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Oscillation Criteria for a Class of First-Order Forced Differential Equations under Impulse Effects

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Abstract

In this paper, we study oscillatory behaviour of all solutions of first-order delay differential equations with a forced term of the form:

$$\begin{cases} \big[x(t) - r(t)x\big(\rho(t)\big)\big]' + p(t)x\big(\tau(t)\big) \\ \qquad\qquad - q(t)x\big(\sigma(t)\big) = f(t), \quad t \in [t_0, \infty)\backslash\{\theta_n\}_{n\in\mathbb{N}}, \\ x(\theta_n^+) = I_n\big(x(\theta_n)\big), \quad n \in \mathbb{N}, \end{cases}$$

where $\{\theta_n\}_{n\in\mathbb{N}}$ is the set of fixed impulse points, ρ, τ, σ are delay functions and r, p, q are nonnegative continuous coefficients, while f is an integrable forcing term. Some examples are given to illustrate the applicability of the new results.

AMS Subject Classifications: 39A10, 34C10.
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1 Introduction

In this paper, we consider the oscillation of all solutions of the following type neutral delay differential equations involving positive and negative coefficients under impulse effects:

$$\begin{cases} \big[x(t) - r(t)x\big(\rho(t)\big)\big]' + p(t)x\big(\tau(t)\big) \\ \qquad\qquad - q(t)x\big(\sigma(t)\big) = f(t), \quad t \in [t_0, \infty)\backslash\{\theta_n\}_{n\in\mathbb{N}}, \\ x(\theta_n^+) = I_n\big(x(\theta_n)\big), \quad n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $t_0 \in \mathbb{R}$ and $\{\theta_n\}_{n\in\mathbb{N}} \subset [t_0, \infty)$ is an increasing divergent sequence of impulse points, with the following primary assumptions:

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(H1) $\rho, \tau, \sigma \in C([t_0, \infty), \mathbb{R})$ are increasing unbounded functions, which satisfy $\rho(t) \leq t$ and $\tau(t) \leq \sigma(t) \leq t$ for all $t \in [t_0, \infty)$. $\alpha \in C^1([t_0, \infty), \mathbb{R})$ is defined by

$$\alpha(t) := \sigma^{-1}\big(\tau(t)\big) \quad \text{for } t \geq t_0 \tag{1.2}$$

and satisfies $\alpha(t) < t$ for all $t \in [t_0, \infty)$. Moreover, ρ, σ satisfy $\{\rho^{-1}(\theta_n)\}_{n\in\mathbb{N}}$, $\{\sigma^{-1}(\theta_n)\}_{n\in\mathbb{N}} \subset \{\theta_n\}_{n\in\mathbb{N}}$. Here, ρ^{-1} and σ^{-1} denote the inverses of the functions ρ and σ, respectively.

(H2) $p, q \in C([t_0, \infty), \mathbb{R}^+)$ satisfy

$$h(t) := \frac{p\big(\alpha^{-1}(t)\big)}{\alpha'\big(\alpha^{-1}(t)\big)} - q(t) \geq 0 \quad \text{for } t \geq t_0, \tag{1.3}$$

where $\alpha \in C^1([t_0, \infty), \mathbb{R})$ is defined in (1.2), and $h \not\equiv 0$ on (θ_n, θ_{n+1}) for all $n \in \mathbb{N}$.

In the case of constant delays, i.e., $\tau(t) = t - \tau_0$ and $\sigma(t) = t - \sigma_0$, where $\tau_0 \geq \sigma_0 \geq 0$, we have $\alpha(t) = t - \tau_0 + \sigma_0$ and $h(t) = p(t - \sigma_0 + \tau_0) - q(t)$ for $t \geq t_0$, while for pantograph equations, i.e., $\tau(t) = t/\tau_0$ and $\sigma(t) = t/\sigma_0$, where $\tau_0 \geq \sigma_0 \geq 1$, we have $\alpha(t) = \sigma_0 t/\tau_0$ and $h(t) = p(\tau_0 t/\sigma_0) - q(t)$ for $t \geq t_0$.

(H3) $I_n \in C(\mathbb{R}, \mathbb{R})$ for all $n \in \mathbb{N}$, and there exists a sequence of positive reals $\{\lambda_n\}_{n\in\mathbb{N}}$ such that $\lambda_n \leq I_n(x)/x \leq 1$ for all $x \in \mathbb{R}\backslash\{0\}$ and $n \in \mathbb{N}$.

The set of piecewise left continuous functions $PLC_\theta([t_0, \infty), \mathbb{R}^+)$ with respect to the impulse sequence $\{\theta_n\}_{n\in\mathbb{N}}$ consists of the functions $\nu \in PLC_\theta([t_0, \infty), \mathbb{R}^+)$ such that the following three properties hold:

(P1) ν is left-continuous on $(t_0, \theta_1]$ and $(\theta_n, \theta_{n+1}]$ for all $n \in \mathbb{N}$,

(P2) $\nu(\theta_n^+) := \lim_{t\to\theta_n^+} \nu(t)$ exists for all $n \in \mathbb{N}$,

(P3) $\nu(\theta_n) = \nu(\theta_n^-) := \lim_{t\to\theta_n^-} \nu(t)$ for all $n \in \mathbb{N}$.

Below, we list some more additional hypothesis required for the study of oscillation of solutions.

