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# Bounded Solutions of Almost Linear Volterra Equations

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**Abstract**

Fixed point theorem of Krasnosel'skii is used as the primary mathematical tool to study the boundedness of solutions of certain Volterra type equations. These equations are studied under a set of assumptions on the functions involved in the equations. The equations will be called almost linear when these assumptions hold.

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## 1 Introduction

Consider the following scalar equations:

$$x'(t) = a(t)h(x(t)) + \int_0^t C(t,s)g(x(s))ds,\ x(0) = x_0,\ t \geq 0, \tag{1.1}$$

and

$$x(t) = a(t) + \int_0^t C(t,s)g(x(s))ds,\ t \geq 0. \tag{1.2}$$

We assume that the functions $h$ and $g$ are continuous and that there exist positive constants $H$, $H^*$, $G$, $G^*$ such that

$$|h(x) - Hx| \leq H^*, \tag{1.3}$$

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and

$$|g(x) - Gx| \leq G^*. \tag{1.4}$$

Throughout this paper we assume $a(t)$ is continuous for $t \geq 0$ and $C(t, s)$ is continuous for $0 \leq s \leq t < \infty$.

Equations (1.1) and (1.2) will be called almost linear if (1.3) and (1.4) hold. In [4] the authors used this concept of almost linear equations and studied boundedness and periodicity properties. In that paper, the authors used the resolvent to obtain the boundedness of solutions of (1.2). In the present paper, we obtain the bounded solutions of (1.1) and (1.2) employing the fixed point theorem of Krasnosel'skii [6, 9].

Krasnosel'skii's theorem continues to receive attention in terms of both extensions [2], and applications [5]. To the best of our knowledge, it has not been applied to Volterra Integro-differential equations or integral equations on unbounded domains to study the existence of bounded solutions. The use of (1.3) and (1.4) into (1.1) and then inverting to an integral equation problem leads to the presence of an integral term which may involve a completely continuous operator, and another term which may involve a contraction operator, provides a natural environment for applying the Krasnosel'skii's fixed point theorem. Recall that an operator is completely continuous if it is continuous and maps bounded sets into relatively compact sets. Normally, Arzela–Ascoli theorem is used to obtain the compactness. But in our case in the present paper, Arzela–Ascoli theorem does not apply because the domain is unbounded. To overcome theses difficulties, we resort to a theorem (Theorem 2.3) which can be found in [1] and [5].

# 2 Bounded Solutions of (1.1)

In this section we employ Krasnosel'skii's fixed point theorem and show, in Theorem 2.5, the existence of a continuous bounded solution of (1.1).

**Theorem 2.1.** *[9] Let $K$ be a closed convex subset of a Banach space $(M, \|\cdot\|)$. Suppose that*

*(i) the mapping $A : K \to M$ is completely continuous;*

*(ii) the mapping $B : K \to K$ is a contraction;*

*(iii) $Au + Bv \in K$ for all $u, v \in K$.*

*Then the mapping $A + B$ has a fixed point in $K$.*

For convenience, we write (1.1) as

$$x'(t) = -a(t)h(x(t)) + \int_0^t C(t, s)g(x(s))ds, \; x(0) = x_0, \; t \geq 0. \tag{2.1}$$

Now we rewrite it as

$$\begin{aligned} x'(t) + Ha(t)x(t) &= Ha(t)x(t) - a(t)h(x(t)) \\ &\quad + \int_0^t C(t,s)(g(x(s)) - Gx(s))ds + \int_0^t C(t,s)Gx(s)ds, \end{aligned}$$

from which we get

$$\begin{aligned} x(t) &= x_0 e^{-H\int_0^t a(s)ds} + \int_0^t e^{-H\int_u^t a(s)ds} a(u)[Hx(u) - h(x(u))]du \\ &\quad + \int_0^t e^{-H\int_u^t a(s)ds} \int_0^u C(u,s)[g(x(s)) - Gx(s)]dsdu \\ &\quad + \int_0^t e^{-H\int_u^t a(s)ds} \int_0^u C(u,s)Gx(s)dsdu. \end{aligned}$$

