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Nonlinear Difference Equations and Stokes Matrices

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Abstract

We study one scalar holomorphic function of finitely many complex variables which, under the assumption that one of the two coefficient matrices has all distinct eigenvalues, allows to calculate the Stokes multipliers of *Okubo's confluent hypergeometric system*. Many properties of this function, including a nonlinear functional equation, are obtained. An open question is whether the function is uniquely determined by this functional equation, after specifying suitable additional conditions.

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1 Introduction

The hypergeometric system and its confluent form have both been investigated by many authors – for a discussion of existing results, and for a representation of its solutions in terms of a single (scalar) function, compare a recent article of *B. and Röscheisen* [7], or the PhD thesis of *C. Röscheisen* [9]. In this paper, we shall mainly concentrate on the confluent system, denoted as

$$zx' = A(z)x, \qquad A(z) = z\Lambda + A_1, \quad \Lambda = \operatorname{diag}[\lambda_1, \ldots, \lambda_n]. \tag{1.1}$$

We shall always assume that the numbers $\lambda_1, \ldots, \lambda_n$ are all distinct, referring to this situation as *the distinct eigenvalue case*. The origin is a singularity of first kind of (1.1), and it is well known that the cases where no two eigenvalues of A_1 differ by a

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nonzero integer are especially convenient for a computation of solutions by a power series "ansatz" at the origin. We refer to such a situation as *a case satisfying the eigenvalue condition at the origin.*

In the theory of formal and proper invariants, presented in work by *Balser, Jurkat, and Lutz* [1,2,8], the diagonal elements of A_1 have been shown to be of a special nature. Therefore, we shall always split $A_1 = \Lambda' + A$, with

$$\Lambda' = \operatorname{diag}[\lambda_1', \ldots, \lambda_n'], \quad A = \begin{bmatrix} 0 & a_{12} & \ldots & a_{1n} \\ a_{21} & 0 & \ldots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & 0 \end{bmatrix}. \tag{1.2}$$

Our goal is to study the so-called *Stokes multipliers* of (1.1) as functions of the parameters contained in the matrices Λ, Λ' and, in particular, in A. In dimension $n = 2$ the multipliers can be explicitly computed, using the classical Gamma function – see (1.4) for the relevant formulas. It is commonly believed, although perhaps not rigorously proven by means of differential Galois theory, that for $n \geq 3$ the Stokes multipliers in general cannot be expressed in terms of "known" higher transcendental functions, but are "new" functions of the parameters in Λ, Λ', and A. What we intend to do in this article is to analyze, as much as possible, the nature of these functions. In particular, we shall show in Section 2 (c) that *one* scalar function $v(\Lambda, \Lambda', A)$ suffices to compute all the entries in the Stokes multipliers, and we shall obtain a nonlinear functional equation for $v(\Lambda, \Lambda', A)$. It is worth emphasizing, although not really surprising, that this functional equation has a natural interpretation with regards to the system (1.1). Roughly speaking, this equation expresses the fact that the Stokes multipliers are invariant with respect to very simple meromorphic transformations of (1.1). For certain concrete constellations of the parameters Λ, Λ' and A, such a transformation may fail to exist, but here we regard the entries of Λ, Λ', and A as variables and show that the transformation matrices are meromorphic functions in these variables. The location of the poles of these functions is also analyzed.

Ideally, we should have liked to prove that the function $v(\Lambda, \Lambda', A)$ is uniquely characterized by the functional equation obtained here plus some additional (initial?) conditions. Unfortunately we have not been able to do this here! In the past, there has been one article by *Y. Sibuya* [10] in which, for a certain second order linear ODE, a functional equation for a Stokes multiplier has been investigated. In oral communications, Sibuya explained to the author that he, too, has been aiming at characterizing the relevant entries in the Stokes multipliers by means of their functional properties. So in a way, this article is a continuation, resp. extension, of his work to a much wider class of equations.

The Stokes multipliers correspond uniquely to a preselected formal fundamental solution for (1.1) – compare, e.g., the papers of *Balser, Jurkat, and Lutz* [1,2,8] for more details. In the distinct eigenvalue case which we investigate here, there is a uniquely defined formal fundamental solution $\hat{X}(z)$ having the form

$$\hat{X}(z) = \hat{F}(z) z^{\Lambda'} e^{z\Lambda}, \tag{1.3}$$

where $\hat{F}(z) = \sum_0^\infty z^{-j} F_j$ is a formal (matrix) power series in $1/z$ beginning with $F_0 = I$. The corresponding family of Stokes multipliers then contains exactly $n\,(n-1)$ nontrivial entries – more precisely, for every pair (j,k) with $j \neq k$, $1 \leq j,k \leq n$, there exists exactly one Stokes multiplier with a nontrivial entry in position (j,k). Therefore, we may combine the collection of these entries as the off-diagonal terms of an $n \times n$ matrix $V = V(\Lambda, \Lambda', A)$, choosing zeros for the values along the diagonal. It is the nature of this matrix, regarded as a function of the entries in Λ, Λ', and A, which we are going to investigate in this article! To emphasize the fact that we consider the matrices Λ, Λ', and A as variables, we shall refer to $V(\Lambda, \Lambda', A)$ as the *Stokes matrix function*. Occasionally we shall consider Λ and Λ' as fixed, and then write $V(A)$ for this function.