(H4) $r \in PLC_\theta([t_0, \infty), \mathbb{R}_0^+)$ satisfies $\lambda_{\kappa(n)} r(\theta_n^+) \geq r(\theta_n)$, where $\kappa : \mathbb{N} \to \mathbb{N}$ satisfies $\kappa(n) < n$ and $\theta_{\kappa(n)} = \rho(\theta_n)$, for all $n \in \mathbb{N}$.

(H5) $f \in L^1([t_0, \infty)) \cap PLC_\theta([t_0, \infty), \mathbb{R})$.

To mention the significance of this work, we would like to continue the paper by giving a short brief on the works, which examine particular equations that can be derived from (1.1).

Let $I_n \equiv \mathrm{I}$, where I is the identity function, and $\rho(t) = t - \rho_0$, $\tau(t) = t - \tau_0$, $\sigma(t) = t - \sigma_0$, where $\rho_0, \tau_0, \sigma_0 \geq 0$, then (1.1) reduces to the following equation:

$$\big[x(t) - r(t)x(t - \rho_0)\big]' + p(t)x(t - \tau_0) - q(t)x(t - \sigma_0) = f(t), \quad t \geq t_0, \tag{1.4}$$

of which oscillatory nature of all solutions have been studied extensively in the literature (see [5–14, 16, 17] and the papers cited therein). Unlike to these mentioned papers, we shall introduce a new type of companion transformation, which involves the positive coefficient as the integrand.

Again, in (1.1), let $I_n \equiv \mathrm{I}$, $f \equiv 0$ and $\rho(t) = t/\rho_0$, $\tau(t) = t/\tau_0$, $\sigma(t) = t/\sigma_0$ with $\rho_0, \tau_0, \sigma_0 \geq 1$, then we obtain the following equation:

$$\big[x(t) - r(t)x(t/\rho_0)\big]' + p(t)x(t/\tau_0) - q(t)x(t/\sigma_0) = 0, \quad t \geq t_0. \tag{1.5}$$

So called Euler-type equation (1.5) is studied in [4]. Clearly, our results not only generalize (to arbitrary delays) but also improve most of the recent results in the literature.

To the best of our knowledge, [1] is the first paper, which attempted to study (1.1) with arbitrary delays, $I_n \equiv \mathrm{I}$ and $r, f \equiv 0$ of the form:

$$x'(t) + p(t)x\big(\tau(t)\big) - q(t)x\big(\sigma(t)\big) = 0, \quad t \geq t_0. \tag{1.6}$$

The method employed in that paper is indeed very interesting and different than the papers mentioned previously as it uses tools from functional analysis, but unfortunately, there are some inconsistencies in the proof of [1, Theorem 1]. To salvage the results in [1], they restricted their attention to those equations of which every nonoscillatory solution is eventually monotonic (see [2]). Consequently, our sufficient conditions on the oscillation of (1.1) seem to be the first ones related to all solutions of the neutral delay differential equations involving opposite signed coefficients, arbitrary delays and impulse effects.

Let $f \equiv 0$, $\rho(t) = t - \rho_0$, $\tau(t) = t - \tau_0$, $\sigma(t) = t - \sigma_0$, then (1.1) reduces to the following form:

$$\begin{cases} \big[x(t) - r(t)x(t - \rho_0)\big]' + p(t)x(t - \tau_0) \\ \qquad\qquad\qquad - q(t)x(t - \sigma_0) = 0, \quad t \in [t_0, \infty)\backslash\{\theta_n\}_{n\in\mathbb{N}}, \\ x(\theta_n^+) = I_n\big(x(\theta_n)\big), \quad n \in \mathbb{N}. \end{cases} \tag{1.7}$$

In [14], (1.7) is studied as well. The authors made a little mistake in the proof of [14, Theorem 2.1] by assuming that the companion functions, which are defined by different parameters, are the same, and the statement of [14, Theorem 2.1] is corrected in [15]. While extending the results in [14] to forced type equations, we also salvage [14, Theorem 2.1] by giving a correct proof for it in the best possible condition, and extend the results given in [17].

As a last note, with all humbleness, we would like to say that our results correct some erroneous results on forced differential equations with positive and negative coefficients, i.e., the proofs of main theorems of [8, 10] yield problems when studying eventually negative solutions, and the forced term is supposed to be eventually positive without being mentioned in the main result of [16].

Let $t_{-1} := \min\{\rho(t_0), \tau(t_0), \sigma(t_0)\}$, with (1.1), we associate an initial condition of the form

$$x(t) = \Psi(t), \quad t \in [t_{-1}, t_0], \tag{1.8}$$

where $\Psi \in C([t_{-1}, t_0], \mathbb{R})$.

Definition 1.1. A real-valued function x is called a *solution* corresponding to t_0 of the initial value problem (1.1) and (1.8) if the following two conditions hold:

(C1) $x(t) = \Psi(t)$ for all $t \in [t_{-1}, t_0]$ and $x \in PLC_\theta([t_0, \infty), \mathbb{R})$,

(C2) $x(t) - r(t)x(\rho(t))$ is continuously differentiable for all $t \in [t_0, \infty)\backslash\{\theta_n\}_{n\in\mathbb{N}}$. Further, x satisfies the differential equation in (1.1) on $[t_0, \infty)\backslash\{\theta_n\}_{n\in\mathbb{N}}$, while satisfies the impulse condition for all $n \in \mathbb{N}$.