Assume

$$a : [0,\infty) \to [0,\infty), \tag{2.2}$$

and for some positive constant $L$,

$$0 \le \int_0^u |C(u,s)|\, ds \le La(u) \text{ for all } u \in [0,\infty), \tag{2.3}$$

such that

$$\int_0^\infty a(s)ds < \infty. \tag{2.4}$$

Moreover, we assume

$$\sup_{t\ge 0} \int_0^t e^{-H\int_u^t a(s)ds} \int_0^u G|C(u,s)|dsdu \le \alpha < 1, \tag{2.5}$$

and

$$\sup_{t\ge 0} \int_0^t e^{-H\int_u^t a(s)ds} \big[a(u)H^* + \int_0^u G^*|C(u,s)|ds\big] du \le \beta < \infty. \tag{2.6}$$

Finally, choose a constant $m > 0$ such that

$$|x_0| e^{-H\int_0^t a(s)ds} + \alpha m + \beta \le m \tag{2.7}$$

for all $t \ge 0$.

Let $M$ be the Banach space of bounded continuous functions $\phi : [0,\infty) \to \mathbb{R}$ with the supremum norm. Let

$$K = \{\psi \in M, \psi(0) = x_0 : ||\psi|| \le m\}.$$

Then $K$ is a closed convex subset of M.

Define mappings $A : K \to M$ and $B : K \to K$ as follows.

$$(A\phi)(t) = \int_0^t e^{-H\int_u^t a(s)ds} a(u)[H\phi(u) - h(\phi(u))]du \qquad (2.8)$$
$$+ \int_0^t e^{-H\int_u^t a(s)ds} \int_0^u C(u,s)[g(\phi(s)) - G\phi(s)]dsdu,$$

and

$$(B\phi)(t) = x_0 e^{-H\int_0^t a(s)ds} \qquad (2.9)$$
$$+ \int_0^t e^{-H\int_u^t a(s)ds} \int_0^u GC(u,s)\phi(s)dsdu.$$

Clearly, both $(A\phi)(t)$ and $(B\phi)(t)$ are continuous in $t$.

**Lemma 2.2.** *Assume* (2.2)*,* (2.5) *and* (2.7)*. The map $B$ is a contraction from $K$ into $K$.*

*Proof.* Let $B$ be given by (2.9). Then for $\phi \in K$, we get from (2.5) and (2.7) that

$$|(B\phi)(t)| \le |x_0| e^{-H\int_0^t a(s)ds} + \alpha m \le m.$$

Also, for $\phi, \psi \in K$, we obtain

$$|(B\phi)(t) - (B\psi)(t)| \le \alpha||\phi - \psi||.$$

This proves that $B$ is a contraction mapping from $K$ into $K$. □

Recall that the mapping is called completely continuous if it is continuous and maps bounded sets into relatively compact sets. We rely on the following theorem for the relative compactness criterion.

**Theorem 2.3.** *[1] Let $M$ be the space of all bounded continuous (vector-valued) functions on $[0,\infty)$ and $S \subset M$. Then $S$ is relatively compact in $M$ if the following conditions hold:*

*(i) $S$ is bounded in $M$;*

*(ii) the functions in $S$ are equicontinuous on any compact interval of $[0,\infty)$;*

*(iii) the functions in $S$ are equiconvergent, that is, given $\epsilon > 0$, there exists a $T = T(\epsilon) > 0$ such that $\|\phi(t) - \phi(\infty)\|_{\mathbb{R}^n} < \epsilon$, for all $t > T$ and all $\phi \in S$.*

By making use of (1.3), (1.4), (2.2) and (2.3) we arrive at

$$|(A\phi)(t)| \le \frac{H^* + LG^*}{H} \int_0^t \frac{d}{du}\left(e^{-H\int_u^t a(s)ds}\right) du \qquad (2.10)$$
$$= \frac{H^* + LG^*}{H}\left(1 - e^{-H\int_0^t a(s)ds}\right), \text{ for all } t \in [0,\infty) \text{ and } \phi \in K.$$

**Lemma 2.4.** *$A(K)$ is relatively compact.*

*Proof.* Proving relative compactness of $A(K)$ is equivalent to showing that all three conditions of Theorem 2.3 hold. That is,

1. $A(K)$ is uniformly bounded,
2. $A(K)$ is equicontinuous,
3. $A(K)$ is equiconvergent.