As has been said above, it is only in dimension $n = 2$ that we can explicitly compute the Stokes matrix function: In this situation, let α, β be so that

$$\alpha + \beta = \lambda_2' - \lambda_1', \quad \alpha\beta = -a_{12}a_{21}.$$

In other words, α and β are the (not necessarily distinct) solutions of a quadratic equation, and $\alpha + \lambda_1'$, $\beta + \lambda_1'$ are the eigenvalues of A_1. Then we have, according to results in the Springer Lecture Notes of *W. B. Jurkat* [8] or the article of *Balser, Jurkat, and Lutz* [2]:

$$\begin{gathered} V = \begin{bmatrix} 0 & v_{12} \\ v_{21} & 0 \end{bmatrix}, \quad v_{21} = \frac{2\pi i a_{21}(\lambda_2 - \lambda_1)^{\lambda_2' - \lambda_1'} e^{i\pi(\lambda_1' - \lambda_2')}}{\Gamma(1+\alpha)\Gamma(1+\beta)}, \\ v_{12} = \frac{2\pi i a_{12}(\lambda_2 - \lambda_1)^{\lambda_1' - \lambda_2'} e^{2\pi i(\lambda_2' - \lambda_1')}}{\Gamma(1-\alpha)\Gamma(1-\beta)}. \end{gathered} \tag{1.4}$$

This case may serve as an example for the results obtained in this article, and shall be considered in some detail in Section 7.

2 Known Results

The following results on the Stokes matrix function $V(\Lambda, \Lambda', A)$ may be easily derived from earlier articles:

(a) It follows from results in articles of *Balser* and *Röscheisen* [3,4,9] that $V(\Lambda, \Lambda', A)$ is an entire function of the elements of A, and the coefficients of its power series expansion may be computed recursively. The dependence upon the other matrices Λ and Λ' is more involved and shall here be investigated to some degree only.

(b) Fixing the matrices Λ and Λ', the Stokes matrix function $V(A)$ is a mapping from $\mathbb{C}^{n(n-1)}$ into itself, and according to results obtained by the author [5] this map can, at every point $A \in \mathbb{C}^{n(n-1)}$ where $A_1 = \Lambda' + A$ satisfies the eigenvalue condition at the origin, be (locally) inverted to obtain an even more interesting *reverse Stokes matrix function* $A(V)$. These "good" points form an open and dense subset of $\mathbb{C}^{n(n-1)}$. At these points, the function $A(V)$ is holomorphic, while the remaining ones in general are branch points.

(c) A transformation $x = P\tilde{x}$, with an arbitrary *permutation matrix* P, may be used to transform (1.1) into a new system of the same form, but with Λ and A_1 replaced by $P^{-1}\Lambda P$ and $P^{-1}A_1P$. To this new system then corresponds the Stokes matrix $P^{-1}V(\Lambda, \Lambda', A)P$. In other words, we obtain the following identity for the Stokes matrix function:

$$V(P^{-1}\Lambda P, P^{-1}\Lambda' P, P^{-1}AP) = P^{-1}V(\Lambda, \Lambda', A)P. \tag{2.1}$$

This transformation behaviour makes it obvious that *we need only find one off-diagonal entry of* $V(\Lambda, \Lambda', A)$*, e.g.,* $v_{21}(\Lambda, \Lambda', A)$, since for every pair (j, k) with $j \neq k$, $1 \leq j, k \leq n$, we can find a permutation matrix P so that

$$v_{jk}(\Lambda, \Lambda', A) = v_{21}(P^{-1}\Lambda P, P^{-1}\Lambda' P, P^{-1}AP).$$

Therefore, analogously to the solutions of (1.1), it is *one scalar Stokes function* $v(\Lambda, \Lambda', A) := v_{21}(\Lambda, \Lambda', A)$ that suffices to compute all elements of the Stokes matrix $V(\Lambda, \Lambda', A)$.

(d) In dimension $n = 2$, the function $v(\Lambda, \Lambda', A)$ can be expressed in terms of Gamma function and other elementary transcendental ones like the exponential function and the logarithm – see (1.4) for the explicit formulas. For $n \geq 3$ we have made it clear in (c) that one higher transcendental function $v(\Lambda, \Lambda', A)$ suffices to compute all Stokes multipliers – an open question is, however, whether this one can be expressed in terms of another (simpler) function which, analogously to the situation of $n = 2$, might be a solution of some difference equation.

(e) Let D be any invertible diagonal matrix. From results in the papers of *Balser, Jurkat, and Lutz* [1, 2, 8] one can derive that

$$V(\Lambda, \Lambda', D^{-1}AD) = D^{-1}V(\Lambda, \Lambda', A)D. \tag{2.2}$$

This identity might be used to restrict to situations where, e.g., the entries in the first row of A are either zero or normalized to equal 1, but this will not be done here.

3 Elementary Transformations

In this section we shall introduce and study some elementary meromorphic transformations that are of great importance for (1.1), and which are the origin of the functional equation for the Stokes matrix function.

