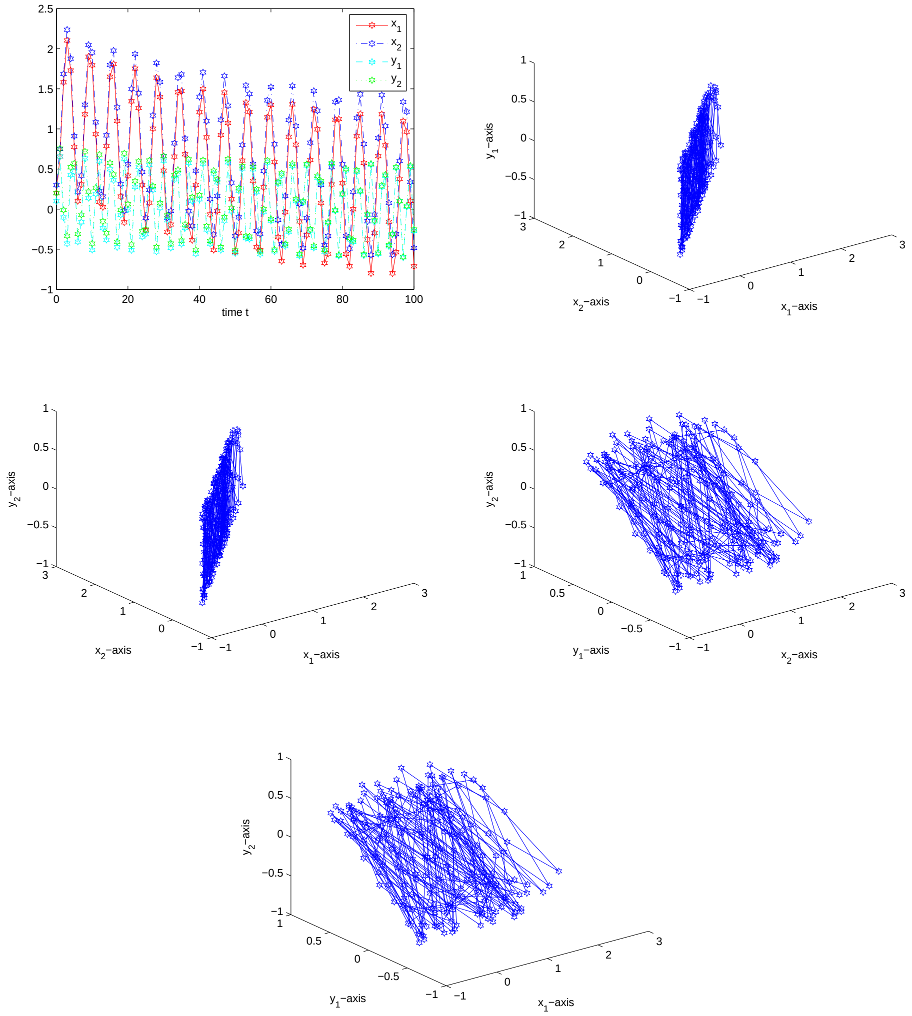

Figure 5.2: Transient responses of states x_1, x_2, y_1, y_2 in Example 5.2

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