To see that $A(K)$ is uniformly bounded, we use (2.10) to obtain

$$\begin{aligned}|(A\phi)(t)| &\leq \frac{H^* + LG^*}{H}\left(1 - e^{-H\int_0^t a(s)ds}\right)\\ &\leq \frac{H^* + LG^*}{H} \quad := Q \text{ for all } t \in [0,\infty) \text{ and } \phi \in K.\end{aligned}$$

To show equicontinuity of $A(K)$, without loss of generality, we let $t_1 > t_2$ for $t_1$,$t_2 \in [0,\infty)$ and use the notations

$$F(\phi(u)) = a(u)[H\phi(u) - h(\phi(u))],$$

and

$$J(\phi(u)) = \int_0^u C(u,s)[g(\phi(s)) - G\phi(s)]ds.$$

Then, we may write

$$(A\phi)(t) = \int_0^t e^{-H\int_u^t a(s)ds}[F(\phi(u)) + J(\phi(u))]du.$$

Hence we have

$$\begin{aligned}|(A\phi)(t_1) - (A\phi)(t_2)| &= \left|\int_0^{t_1} e^{-H\int_u^{t_1} a(s)ds}[F(\phi(u)) + J(\phi(u))]du\right.\\ &\quad \left. - \int_0^{t_2} e^{-H\int_u^{t_2} a(s)ds}[F(\phi(u)) + J(\phi(u))]du\right|\\ &= \left|\int_0^{t_2}[e^{-H\int_u^{t_1} a(s)ds} - e^{-H\int_u^{t_2} a(s)ds}][F(\phi(u)) + J(\phi(u))]du\right|\\ &\quad + \left|\int_{t_2}^{t_1} e^{-H\int_u^{t_1} a(s)ds}[F(\phi(u)) + J(\phi(u))]du\right|\\ &= \int_0^{t_2} |e^{-H\int_u^{t_2} a(s)ds} - e^{-H\int_u^{t_1} a(s)ds}||F(\phi(u)) + J(\phi(u))|du\\ &\quad + \int_{t_2}^{t_1} e^{-H\int_u^{t_1} a(s)ds}|F(\phi(u)) + J(\phi(u))|du\end{aligned}$$

$$
\begin{aligned}
&\le HQ\int_0^{t_2}[e^{-H\int_u^{t_2}a(s)ds}-e^{-H\int_u^{t_1}a(s)ds}]a(u)du \\
&\quad +Q[1-e^{-H\int_{t_2}^{t_1}a(s)ds}] \\
&\le Q[2-2e^{-H\int_{t_2}^{t_1}a(s)ds}-e^{-H\int_0^{t_2}a(s)ds}+e^{-H\int_0^{t_1}a(s)ds}] \\
&\le 2Q[1-e^{-H\int_{t_2}^{t_1}a(s)ds}]\to 0 \text{ as } t_2\to t_1.
\end{aligned}
$$

This shows that $A(K)$ is equicontinuous.

To see that $A(K)$ is equiconvergent, we have

$$
\begin{aligned}
|(A\phi)(\infty)-(A\phi)(t)| &= \left|\int_0^\infty e^{-H\int_u^\infty a(s)ds}\left[F(\phi(u))+J(\phi(u))\right]du\right. \\
&\quad \left.-\int_0^t e^{-H\int_u^t a(s)ds}\left[F(\phi(u))+J(\phi(u))\right]du\right| \\
&\le \left|\int_0^t \left[e^{-H\int_u^\infty a(s)ds}-e^{-H\int_u^t a(s)ds}\right]\left[F(\phi(u))+J(\phi(u))\right]du\right| \\
&\quad +\left|\int_t^\infty e^{-H\int_u^\infty a(s)ds}\left[F(\phi(u))+J(\phi(u))\right]du\right| \\
&\le \int_0^t \left|e^{-H\int_u^\infty a(s)ds}-e^{-H\int_u^t a(s)ds}\right|\left|F(\phi(u))+J(\phi(u))\right|du \\
&\quad +\int_t^\infty e^{-H\int_u^\infty a(s)ds}\left|F(\phi(u))+J(\phi(u))\right|du \\
&\le HQ\int_0^t \left[e^{-H\int_u^t a(s)ds}-e^{-H\int_u^\infty a(s)ds}\right]a(u)du \\
&\quad +Q\left[1-e^{-H\int_t^\infty a(s)ds}\right] \\
&\le Q\left[2-2e^{-H\int_t^\infty a(s)ds}+e^{-H\int_0^\infty a(s)ds}-e^{-H\int_0^t a(s)ds}\right] \\
&\le 2Q\left[1-e^{-H\int_t^\infty a(s)ds}\right]\to 0 \text{ as } t\to\infty,
\end{aligned}
$$