As in the author's book [6], a square matrix $T(z)$, whose elements are holomorphic outside of a sufficiently large disc about the origin, shall be called a *meromorphic transformation matrix* (near the point infinity), provided that its entries have at most poles at infinity and its determinant is not identically zero – then, the inverse matrix again is a meromorphic transformation matrix. Setting $x = T(z)\tilde{x}$, we see that x is a solution of (1.1) if, and only if, $\tilde{x}$ satisfies the system $z\tilde{x} = \tilde{A}(z)\tilde{x}$, with $\tilde{A}(z)$ and $A(z)$ linked by the identity

$$zT'(z) = A(z)T(z) - T(z)\tilde{A}(z). \tag{3.1}$$

For a confluent hypergeometric system (1.1) it is not clear whether, aside from the trivial case of $T(z) = I$, we may choose a meromorphic transformation matrix so that the coefficient matrix of the transformed system is again of the form, say, $\tilde{A}(z) = z\tilde{\Lambda} + \tilde{A}_1$, with $\tilde{A}_1 = \tilde{\Lambda}' + \tilde{A}$ and the diagonal terms of $\tilde{A}$ vanishing, but let us assume that this is the case. Then, the general theory of invariants presented by *Balser, Jurkat, and Lutz* [1,2] tells us that $\tilde{\Lambda}$, up to the ordering of the diagonal elements, coincides with Λ. Hence, replacing $T(z)$ by $T(z)P$, with a suitable permutation matrix P, we may assume that $\tilde{\Lambda} = \Lambda$. Having done so, one obtains from the same articles quoted above that $\tilde{\Lambda}' = \Lambda' + K$, with a diagonal matrix K of integer diagonal entries. It is this possibility to change the numbers λ'_ν by integers that leads to interesting identities for the Stokes matrix function. Aside from the diagonal values, a meromorphic transformation may also change the eigenvalues of A_1 by integers. In the sequel, we wish to find the most elementary transformations that result in such a change. In order to do so, we consider two hypergeometric systems with coefficient matrices $A(z) = z\Lambda + A_1$ and $\tilde{A}(z) = z\Lambda + \tilde{A}_1$, using the following modified notation:

(A) Let some natural number $m < n$ be given, and block the coefficient matrices in the form

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}, \qquad \Lambda' = \begin{bmatrix} \Lambda'_1 & 0 \\ 0 & \Lambda'_2 \end{bmatrix}, \qquad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \tag{3.2}$$

(and analogously for $\tilde{\Lambda}'$ and $\tilde{A}$), with square diagonal blocks of dimension m resp. $n - m$. In the distinct eigenvalue situation, both matrices Λ_j are diagonal, and in particular *their spectra are disjoint, or in other words:* Λ_1 *and* Λ_2 *do not have an eigenvalue in common.*

Using this notation, we intend to find a meromorphic transformation matrix of the special form[1]

$$T(z)=\begin{bmatrix} z & T_{12}\\ 0 & I\end{bmatrix}\begin{bmatrix} I & 0\\ T_{21} & I\end{bmatrix}=\begin{bmatrix} z+T_{12}T_{21} & T_{12}\\ T_{21} & I\end{bmatrix}, \tag{3.3}$$

so that (3.1) is satisfied. If this is so, then we shall say that $T(z)$ is *admissible for* the hypergeometric system (1.1) – observe that, given an admissible transformation matrix for (1.1), the matrices $\tilde\Lambda'$ and $\tilde A$ can be computed from (3.1).

An elementary computation shows that existence of an admissible transformation is equivalent to the following seven (nonlinear) equations for the various blocks of $T(z)$ and $\tilde A(z)$:

$$\begin{array}{rcl}
I &=& \Lambda_1 T_{12}T_{21} + A_{11} + \Lambda_1' - (\tilde A_{11} + \tilde\Lambda_1') - T_{12}T_{21}\Lambda_1,\\
0 &=& (A_{11}+\Lambda_1')T_{12}T_{21} + A_{12}T_{21} - T_{12}T_{21}(\tilde A_{11}+\tilde\Lambda_1') - T_{12}\tilde A_{21},\\
0 &=& \Lambda_1 T_{12} - \tilde A_{12} - T_{12}\Lambda_2,\\
0 &=& (A_{11}+\Lambda_1')T_{12} + A_{12} - T_{12}T_{21}\tilde A_{12} - T_{12}(\tilde A_{22}+\tilde\Lambda_2'),\\
0 &=& A_{21} + \Lambda_2 T_{21} - T_{21}\Lambda_1,\\
0 &=& A_{21}T_{12}T_{21} + (A_{22}+\Lambda_2')T_{21} - T_{21}(\tilde A_{11}+\tilde\Lambda_1') - \tilde A_{21},\\
0 &=& A_{21}T_{12} + A_{22} + \Lambda_2' - T_{21}\tilde A_{12} - (\tilde A_{22}+\tilde\Lambda_2').
\end{array}$$

Owing to the disjointness of the spectra of Λ_1 and Λ_2, we conclude that there exists exactly one solution T_{21} of the (linear) equation

$$A_{21} = T_{21}\Lambda_1 - \Lambda_2 T_{21}, \tag{3.4}$$

and then the fifth one of the above identities holds. Let us for the moment assume that we know T_{12}. Then we may use the first, third, sixth, and seventh equation, in this order, to compute the blocks $\tilde A_{11}+\tilde\Lambda_1$, $\tilde A_{12}$, $\tilde A_{21}$, and $\tilde A_{22}+\tilde\Lambda_2$. Inserting previously computed blocks, and observing that the matrices Λ_1 and Λ_2 both are diagonal while the diagonal entries of A_{11}, $\tilde A_{11}$, A_{22}, and $\tilde A_{22}$ vanish, we obtain the following results:

$$\tilde\Lambda_1' = \Lambda_1' - I, \qquad \tilde A_{11} = A_{11} - T_{12}T_{21}\Lambda_1 + \Lambda_1 T_{12}T_{21}, \tag{3.5}$$

$$\tilde A_{12} = \Lambda_1 T_{12} - T_{12}\Lambda_2, \tag{3.6}$$

$$\tilde A_{21} = (\Lambda_2' + A_{22})T_{21} - T_{21}(A_{11}+\Lambda_1' - I) + T_{21}T_{12}T_{21}\Lambda_1 - \Lambda_2 T_{21}T_{12}T_{21}, \tag{3.7}$$

$$\tilde\Lambda_2' = \Lambda_2', \qquad \tilde A_{22} = A_{22} - \Lambda_2 T_{21}T_{12} + T_{21}T_{12}\Lambda_2. \tag{3.8}$$

[1]Observe that for convenience we shall write z instead of zI, for $z\in\mathbb{C}$ and an identity matrix of appropriate size.

Inserting into the fourth equation, we then obtain the following identity for the, still undetermined, matrix T_{12}, which shall be referred to as *the main equation* in this article:

$$T_{12}(A_{22} + \Lambda_2') - (A_{11} + \Lambda_1')T_{12} = A_{12} - T_{12}A_{21}T_{12}. \tag{3.9}$$

If this equation holds, we can check that then the one remaining identity (the second one of the original seven equations) is satisfied as well. So *whether or not an admissible meromorphic transformation matrix for* (1.1) *exists is completely equivalent to the question of whether* (3.9) *admits a solution* T_{12}. Such *Riccati-type matrix equations* have been investigated in great detail. Here, we shall be content with the following result that is an easy application of the implicit function theorem:

Theorem 3.1 (Existence of the transformation)**.** *Assume that the two diagonal blocks of A_1 have disjoint spectra. Then there exist $\varepsilon, \delta > 0$ such that for all blocks A_{12} with $\|A_{12}\| < \delta$ there exists a unique matrix T_{12} with $\|T_{12}\| < \varepsilon$ satisfying* (3.9)*. Consequently, an admissible transformation exists for all such A_{12}, and even is unique when T_{12} is chosen accordingly.*

Proof. The mapping $T_{12} \mapsto T_{12}(A_{22} + \Lambda_2') - (A_{11} + \Lambda_1')T_{12} + T_{12}A_{21}T_{12}$ is (arbitrarily often) continuously differentiable from $\mathbb{C}^{m\times(n-m)}$ into itself and has the origin as a fixed point. Arranging the elements of T_{12} into a column vector of length $m(n-m)$, we find that the derivative of this map at the origin is the same as the coefficient matrix of the corresponding linear mapping $T_{12} \mapsto T_{12}(A_{22} + \Lambda_2') - (A_{11} + \Lambda_1')T_{12}$. Due to the assumption of disjoint spectra, this linear map is bijective, and therefore the determinant of the derivative of the nonlinear map cannot vanish at the origin. Hence the statement follows from the inverse mapping theorem. □

Observe that in Theorem 4.2 we shall even give necessary and sufficient conditions for the existence of an admissible transformation. For $n = 2$ we necessarily have $m = 1$, and this case shall be studied in more detail in Section 7, even for general values of n.

4 Structure of the Main Equation

As we have said before, a meromorphic transformation matrix may alter the diagonal elements and the eigenvalues of A_1, but by integer values only. In addition, it is clear that the trace of A_1 equals the sum of its eigenvalues. A transformation $T(z)$ which is admissible for (1.1) lowers the first m diagonal elements of A_1 by 1, and its effect upon the eigenvalues of A_1 can be deduced from the following observations:

- The transformation (3.3) can be written as follows:

$$T(z) = T_u \begin{bmatrix} z & 0 \\ 0 & I \end{bmatrix} T_\ell, \quad T_u = \begin{bmatrix} I & T_{12} \\ 0 & I \end{bmatrix}, \quad T_\ell = \begin{bmatrix} I & 0 \\ T_{21} & I \end{bmatrix}. \tag{4.1}$$

Moreover, one can check that equation (3.9) holds if, and only if, we have

$$A_1 T_u = T_u \begin{bmatrix} A_{11} + \Lambda_1' - T_{12} A_{21} & 0 \\ A_{21} & A_{22} + \Lambda_2' + A_{21} T_{12} \end{bmatrix}.$$

Therefore, the characteristic polynomial of A_1 factors as

$$\det(A_1 - \mu) = \det(A_{11} + \Lambda_1' - T_{12} A_{21} - \mu)\ \det(A_{22} + \Lambda_2' + A_{21} T_{12} - \mu). \quad (4.2)$$

Hence, the eigenvalues of A_1 split into two sets corresponding to those of the left, resp. right, factor. Moreover, the mapping $x \mapsto A_1 x$ has an invariant subspace of dimension $n-m$, namely the linear hull of the last $n-m$ column vectors of T_u. In the special situation of $m = n-1$, the (one-dimensional) block $A_{22}+\Lambda_2'+A_{21}T_{12}$ is an eigenvalue of A_1, and the last column of T_u is the corresponding eigenvector, uniquely normalized by the fact that its last coordinate equals 1. Finally, the mapping $x^\tau \mapsto x^\tau A_1$ has an invariant subspace of dimension m, namely the linear hull of the first m row vectors of T_u^{-1}. In the special situation of $m = 1$, the (one-dimensional) block $A_{11} + \Lambda_1' - T_{12} A_{21}$ is an eigenvalue of A_1, and the first row of T_u^{-1} is a corresponding (row) eigenvector, normalized by the fact that its first coordinate equals 1.