Definition 1.2. A solution x of (1.1) is called *oscillatory* if there exists an increasing divergent sequence $\{\xi_n\}_{n\in\mathbb{N}}$ such that $x(\xi_n)x(\xi_n^+) \leq 0$ for all $n \in \mathbb{N}$; otherwise, the solution is called *nonoscillatory*.

Definition 1.3. (1.1) is called *oscillatory* if it possesses oscillatory solutions for every initial function Ψ given by (1.8).

Note that all solutions of (1.1) are oscillatory provided that there exists an increasing divergent subsequence $\{n_k\}_{k\in\mathbb{N}} \subset \mathbb{N}$ satisfying $I_{n_k}(x)/x \leq 0$ for all $x \in \mathbb{R}\backslash\{0\}$ and all $k \in \mathbb{N}$. Therefore, examining (1.1) under (H3) makes sense.

2 Main Results

In this section, we always suppose that (H1)–(H5) hold. We introduce

$$\beta(t) := \begin{cases} \tau(t), & r \equiv 0 \\ \min\{\rho(t), \tau(t)\}, & r \not\equiv 0 \end{cases} \text{ and } \gamma(t) := \begin{cases} \sigma(t), & r \equiv 0,\ q \not\equiv 0 \\ \rho(t), & r \not\equiv 0,\ q \equiv 0 \\ \max\{\rho(t), \sigma(t)\}, & r \not\equiv 0,\ q \not\equiv 0 \end{cases}$$

for $t \in [t_0, \infty)$. Throughout the paper, h defined by (1.3) is assumed to satisfy $h \not\equiv 0$.

Lemma 2.1. *Assume that*

$$r(t) + \int_{\alpha(t)}^{t} q(\eta)\,\mathrm{d}\eta \leq 1 \quad \textit{for all sufficiently large } t. \tag{2.1}$$

Let x be a solution of (1.1)*, and set the companion function z_x of x by*

$$z_x(t) := x(t) - r(t)x\big(\rho(t)\big) - \int_{\alpha(t)}^{t} \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)} x\big(\sigma(\eta)\big)\,\mathrm{d}\eta + \int_{t}^{\infty} f(\eta)\,\mathrm{d}\eta \tag{2.2}$$

for $t \in [\beta^{-1}(t_0), \infty)$. Then, the followings hold:

(i) if x is an eventually positive solution of (1.1)*, then z_x is of positive sign eventually,*

(ii) if x is an eventually negative solution of (1.1)*, then z_x is of negative sign eventually.*

Before starting the proof, we would like to mention that, z_x left-continuous on $[\beta^{-1}(t_0), \infty)\backslash\{\theta_n\}_{n\in\mathbb{N}}$. Since $x \circ \rho$ and $x \circ \sigma$ have discontinuities at $\{\rho^{-1}(\theta_n)\}_{n\in\mathbb{N}}$ and $\{\sigma^{-1}(\theta_n)\}_{n\in\mathbb{N}}$, respectively, and $\{\rho^{-1}(\theta_n)\}_{n\in\mathbb{N}}, \{\sigma^{-1}(\theta_n)\}_{n\in\mathbb{N}} \subset \{\theta_n\}_{n\in\mathbb{N}}$ holds by (H1), z_x is only discontinuous at the points $\{\theta_n\}_{n\in\mathbb{N}}$.

Proof. (i) There exists $n_1 \in \mathbb{N}(n_0)$ such that $x(t), x(\rho(t)), x(\tau(t)), x(\sigma(t)) > 0$ and (2.1) hold for all $t \in [\theta_{n_1}, \infty)$. From (1.1) and (2.2), we get

$$\begin{aligned} z_x'(t) &= \big[x(t) - r(t)x\big(\rho(t)\big)\big]' - \frac{p\big(\alpha^{-1}(t)\big)}{\alpha'\big(\alpha^{-1}(t)\big)} x\big(\sigma(t)\big) - p(t)x\big(\tau(t)\big) - f(t) \\ &\leq -h(t)x\big(\sigma(t)\big) \leq 0 \end{aligned} \tag{2.3}$$

for all $t \in (\theta_n, \theta_{n+1}]$ and all $n \in \mathbb{N}(n_1)$. Now, we prove that z_x is nonincreasing on $[\theta_{n_1}, \infty)$. Note that $x(\theta_n^+) = I_n\big(x(\theta_n)\big) \leq x(\theta_n)$ for all $n \in \mathbb{N}(n_1)$ by (H3). Considering (H3) and (H4), we have

$$\begin{aligned} r(\theta_n^+)x(\theta_{\kappa(n)}^+) &= r(\theta_n^+)I_{\kappa(n)}\big(x(\theta_{\kappa(n)})\big) \geq \lambda_{\kappa(n)} r(\theta_n^+)x(\theta_{\kappa(n)}) \\ &\geq r(\theta_n)x(\theta_{\kappa(n)}), \end{aligned} \tag{2.4}$$