where we used (2.4) which yields $\lim_{t\to\infty}\int_t^\infty a(u)du=0$. □

We are now ready to use the fixed point theorem of Krasnosel'skii (Theorem 2.1) to show the existence of a bounded continuous solution of (1.1), which is equivalent to (2.1).

**Theorem 2.5.** *Assume* (1.3), (1.4), (2.2), (2.3), (2.4), (2.5), (2.6) *and* (2.7) *hold. Then* (1.1) *has a bounded continuous solution.*

*Proof.* For $\phi,\psi\in K$, we get

$$|(A\phi)(t)+(B\psi)(t)|\le|x_0|e^{-H\int_0^t a(s)ds}+\alpha m+\beta\le m,$$

which proves that $A\phi + B\psi \in K$. Moreover, Lemma 2.2 and Lemma 2.4 satisfy the requirements of Krasnosel'skii's fixed point theorem and hence there exists a function $x(t) \in K$ such that

$$x(t) = Ax(t) + Bx(t).$$

This proves that (1.1) has a bounded continuous solution $x(t)$. □

# 3 Bounded Solutions of (1.2)

Now, we turn our attention to the integral equation given by (1.2). We employ the same method as we did in the previous section. To set up our mappings, we rewrite (1.2),

$$x(t) = a(t) + \int_0^t C(t,s)[g(x(s)) - Gx(s)]ds + \int_0^t C(t,s)Gx(s)ds. \tag{3.1}$$

Let $M$ be the Banach space of bounded continuous functions $\phi : [0,\infty) \to (-\infty,\infty)$ with the supremum norm. Let

$$K = \{\psi \in M : ||\psi|| \le m\},$$

where $m$ is a constant defined later in (3.2). Then $K$ is a closed convex subset of M.

Using (3.1), we define mappings $A : K \to M$ and $B : K \to K$ as follows.

$$(A\phi)(t) = \int_0^t C(t,s)[g(\phi(s)) - G\phi(s)]ds,$$

and

$$(B\phi)(t) = \int_0^t GC(t,s)\phi(s)ds + a(t).$$

Clearly, both $(A\phi)(t)$ and $(B\phi)(t)$ are continuous in $t$. Choose $m$ such that

$$(G^* + mG)\int_0^t |C(t,s)|\, ds + |a(t)| \le m. \tag{3.2}$$

Finally, to prove the map $B$ is a contraction, we ask that there exists an $\alpha \in (0,1)$ such that

$$G\int_0^t |C(t,s)|\, ds \le \alpha, \text{ for all } t \ge 0. \tag{3.3}$$

It is obvious that condition (3.3) implies the map $B$ is a contraction on $K$.

**Lemma 3.1.** *$A(K)$ is relatively compact.*

*Proof.* By (3.2), $A(K)$ is uniformly bounded. For showing $A(K)$ is equicontinuous, we let $t_1 > t_2$ for $t_1, t_2 \in [0, \infty)$. Then,

$$\begin{aligned} |(A\phi)(t_1) - (A\phi)(t_2)| &\le \int_0^{t_2} |C(t_1, s) - C(t_2, s)||g(\varphi(s))) - G\varphi(s)|ds \\ &+ \int_{t_1}^{t_2} |C(t_1, s)||g(\varphi(s))) - G\varphi(s)|ds \\ &\le G^*[\int_0^{t_2} |C(t_1, s) - C(t_2, s)|ds + \int_{t_1}^{t_2} |C(t_1, s)|ds]. \end{aligned}$$