- Similar observations can be made for the matrix $\tilde{A}_1$: From the identities obtained in Section 3, we see that

$$T_\ell \tilde{A}_1 = \begin{bmatrix} \tilde{A}_{11} + \Lambda_1' - I - \tilde{A}_{12} T_{21} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} + \Lambda_2' + T_{21} \tilde{A}_{12} \end{bmatrix} T_\ell,$$

$$A_{11} - T_{12} A_{21} = \tilde{A}_{11} - \tilde{A}_{12} T_{21}, \qquad A_{22} + A_{21} T_{12} = \tilde{A}_{22} + T_{21} \tilde{A}_{12}.$$

Therefore, the characteristic polynomial of $\tilde{A}_1$ factors analogously to (4.2), and the eigenvalues split into corresponding sets. Those in the first set are obtained by subtracting 1's from those eigenvalues of A_1 corresponding to the left factor in (4.2), while the others agree with the ones for the right factor. The mapping $x^\tau \mapsto x^\tau \tilde{A}_1$ has an invariant subspace of dimension $n-m$ spanned by the last $n-m$ rows of T_ℓ. In the special situation of $m = n-1$, the (one-dimensional) block $\tilde{A}_{22} + \Lambda_2' + T_{21}\tilde{A}_{12}$ is an eigenvalue of $\tilde{A}_1$, and the last row of T_ℓ is the corresponding (row) eigenvector, uniquely normalized by the fact that its last component equals 1. Moreover, the mapping $x \mapsto \tilde{A}_1 x$ has an invariant subspace of dimension m, namely the linear hull of the first m column vectors of T_ℓ^{-1}. In the special situation of $m = 1$, the (one-dimensional) block $\tilde{A}_{11} + \Lambda_1' - I - \tilde{A}_{12} T_{21}$ is an eigenvalue of $\tilde{A}_1$, and the first column of T_ℓ^{-1} is a corresponding eigenvector, normalized by the fact that its first component equals 1.

Remark 4.1. As follows from the discussion above, the transformation $T(z)$ decreases m eigenvalues of A_1 by 1, and also subtracts 1 from the first m diagonal elements of

A_1. Recall that one may apply a transformation $x = P\tilde{x}$, with a permutation matrix P, to transform the system (1.1) into another one of the same form, but with the diagonal elements of Λ and Λ' permuted in any prescribed way. Therefore we may also consider meromorphic transformation matrices that, instead of the first, decrease any prescribed set of m diagonal elements of A_1. Note that we could even restrict ourselves to the case $m = 1$, since the more general transformations may be built by combining a finite number of these elementary ones, provided they all exist.

In order to formulate a necessary and sufficient condition for the existence of an admissible transformation for (1.1), we use the following terminology:

(N) We say that an $n - m$-dimensional subspace $U \subset \mathbb{C}^n$ *has a normalized basis* $(b_1, \ldots, b_{n-m})$, provided that $b_j = (b_{1j}, \ldots, b_{nj})^\tau, 1 \le j \le n - m$, with

$$\begin{bmatrix} b_{11} & \ldots & b_{1,n-m} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{n,n-m} \end{bmatrix} = \begin{bmatrix} B \\ \\ I \end{bmatrix}. \tag{4.3}$$

Similarly, we say that a rowspace $V \subset \mathbb{C}^{1\times m}$ of dimension m has a *normalized basis* $(b_1, \ldots, b_{n-m})$, provided that $b_j = (b_{j1}, \ldots, b_{jn})^\tau, 1 \le j \le m$, with

$$\begin{bmatrix} b_{11} & \ldots & b_{1,n} \\ \vdots & & \vdots \\ b_{m1} & \ldots & b_{mn} \end{bmatrix} = \begin{bmatrix} I & B \end{bmatrix}. \tag{4.4}$$

Note that such a basis, in case it exists, is uniquely defined.

Observe that we have shown above that existence of an admissible transformation for (1.1) implies existence of invariant subspaces for $x \mapsto A_1 x$ (of dimension $n - m$), as well as for $x^\tau \mapsto x^\tau A_1$ (of dimension m), having a normalized basis with $B = T_{12}$ resp. $B = -T_{12}$. The opposite implication shall be shown now:

Theorem 4.2. *Given a system* (1.1)*, the following conditions are equivalent:*

(a) *An admissible transformation for* (1.1) *exists.*

(b) *The mapping $x \mapsto A_1 x$ has an invariant subspace U of dimension $n - m$ having a normalized basis.*

(c) *The mapping $x^\tau \mapsto x^\tau A_1$ has an invariant subspace V of dimension m having a normalized basis.*

These conditions always hold when A_1 is diagonalizable.