for all $n \in \mathbb{N}(n_1)$. In view of (2.2) and (2.4), we obtain

$$\begin{aligned} z_x(\theta_n^+) =& x(\theta_n^+) - r(\theta_n^+)x(\theta_{\kappa(n)}^+) - \int_{\alpha(\theta_n)}^{\theta_n} \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)} x\big(\sigma(\eta)\big)\,\mathrm{d}\eta + \int_{\theta_n}^{\infty} f(\eta)\,\mathrm{d}\eta \\ \leq& x(\theta_n) - r(\theta_n)x(\theta_{\kappa(n)}) - \int_{\alpha(\theta_n)}^{\theta_n} \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)} x\big(\sigma(\eta)\big)\,\mathrm{d}\eta + \int_{\theta_n}^{\infty} f(\eta)\,\mathrm{d}\eta \\ =& z_x(\theta_n), \end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}(n_1)$, which implies together with (2.3) that z_x is nonincreasing on $[\theta_{n_1}, \infty)$. By (H2), z_x is of constant sign on $[\theta_{n_2}, \infty)$ for some $n_2 \in \mathbb{N}(n_1)$. To prove that z_x is positive on $[\theta_{n_2}, \infty)$, suppose on the contrary that $z_x(\theta_{n_2}) \leq 0$. In view of (H2) and (H5), there exists $n_3 \in \mathbb{N}(n_2)$ satisfying

$$z_x(\theta_{n_3}) < 0 \quad \text{and} \quad \left| \int_t^\infty f(\eta)\,\mathrm{d}\eta \right| < -\frac{1}{2} z_x(\theta_{n_3}) \quad \text{for all } t \in [\theta_{n_3}, \infty).$$

Therefore, considering nonincreasing nature of z_x and integrating the inequality (2.3) over $[\theta_{n_3}, t)$, where $t \geq \alpha^{-1}(\theta_{n_3})$, we obtain

$$\begin{aligned} z_x(t) &= z_x(\theta_{n_3}) - \int_{\theta_{n_3}}^t h(\eta) x\big(\sigma(\eta)\big)\,\mathrm{d}\eta - \sum_{\theta_{n_3} < \theta_k < t} \big[z_x(\theta_k) - z_x(\theta_k^+)\big] \\ &\leq z_x(\theta_{n_3}) - \int_{\theta_{n_3}}^t h(\eta) x\big(\sigma(\eta)\big)\,\mathrm{d}\eta, \end{aligned}$$

or equivalently

$$\begin{aligned} x(t) \leq & z_x(\theta_{n_3}) + r(t) x\big(\rho(t)\big) - \int_{\theta_{n_3}}^t h(\eta) x\big(\sigma(t)\big)\,\mathrm{d}\eta \\ & + \int_{\alpha(t)}^t \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)} x\big(\sigma(\eta)\big)\,\mathrm{d}\eta - \int_t^\infty f(\eta)\,\mathrm{d}\eta \\ = & z_x(\theta_{n_3}) + r(t) x\big(\rho(t)\big) - \int_{\theta_{n_3}}^{\alpha(t)} h(\eta) x\big(\sigma(\eta)\big)\,\mathrm{d}\eta \\ & + \int_{\alpha(t)}^t q(\eta) x\big(\sigma(\eta)\big)\,\mathrm{d}\eta - \int_t^\infty f(\eta)\,\mathrm{d}\eta \\ \leq & \frac{1}{2} z_x(\theta_{n_3}) + r(t) x\big(\rho(t)\big) + \int_{\alpha(t)}^t q(\eta) x\big(\sigma(\eta)\big)\,\mathrm{d}\eta \end{aligned} \tag{2.6}$$

for all $t \in [\theta_{n_4}, \infty)$, where $\theta_{n_4} \geq \alpha^{-1}(\theta_{n_3})$. Now, set $L := \limsup_{t\to\infty} x(t)$ and $y(t) := \sup\{x(\eta) : \ \eta \in [\beta(t), \gamma(t))\}$ for $t \in [\theta_{n_4}, \infty)$. Clearly, $\limsup_{t\to\infty} y(t) = L$ holds. Hence, from (2.1) and (2.6), we have

$$\begin{aligned} x(t) &\leq \frac{1}{2} z_x(\theta_{n_3}) + \left(r(t) + \int_{\alpha(t)}^t q(\eta)\,\mathrm{d}\eta \right) y(t) \\ &\leq \frac{1}{2} z_x(\theta_{n_3}) + y(t) \end{aligned} \tag{2.7}$$

for all $t \in [\theta_{n_4}, \infty)$. We claim that L is finite. That is, x is bounded. If not, there exists $T \in [\theta_{n_4}, \infty)$ such that $x(T^+) = y(T^+)$ holds. This indicates from (2.7) that

$$
\begin{aligned}
x(T^+) &\le \frac{1}{2} z_x(\theta_{n_3}) + y(T^+) \\
&= \frac{1}{2} z_x(\theta_{n_3}) + x(T^+),
\end{aligned}
$$

which is a contradiction to $z_x(\theta_{n_3}) < 0$. Therefore, L is a finite constant. Taking upper limit on both sides of (2.7) as $t \to \infty$, we see that $L \le z_x(\theta_{n_3})/2 + L$, which is also a contradiction. This contradiction proves that z_x is an eventually positive function.

(ii) Following the proof of (i), one can easily prove (ii). Thus, we omit.

□

Now, for a nondecreasing arbitrary function $\varphi \in C([t_0, \infty), \mathbb{R}^+)$, we define the function $\Phi : [t_0, \infty)^2 \to \mathbb{R}^+$ by

$$
\Phi(t, s) := \max \left\{ \int_{\xi}^{\beta^{-1}(\xi)} \frac{\mathrm{d}\eta}{\varphi(\eta)} \,:\, \xi \in [s, t] \right\} \quad \text{for } s, t \in [t_0, \infty). \tag{2.8}
$$

Note that the function $\Phi(\cdot, s)$, which plays the major role in the proof of the following results, is increasing with respect to its first component on $[s, \infty)$ for each fixed $s \in [t_0, \infty)$.