Due to the continuity of $C$, we have $|C(t_1, s) - C(t_2, s)| \to 0,$ as $t_2 \to t_1$. Also, due to (3.2), we have $\int_{t_1}^{t_2} |C(t_1, s)|ds \to 0$ as $t_2 \to t_1$. Thus,

$$|(A\phi)(t_1) - (A\phi)(t_2)| \to 0, \text{ as } t_2 \to t_1.$$

Now, we show

$$|(A\phi)(\infty) - (A\phi)(t)| \to 0, \text{ as } t \to \infty.$$

Using condition (3.2) and the continuity of $C$, we obtain

$$\begin{aligned} |(A\phi)(\infty) - (A\phi)(t)| &\le G^*[\int_0^{\infty} |C(\infty, s) - C(t, s)|ds + \int_t^{\infty} |C(t, s)|ds] \\ &\to 0 \text{ as } t \to \infty. \end{aligned}$$

□

**Theorem 3.2.** *Assume* (1.4)*,* (3.2) *and* (3.3)*. Then* (1.2) *has a bounded continuous solution.*

The proof of Theorem 3.2 is very similar to the proof of Theorem 2.5, and hence we omit it.

# 4 An Example

Consider the Volterra integro-differential equation

$$x'(t) = \frac{1}{(t+1)^2}h(x(t)) + \int_0^t \frac{1}{(t+s+1)^3}g(x(s))ds, \; x(0) = x_0, \; t \ge 0, \tag{4.1}$$

where the functions $h$ and $g$ satisfy conditions (1.3) and (1.4), respectively. Let $H$, $G$, $H^*$, and $G^*$ be positive constants with $\frac{G}{2} < 1$. We choose a constant $m > 0$ such that for any initial point $x_0$, the inequality

$$|x_0|e^{-H} + m\frac{G}{2} + (H^* + G^*) \le m$$

holds. Then, (4.1) has a continuous bounded solution $x(t)$ satisfying

$$||x|| \leq m.$$

To see this, we let $a(t) = \dfrac{1}{(t+1)^2}$, and $C(t,s) = \dfrac{1}{(t+s+1)^3}$. A simple calculation leads to

$$\begin{aligned}\int_0^u |C(u,s)|ds &= \frac{1}{2(u+1)^2} - \frac{1}{2(2u+1)^2}\\ &\leq \frac{1}{2(u+1)^2} + \frac{1}{2(2u+1)^2}\\ &\leq \frac{1}{(u+1)^2}.\end{aligned}$$

This shows that condition (2.3) is satisfied with $L = 1$. It is an easy exercise to see that (2.4) holds. Next, we verify (2.5).

$$\begin{aligned}\sup_{t\geq 0}\int_0^t e^{-H\int_u^t a(s)ds}\int_0^u G|C(u,s)|dsdu &\leq \sup_{t\geq 0}\int_0^t\int_0^u G|C(u,s)|dsdu\\ &\leq \sup_{t\geq 0} G\int_0^t \big(\frac{1}{2(u+1)^2} - \frac{1}{2(2u+1)^2}\big)du\\ &\leq \sup_{t\geq 0}\frac{G}{2}\int_0^t \frac{1}{(u+1)^2}du\\ &\leq \frac{G}{2} < 1.\end{aligned}$$

Now, we verify (2.6).

$$\begin{aligned}&\sup_{t\geq 0}\int_0^t e^{-H\int_u^t a(s)ds}\big[a(u)H^* + \int_0^u G^*|C(u,s)|ds\big]du\\ \leq\ &\sup_{t\geq 0}\int_0^t \big(H^*a(u) + G^*\frac{1}{(u+1)^2}\big)du\\ =\ &(H^* + G^*)\sup_{t\geq 0}\int_0^t \frac{1}{(u+1)^2}du\\ \leq\ &(H^* + G^*).\end{aligned}$$

Thus, as an application of Theorem 2.5, we see that (4.1) has a continuous bounded solution $x(t)$ satisfying

$$||x|| \leq m.$$

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