Proof. By definition, a diagonalizable A_1 has n linearly independent eigenvectors, say $\{b_j = (b_{1j}, \ldots, b_{nj})^T, 1 \le j \le n\}$. The $(n-m) \times n$-matrix

$$B = \begin{bmatrix} b_{m+1,1} & \ldots & b_{m+1,n} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{nn} \end{bmatrix}$$

then has maximal rank, hence contains an $(n-m) \times (n-m)$ submatrix C whose determinant does not vanish. By renumeration of the eigenvectors we may arrange this to be the one corresponding to the first $n-m$ eigenvectors. They then span an invariant subspace for $x \mapsto A_1 x$, and the columns of

$$\begin{bmatrix} b_{11} & \ldots & b_{1,n-m} \\ \vdots & & \vdots \\ b_{n1} & \ldots & b_{n,n-m} \end{bmatrix} C^{-1}$$

are a normalized basis. So for a diagonalizable matrix A_1 condition (b) holds. As we stated above, (a) implies both (b) and (c). Next, assume (b) and let B be as in (4.3). The fact that U is invariant is equivalent to existence of an invertible square matrix C for which the identity

$$A_1 \begin{bmatrix} B \\ I \end{bmatrix} = \begin{bmatrix} B \\ I \end{bmatrix} C$$

holds. This implies $C = A_{21} B + \Lambda_2' + A_{22}$. From this, we then conclude that $T_{21} := B$ satisfies (3.9), so (a) follows. Analogously one can conclude (a) from (c). Hence the proof is completed $\square$

Remark 4.3. Owing to Remark 4.1 and Theorem 4.2, we find that if all eigenvalues of A_1 are distinct, and in addition the difference of any two of them never is an integer, then for arbitrary $m \in \{1, \ldots, n\}$ a transformation $T(z)$ of the above form always exists (but may not be unique). Moreover, the new matrix $\tilde{A}_1$ again satisfies this eigenvalue condition, so that a second transformation (corresponding to the same m, or even to any other one) exists, and so on. However, the second transformation may alter other eigenvalues by 1 than the first one. So the more important question shall be whether a second transformation exists that changes the same eigenvalues as the first one. This shall not be discussed at this time!

5 The Functional Equation

For what follows, it is important to study the effect of admissible transformations, introduced in the previous section, on the Stokes matrix function. Roughly speaking, the Stokes multipliers, and therefore the Stokes matrix function as well, are invariant with respect to meromorphic transformations, but we have to take into account that they correspond to the formal fundamental solution (1.3) rather than the system (1.1). Therefore, it is important to find out the influence of a meromorphic transformation matrix on the form of this formal solution. This is what has to be done to prove the next theorem:

Theorem 5.1 (Invariance of the Stokes matrix)**.** *Let two confluent hypergeometric systems $zx' = A(z)X$ and $z\tilde{x} = \tilde{A}(z)\tilde{x}$ be given, with coefficient matrices $A(z) = z\Lambda + \Lambda' + A$ and $\tilde{A}(z) = z\Lambda + \tilde{\Lambda}' + \tilde{A}$, linked by an admissible transformation $x = T(z)\tilde{x}$. Then*

$$V(\Lambda, \Lambda', A) = V(\Lambda, \tilde{\Lambda}', \tilde{A}).$$

Proof. Let $\hat{X}(z)$ be as in (1.3). Then $\tilde{X}(z) := T^{-1}(z)\hat{X}(z)$ is a formal fundamental solution of the transformed equation $z\tilde{x} = \tilde{A}(z)\tilde{x}$, and it can be verified that $\tilde{X}(z) = \hat{G}(z)z^{\tilde{\Lambda}'}e^{z\Lambda}$, with a formal power series $\hat{G}(z)$ (in $1/z$) starting with I as its constant term. From the papers of *Balser, Jurkat, and Lutz* [1,2] we then obtain that both systems have the same Stokes multipliers, and consequently the Stokes matrix functions are the same, too. □

As we have seen above, the blocks $A_{11} - T_{12}\,A_{21}$ and $A_{22} + A_{21}\,T_{12}$ play a special role, since their eigenvalues remain unchanged by the admissible transformation, and therefore we set, in view of (3.4):

$$T_{11} := A_{11} - T_{12}(T_{21}\Lambda_1 - \Lambda_2 T_{21}), \quad T_{22} := A_{22} + (T_{21}\Lambda_1 - \Lambda_2 T_{21})T_{12}.$$

The fact that the Stokes matrix function is invariant under an admissible transformation can be viewed in a more symmetric way, parameterizing A and $\tilde{A}$ in terms of the four blocks T_{12}, T_{21}, T_{11}, and T_{22}. However, observe that the diagonal elements of A_{11} and A_{22} vanish, and therefore we need to restrict the diagonal elements of T_{11}, resp. of T_{22}, to be equal to those of the matrix $T_{12}(T_{21}\Lambda_1 - \Lambda_2 T_{21})$, resp. those of $-(T_{21}\Lambda_1 - \Lambda_2 T_{21})T_{12}$. Altogether, these four blocks contain $n^2 - n$ independent parameters. Given four such matrices, we can now *define* (with T_u, T_ℓ as in (4.1)):

$$\begin{aligned} A &= T_u \begin{bmatrix} T_{11} & T_{12}\Lambda_2' - \Lambda_1' T_{12} \\ T_{21}\Lambda_1 - \Lambda_2 T_{21} & T_{22} \end{bmatrix} T_u^{-1}, \\ \tilde{A} &= T_\ell^{-1} \begin{bmatrix} T_{11} & \Lambda_1 T_{12} - T_{12}\Lambda_2 \\ (\Lambda_2' + I)T_{21} - T_{21}\Lambda_1' & T_{22} \end{bmatrix} T_\ell \end{aligned} \tag{5.1}$$