Lemma 2.2. *Suppose that*

$$
r(t) + \int_{\alpha(t)}^{t} \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)} \, \mathrm{d}\eta \ge 1 \quad \textit{for all sufficiently large } t \tag{2.9}
$$

and that

$$
\lim_{n \to \infty} f_n\big(\beta^{-n}(t)\big) = 0 \quad \textit{and} \quad \lim_{n \to \infty} f_n\big(\gamma^{-n}(t)\big) = 0 \tag{2.10}
$$

for each fixed sufficiently large t, where the recursion $f_n : [t_0, \infty) \to \mathbb{R}^+$ is defined by

$$
f_n(t) := \begin{cases} 0, \quad n = 0 \\ r(t) f_{n-1}\big(\rho(t)\big) + \displaystyle\int_{\alpha(t)}^{t} \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)} f_{n-1}\big(\sigma(\eta)\big) \, \mathrm{d}\eta \\ \qquad\qquad + \displaystyle\int_{t}^{\infty} f(\eta) \, \mathrm{d}\eta, \quad n \in \mathbb{N} \end{cases} \tag{2.11}
$$

for $t \in [\beta^{-n}(t_0), \infty)$. Further, suppose that there exists a nondecreasing function $\varphi \in C([t_0, \infty), \mathbb{R}^+)$ such that the second-order impulsive differential inequality

$$
\begin{cases} x''(t) + \dfrac{\lambda h(t)}{\varphi(t)\Phi(t, s)} x(t) \le 0, & t \in [t_0, \infty) \backslash \{\theta_n\}_{n \in \mathbb{N}}, \\ x'(\theta_n^+) \le x'(\theta_n), & n \in \mathbb{N}, \\ x(\theta_n^+) = x(\theta_n), & n \in \mathbb{N}, \end{cases} \tag{2.12}
$$

where $s \in [t_0, \infty)$*,* $\lambda \in (0,1)$ *and* $\Phi(\cdot, s)$ *is defined by* (2.8)*, has no eventually positive solutions. Then, the followings are true:*

(i) if x *is an eventually positive solution of* (1.1)*, then* z_x *introduced in* (2.2) *is of negative sign eventually.*

(ii) if x *is an eventually negative solution of* (1.1)*, then* z_x *is of positive sign eventually.*

Proof. (i) Similar to the proof of Lemma 2.1, it follows that there exists $n_1 \in \mathbb{N}(n_0)$ such that $\theta_{n_1} \geq s$, $x(t), x(\rho(t)), x(\beta(t)), x(\sigma(t)) > 0$, $z_x(t)$ is nonincreasing, (2.3), (2.9) and (2.12) hold for all $t \in [\theta_{n_1}, \infty)$. Therefore, z_x is of constant sign eventually. To prove z_x is of negative sign eventually, suppose on contrary that there exists $n_2 \in \mathbb{N}(n_1)$ such that $\theta_{n_2} \geq \beta^{-1}(\theta_{n_1})$ and $z_x(t) > 0$ for all $t \in [\theta_{n_2}, \infty)$. Set $\mu := \inf\{x(\eta) : \eta \in [\beta(\theta_{n_2}), \gamma(\theta_{n_2})]\} > 0$. From (2.2) and (2.9), we have

$$\begin{aligned} x(t) &\geq z_x(t) + \left(r(t) + \int_{\alpha(t)}^{t} \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)} \, \mathrm{d}\eta \right) \mu - \int_{t}^{\infty} f(\eta) \, \mathrm{d}\eta \\ &> \mu - f_1(t) \end{aligned}$$

for all $t \in [\gamma^{-1}(\beta(\theta_{n_2})), \theta_{n_2}]$. By iterating the above procedure and considering (2.11), we see that

$$x(t) > \mu - f_n(t) \quad \text{for all } t \in [\gamma^{-n}(\beta(\theta_{n_2})), \gamma^{-(n-1)}(\theta_{n_2})] \text{ and all } n \in \mathbb{N}. \tag{2.13}$$

Hence, (2.10) and (2.13) ensures existence of $n_3 \in \mathbb{N}(n_2)$ satisfying $x(t) > \mu/2$ holds for all $t \in [\theta_{n_3}, \infty)$. Set $L := \lim_{t \to \infty} z_x(t)$. Now, we consider the following possible ranges of L:

Case 1. $L = 0$. In this case, there exists, $T \in [\theta_{n_3}, \infty)$ such that

$$x(t) > \frac{1}{\Phi(t,s)} \int_{T}^{\beta^{-1}(t)} \frac{z_x(\eta)}{\varphi(\eta)} \, \mathrm{d}\eta \quad \text{for all } t \in [T, \beta^{-1}(T)]. \tag{2.14}$$

Case 2. $L > 0$. In this case, we see that $z_x(t) > L$ for all $t \in [\theta_{n_3}, \infty)$, since z_x is nonincreasing. From (2.2) and (2.9), we have

$$x(t) > L + \frac{\mu}{2} - \int_{t}^{\infty} f(\eta) \, \mathrm{d}\eta \quad \text{for all } t \in [\beta^{-1}(\theta_{n_3}), \infty).$$

Repeating the above procedure, we have

$$x(t) > nL + \frac{\mu}{2} - f_n(t) \quad \text{for all } t \in [\beta^{-n}(\theta_{n_3}), \infty) \text{ and all } n \in \mathbb{N}. \tag{2.15}$$

(2.10) and (2.15) shows that $\lim_{t \to \infty} x(t) = \infty$. Thus, there exists $T \in [\theta_{n_3}, \infty)$ such that (2.14) holds for all $t \in [T, \beta^{-1}(T)]$.

Set

$$y(t) := \int_T^t \frac{z_x(\eta)}{\varphi(\eta)}\,\mathrm{d}\eta > 0 \quad \text{for } t \in [T, \infty). \tag{2.16}$$

By the discussion made in Cases 1 and 2, we have

$$x(t) > \frac{1}{\Phi(t,s)} y\big(\beta^{-1}(t)\big) \quad \text{for all } t \in [T, \beta^{-1}(T)]. \tag{2.17}$$

By (2.2), (2.9) and (2.17), we have

$$\begin{aligned} x(t) &> z_x(t) + \left(r(t) + \int_{\alpha(t)}^t \frac{p\big(\alpha^{-1}(\eta)\big)}{\alpha'\big(\alpha^{-1}(\eta)\big)}\,\mathrm{d}\eta \right) \frac{1}{\Phi(t,s)} y(t) - \int_t^\infty f(\eta)\,\mathrm{d}\eta \\ &> \frac{1}{\Phi(t,s)} \int_t^{\beta^{-1}(t)} \frac{z_x(\eta)}{\varphi(\eta)}\,\mathrm{d}\eta + \frac{1}{\Phi(t,s)} y(t) - \int_t^\infty f(\eta)\,\mathrm{d}\eta \\ &= \frac{1}{\Phi(t,s)} y\big(\beta^{-1}(t)\big) - f_1(t) \end{aligned} \tag{2.18}$$

for all $t \in [\gamma^{-1}(T), \gamma^{-1}(\beta^{-1}(T))]$. Applying induction to (2.18), we see that

$$x(t) > \frac{1}{\Phi(t,s)} y\big(\beta^{-1}(t)\big) - f_n(t) \tag{2.19}$$

for all $t \in [\gamma^{-n}(T), \gamma^{-n}(\beta^{-1}(T))]$ and all $n \in \mathbb{N}$. Considering (2.10) and (2.19), there exists $n_4 \in \mathbb{N}(n_3)$ such that

$$x\big(\sigma(t)\big) > \frac{\lambda}{\Phi(t,s)} y(t) \quad \text{for all } t \in [\theta_{n_4}, \infty). \tag{2.20}$$

Now, set

$$w(t) := \int_T^t z_x(\eta)\,\mathrm{d}\eta \quad \text{for all } t \in [\theta_{n_4}, \infty). \tag{2.21}$$

In view of (2.16), (2.20), (2.21) and the nondecreasing nature of φ, we have

$$x\big(\sigma(t)\big) > \frac{\lambda}{\Phi(t,s)} y(t) \geq \frac{\lambda}{\varphi(t)\Phi(t,s)} w(t) \quad \text{for all } t \in [\theta_{n_4}, \infty). \tag{2.22}$$

Note that $w'(t) = z_x(t)$ and $w''(t) = z_x'(t)$ hold for all $t \in [\theta_{n_4}, \infty) \backslash \{\theta_n\}_{n\in\mathbb{N}}$. Hence, we have $w(\theta_n^+) = w(\theta_n)$ and $w'(\theta_n^+) = z_x(\theta_n^+) \leq z_x(\theta_n) = w'(\theta_n)$ for all $n \in \mathbb{N}(n_5)$, where $\theta_{n_5} \geq \theta_{n_4}$. Thus, (2.3), (2.21) and (2.22) show that

$$w''(t) + \frac{\lambda h(t)}{\varphi(t)\Phi(t,s)} w(t) \leq 0 \quad \text{for all } t \in [\theta_{n_5}, \infty) \backslash \{\theta_n\}_{n\in\mathbb{N}}. \tag{2.23}$$

This contradicts to our assumption that (2.12) has no eventually positive solutions, and proves that z_x is of negative sign eventually.

(ii) In this case, proof is very similar to the proof of (i), and thus is omitted. □

Now, we give the following lemma which is extracted from [14, Lemma 2.4] (see also [3]).

Lemma 2.3. *Consider the second-order impulsive differential inequality*

$$\begin{cases} x''(t) + a(t)x(t) \leq 0, & t \in [t_0, \infty)\backslash\{\theta_n\}_{n\in\mathbb{N}}, \\ x'(\theta_n^+) \leq \gamma_n x'(\theta_n), & n \in \mathbb{N}, \\ x(\theta_n^+) \geq x(\theta_n), & n \in \mathbb{N}, \end{cases} \tag{2.24}$$

where $a \in PLC_\theta([t_0, \infty), \mathbb{R}^+)$ *and* $\{\gamma_n\}_{n\in\mathbb{N}}$ *is a positive sequence of reals. If*

$$\sum_{n=1}^{\infty} \left(\prod_{k=1}^{n} \frac{1}{\gamma_k} \right) \int_{\theta_n}^{\theta_{n+1}} a(\eta)\, \mathrm{d}\eta = \infty,$$

then (2.24) *has no eventually positive solutions.*

Now, we state our main results, which can be regarded as generalized corrections of [14, Theorem 2.1, Theorem 2.2] respectively.