and verify that then the identities (3.4) – (3.9) hold. Parameterizing the matrices in this fashion, the question of existence of an admissible transformation becomes void! Fixing Λ and Λ_2', and writing $V(\Lambda_1', A)$ for the Stokes matrix function, we may express its invariance with respect to an admissible transformation as

$$\begin{aligned} & V\left(\Lambda_1', T_u \begin{bmatrix} T_{11} & T_{12}\Lambda_2' - \Lambda_1' T_{12} \\ T_{21}\Lambda_1 - \Lambda_2 T_{21} & T_{22} \end{bmatrix} T_u^{-1}\right) \\ & \quad = V\left(\Lambda_1' - I, T_\ell^{-1} \begin{bmatrix} T_{11} & \Lambda_1 T_{12} - T_{12}\Lambda_2 \\ T_{21}\Lambda_1' - (\Lambda_2' + I)T_{21} & T_{22} \end{bmatrix} T_\ell\right). \end{aligned} \tag{5.2}$$

Whether or not this identity appears more natural than the one in the above theorem may be a matter of taste! In any case, it is an important functional equation satisfied by the Stokes matrix function.

6 The Characteristic Sequence

The functional equation which we derived earlier becomes easier to interpret when, instead of the Stokes matrix function, we consider the reverse function introduced in Section 2. To do this, we introduce a sequence of systems of the form (1.1), of which two consecutive ones are related by a transformation as in the previous section. In detail, let a system (1.1) be given, and define a sequence of matrices $A(z;k) = z\Lambda + A_1(k)$, together with a sequence of transformations $T(z;k)$ of a form analogous to (3.3), but with T_{21} and T_{12} depending on k, so that $A_1(0)$ equals the (given) matrix A_1, while the remaining matrices satisfy the following identities:

- The matrices $A_1(k)$ are written in the form
$$A_1(k) = \begin{bmatrix} \Lambda_1' + A_{11}(k) - k & A_{12}(k) \\ A_{21}(k) & \Lambda_2' + A_{22}(k) \end{bmatrix}, \tag{6.1}$$
with the diagonal elements of $A_{\nu\nu}(k)$ all vanishing.
- Replacing A_1 by $A_1(k)$, $\tilde{A}_1$ by $A_1(k{+}1)$, and $T(z)$ by $T(z;k)$, we obtain from (3.4) – (3.9) that
$$A_{21}(k) = T_{21}(k)\Lambda_1 - \Lambda_2 T_{21}(k) \tag{6.2}$$

$$A_{11}(k{+}1) = A_{11}(k) - T_{12}(k)T_{21}(k)\Lambda_1 + \Lambda_1 T_{12}(k)T_{21}(k) \tag{6.3}$$

$$A_{12}(k{+}1) = \Lambda_1 T_{12}(k) - T_{12}(k)\Lambda_2 \tag{6.4}$$

$$\begin{aligned} A_{21}(k{+}1) = {} & \big(\Lambda_2' + A_{22}(k)\big)T_{21}(k) - T_{21}(k)\big(A_{11}(k) + \Lambda_1' - k{-}1\big) \\ & + T_{21}(k)T_{12}(k)T_{21}(k)\Lambda_1 - \Lambda_2 T_{21}(k)T_{12}(k)T_{21}(k) \end{aligned} \tag{6.5}$$

$$A_{22}(k{+}1) = A_{22}(k) - \Lambda_2 T_{21}(k) T_{12}(k) \ + T_{21}(k) T_{12}(k) \Lambda_2 \tag{6.6}$$

$$\begin{aligned} T_{12}(k)\big(A_{22}(k) + \Lambda_2'\big) - \big(A_{11}(k) + \Lambda_1' - k\big) T_{12}(k) \\ = A_{12}(k) - T_{12}(k) A_{21}(k) T_{12}(k) \end{aligned} \tag{6.7}$$

with the last equation again being the main one. Observe that it follows from the results in the previous sections that the following statements hold true:

- The diagonal elements of all the matrices $A_{\nu\nu}(k)$ vanish.
- Abbreviating

  $$T_{11}(k) := A_{11}(k) - T_{12}(k) A_{21}(k) \quad \text{and} \quad T_{22}(k) := A_{22}(k) + A_{21}(k) T_{12}(k),$$

  the matrices $T_{\nu\nu} + \Lambda_\nu'$ have eigenvalues that are independent of k.
- The eigenvalues of $A_1(k)$ split into two subsets, which are equal to the eigenvalues of the matrices $\Lambda_1' {+} T_{11}(k) {-} k$ and $\Lambda_2' {+} T_{11}(k)$, resp. In particular, the eigenvalues in the second set do not depend on k, while those in the first all are decreased by 1 when proceeding from k to $k + 1$.
- The main equation (6.7) can be rewritten as

  $$T_{12}(k)\big(k + T_{22}(k) + \Lambda_2'\big) - \big(A_{11}(k) + \Lambda_1'\big) T_{12}(k) = A_{12}(k). \tag{6.8}$$

  In this form, the equation is linear in the entries of $T_{12}(k)$, provided that we know $T_{22}(k)$.