Theorem 2.4. *Assume that* (2.1)*,* (2.9)*,* (2.10) *hold. If* (2.12) *has no eventually positive solutions, for some nondecreasing* $\varphi \in C([t_0, \infty), \mathbb{R}^+)$*, some* $s \in [t_0, \infty)$ *and some* $\lambda \in (0, 1)$*, then every solution of* (1.1) *is oscillatory.*

Proof. Suppose on contrary that x is a nonoscillatory solution of (1.1). Let x be an eventually positive(negative) solution. Lemma 2.1 contradicts Lemma 2.2 about the sign of z_x. Therefore, (1.1) can not have nonoscillatory solutions. That is, (1.1) is oscillatory, and the proof is done. □

Remark 2.5. In Theorem 2.4, we may let $\lambda = 1$, when $f \equiv 0$.

Corollary 2.6. *Assume that* (2.1)*,* (2.9)*,* (2.10) *hold. If* $h/(\varphi\Phi(\cdot, s)) \notin L^1([t_0, \infty))$ *for some nondecreasing* $\varphi \in C([t_0, \infty), \mathbb{R}^+)$ *and some* $s \in [t_0, \infty)$*, where* $\Phi(\cdot, s)$ *is defined by* (2.8)*, then every solution of* (1.1) *is oscillatory.*

Proof. It is obvious that $h/(\varphi\Phi(\cdot, s)) \notin L^1([t_0, \infty))$ implies that

$$\frac{1}{2} \sum_{n=1}^{\infty} \int_{\theta_n}^{\theta_{n+1}} \frac{h(\eta)}{\varphi(\eta)\Phi(\eta, s)}\, \mathrm{d}\eta = \frac{1}{2} \int_{\theta_1}^{\infty} \frac{h(\eta)}{\varphi(\eta)\Phi(\eta, s)}\, \mathrm{d}\eta = \infty.$$

Applying Theorem 2.4 and Lemma 2.3 with $\lambda = 1/2$ and $\gamma_n = 1$ for all $n \in \mathbb{N}$, we see that (1.1) is oscillatory. □

3 Applications and Conclusions

In this section, we shall provide some examples for impulsive delay differential equations with a nontrivial forcing term. Because of the nontrivial forcing term, none of the results in the literature are applicable for the equations in below.

Example 3.1. Consider the following differential equation

$$\begin{cases} \left[x(t)-\dfrac{1}{2}(\lceil t\rceil-t)x(t-1)\right]'+\left(\dfrac{3}{2}+\dfrac{1}{t-1}\right)x(t-2) \\ \qquad\qquad -\left(\dfrac{1}{2}-\dfrac{1}{t}\right)x(t-1)=\dfrac{1}{t^4},\ t\in[3,\infty)\backslash\{n+2\}_{n\in\mathbb{N}}, \\ x((n+2)^+)=\dfrac{n+1}{n+3}x(n+2),\ n\in\mathbb{N}, \end{cases} \tag{3.1}$$

where $\lceil\cdot\rceil$ denotes the smallest integer function, $\theta_n=n+2$ and $\lambda_n=(n+1)/(n+3)$ for $n\in\mathbb{N}$. For (3.1), we have $\alpha(t)=t-1$, $\beta(t)=t-2$, $\gamma(t)=t-1$ and $h(t)=1+2/t$ for $t\geq 3$. Letting $\varphi\equiv 1$, we see that $\Phi(\cdot,3)\equiv 2$ holds. Clearly, (H1)–(H5) are satisfied and that $h\notin L^1([3,\infty))$. In fact, $n\in\mathbb{N}$, we have $\kappa(n)=n-1$, $(\lceil n+2\rceil-(n+2))^+/2=1/2$ and $(\lceil n+2\rceil-(n+2))/2=0$, which indicates that

$$\frac{n}{n+2}\frac{1}{2}\geq 0 \quad \text{for all } n\in\mathbb{N}.$$

For all sufficiently large t, we deduce that

$$\frac{1}{2}(\lceil t\rceil-t)+\int_{t-1}^{t}\left(\frac{1}{2}-\frac{1}{\eta}\right)\mathrm{d}\eta\leq 1,\quad \frac{1}{2}(\lceil t\rceil-t)+\int_{t-1}^{t}\left(\frac{3}{2}+\frac{1}{\eta-2}\right)\mathrm{d}\eta\geq 1,$$

and, we get

$$\frac{1}{2}\int_{3}^{\infty}\left(1+\frac{2}{\eta}\right)\mathrm{d}\eta=\infty.$$

Also it can be shown that (2.10) is satisfied for (3.1), because the forced term $1/t^4$ decreases to zero sufficiently fast at infinity and r,p are bounded. All the conditions of Corollary 2.6 are satisfied, thus every solution of (3.1) is oscillatory.

Example 3.2. Consider the following differential equation

$$\begin{cases} \left[x(t)-\left(\dfrac{\mathrm{e}-\mathrm{e}^{\lceil\ln(t)\rceil}}{\mathrm{e}^{2\lceil\ln(t)\rceil}(\mathrm{e}-1)}(t-\mathrm{e}^{\lceil\ln(t)\rceil})+\dfrac{1}{\mathrm{e}^{\lceil\ln(t)\rceil}}\right)x(t/\mathrm{e})\right]' \\ \qquad\qquad +x(t/\mathrm{e}^2)-\dfrac{1}{\mathrm{e}t}x(t/\mathrm{e})=\dfrac{1}{\mathrm{e}^t},\quad t\in[1,\infty)\backslash\{\mathrm{e}^n\}_{n\in\mathbb{N}}, \\ x((\mathrm{e}^n)^+)=\dfrac{1}{\mathrm{e}^{n-1}}x(\mathrm{e}^n),\quad n\in\mathbb{N}, \end{cases} \tag{3.2}$$

where $\theta_n = \mathrm{e}^n$ and $\lambda_n = 1/\mathrm{e}^{n-1}$ for $n \in \mathbb{N}$. It is easy to see that $\alpha(t) = t/\mathrm{e}$, $\beta(t) = t/\mathrm{e}^2$, $\gamma(t) = t/\mathrm{e}$ and $h(t) = (1 - 1/t)/\mathrm{e}$ for $t \geq 1$. Letting $\varphi \equiv 1$, we have that $\Phi(t, 1) = t/\mathrm{e}$ for $t \geq 1$. (H1)–(H5) hold and that $h/g(\cdot, 1) \notin L^1([1, \infty))$. Note that $1/\mathrm{e} \geq r(t) \geq 1/\mathrm{e}^{\lceil \ln(t) \rceil}$ holds for all $t \geq 1$. Also, it is obvious that $\kappa(n) = n - 1$, $r((\mathrm{e}^n)^+) = 1/\mathrm{e}$ and $r(\mathrm{e}^n) = 1/\mathrm{e}^n$ for all $n \in \mathbb{N}$, which shows that (H4) holds, i.e. $1/\mathrm{e}^{n-1} \geq 1/\mathrm{e}^n$ for all $n \in \mathbb{N}$. Clearly, for all sufficiently large t, we have

$$\frac{1}{\mathrm{e}} + \int_{t/\mathrm{e}}^{t} \frac{1}{\mathrm{e}\eta} \,\mathrm{d}\eta \leq 1, \quad \frac{1}{\mathrm{e}^{\lceil \ln(t) \rceil}} + \int_{t/\mathrm{e}}^{t} \frac{1}{\mathrm{e}} \,\mathrm{d}\eta \geq 1,$$

and, we get

$$\int_{1}^{\infty} \frac{\mathrm{e}}{\eta}\left(1 - \frac{1}{\eta}\right) \mathrm{d}\eta = \infty.$$

As an easy exercise, the readers may verify that (2.10) also holds. All the conditions of Corollary 2.6 are held, hence every solution of (3.2) oscillates.

We would like to mention at this point that Theorem 2.4 for homogeneous equations ($f = 0$) corrects the main result of [14] (see [15]). Our results for nonimpulsive equations (1.4) with constant delays generalize and improve the main results of the papers [10,16]. Next, we focus our attention to equations of the form (1.4) and (1.5), when there is no forcing term and no impulse effect. In this case, we can obtain the following results. We first consider equations with constant delays.

Corollary 3.3. *Assume that* $\rho_0 \geq 0$, $\tau_0 \geq \sigma_0 \geq 0$, $p(t - \sigma_0 + \tau_0) - q(t) \geq 0$ $(\not\equiv 0)$ *for* $t \geq t_0$ *and*

$$r(t) + \int_{t-\tau_0+\sigma_0}^{t} q(\eta) \,\mathrm{d}\eta \leq 1, \quad r(t) + \int_{t-\tau_0+\sigma_0}^{t} p(\eta - \sigma_0 + \tau_0) \,\mathrm{d}\eta \geq 1 \quad \textit{for all } t \geq t_0$$

and

$$x''(t) + \frac{1}{\max\{\rho_0, \tau_0\}}\big(p(t - \sigma_0 + \tau_0) - q(t)\big)x(t) = 0 \quad \textit{for all } t \geq t_0$$

is oscillatory. Then, (1.4) *with* $f = 0$ *is also oscillatory.*

Corollary 3.3 follows by letting $\varphi(t) \equiv 1$ for $t \geq t_0$. This result improves and corrects the main results of [7,9], respectively. Next, we consider pantograph equations.

Corollary 3.4. *Assume that* $\rho_0 \geq 1$, $\tau_0 \geq \sigma_0 \geq 1$, $p(\tau_0 t/\sigma_0) - q(t) \geq 0$ $(\not\equiv 0)$ *for* $t \geq t_0$ *and*

$$r(t) + \int_{\sigma_0 t/\tau_0}^{t} q(\eta) \,\mathrm{d}\eta \leq 1, \quad r(t) + \int_{\sigma_0 t/\tau_0}^{t} p(\tau_0\eta/\sigma_0) \,\mathrm{d}\eta \geq 1 \quad \textit{for all } t \geq t_0$$

and

$$x''(t) + \frac{1}{\ln\big(\max\{\rho_0, \tau_0\}\big)} \frac{1}{t}\left(\frac{\tau_0}{\sigma_0} p(\tau_0 t/\sigma_0) - q(t)\right)x(t) = 0 \quad \textit{for all } t \geq t_0$$

is oscillatory. Then, (1.5) *is also oscillatory.*

We finalize the paper by mentioning that Corollary 3.4 improves the main results of [4]. To obtain this result, we let $\varphi(t) = t$ for $t \geq t_0$.

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