Remark 6.1. Equations (6.2) – (6.7) may be viewed as a nonlinear system of difference equations for the blocks $A_{\nu\mu}(k)$ – however, this system is given in an implicit form. To make it explicit, one has to solve equations (6.2) and (6.7) for $T_{21}(k)$ and $T_{12}(k)$, resp., and then insert into the remaining identities. In generic situations, a solution of (6.7) always exists, but may not be unique, and it is not obvious which one to select. We shall not go into detail about this here. Instead, we shall now restrict ourselves to the simpler situation of $m = 1$, bearing in mind that the general case can be built with help of several such elementary transformations – assuming their existence. Indeed, it shall turn out that in this case of $m = 1$, there is a natural way of selecting a solution of (6.7), and we shall obtain additional information on the sequence of matrices $A_1(k)$.

7 A Special Case

We shall now investigate the characteristic sequence for $m = 1$, under some additional assumptions and using a more appropriate notation:

- In what follows, we shall restrict to the case $m = 1$, so that the first diagonal blocks are one-dimensional. In fact, we shall from now on always assume that $\Lambda_1' = \Lambda_1 = 0$. This can w. l. o. g. be made to hold by a transformation $x = \mathrm{e}^{\lambda z} z^{\alpha} \tilde{x}$ with suitable $\lambda, \alpha \in \mathbb{C}$.

- In this situation, the off-diagonal blocks of $A_1(k)$ are (row, resp. column) vectors, and therefore we shall from now on denote them as
$$A_{12}(k) =: a_1^{\tau}(k), \quad A_{21}(k) =: a_2(k), \qquad a_1(k), a_2(k) \in \mathbb{C}^{n-1}.$$
Similarly, the parameters of the corresponding admissible transformations shall be written as
$$T_{12}(k) =: t_1^{\tau}(k), \quad T_{21}(k) =: t_2(k), \qquad t_1(k), t_2(k) \in \mathbb{C}^{n-1}.$$

- For simplicity of notation, we shall from now on write for the second diagonal blocks
$$A(k) := A_{22}(k), \qquad \Lambda' := \Lambda_2', \qquad \Lambda := \Lambda_2,$$
hence $\Lambda' = \operatorname{diag}[\lambda_2', \ldots, \lambda_n']$, $\Lambda = \operatorname{diag}[\lambda_2, \ldots, \lambda_n]$. Make sure to distinguish these matrices from the (n-dimensional) ones previously denoted by the same symbols!

In this special case, equation (6.3) becomes trivial, since all $A_{11}(k)$ vanish. The remaining ones of formulas (6.2) – (6.7) simplify, and for convenience of the reader we display them here, using the new notation:

$$a_2(k) = -\Lambda t_2(k) \tag{7.1}$$

$$a_1^{\tau}(k+1) = -t_1^{\tau}(k)\Lambda \tag{7.2}$$

$$a_2(k+1) = \big(k + 1 + \Lambda' + A(k)\big)t_2(k) - \Lambda t_2(k) t_1^{\tau}(k) t_2(k) \tag{7.3}$$

$$A(k+1) = A(k) - \Lambda t_2(k) t_1^{\tau}(k) \ + t_2(k) t_1^{\tau}(k)\Lambda \tag{7.4}$$

$$t_1^{\tau}(k)\big(k + \Lambda' + A(k)\big) = a_1^{\tau}(k) - t_1^{\tau}(k) a_2(k) t_1^{\tau}(k). \tag{7.5}$$

Let $A_1(k)$, for some $k \in \mathbb{N}_0$, be given, and suppose that we have computed $t_2(k)$ and $t_1(k)$ so that (7.1) and (7.5) hold. We then define $a_1(k+1)$, $a_2(k+1)$, and $A(k+1)$ by means of (7.2), (7.3), and (7.4), resp. From results in the previous sections, or by direct verification, we obtain the following:

- Existence of a solution of (7.5) is equivalent to the existence of a row-eigenvector $e^\tau(k)$ of A_1 (corresponding to an eigenvalue $\mu(k)$, say) whose first coordinate equals 1, and then this eigenvector is of the form
$$e(k) = \begin{bmatrix} 1 \\ -t_1(k) \end{bmatrix}$$
with $t_1^\tau(k)$ solving (7.5). Moreover $e^\tau(k) A_1(k) = \mu(k) e^\tau(k)$ holds if, and only if, the two equations
$$\mu(k) = -k - t_1^\tau(k) a_2(k), \quad t_1^\tau(k)\big(\Lambda' + A(k) - \mu(k)\big) = a_1^\tau(k) \tag{7.6}$$
are satisfied.

- The number $\mu(k) := -k - a_1^\tau(k) t_2(k)$ is an eigenvalue of $A_1(k)$, while the other ones are equal to the eigenvalues of
$$T(k) := A(k) + a_2(k) t_1^\tau(k),$$
the latter ones being independent of k.

- Assume that an eigenvalue $\mu(k)$ of $A_1(k)$ exists for which $\det(\Lambda' + A(k) - \mu(k)) \neq 0$. Then the last $n-1$ columns of $A_1(k) - \mu(k)I$ are linearly independent, while $\det(A_1(k) - \mu(k)I) = 0$. This implies that the first column of this matrix is a linear combination of the other ones. Moreover, if $e^\tau(k)$ is a corresponding row-eigenvector, then its first coordinate cannot vanish, and hence we may assume it to be equal to 1. So in this case, the first equation in (7.6) can be ignored, while the remaining one has a unique solution $t_1^\tau(k)$. In other words, if we select an eigenvalue $\mu(k)$ of $A_1(k)$ which is not an eigenvalue of $\Lambda' + A(k)$, assuming such an eigenvalue exists, then we obtain
$$t_1^\tau(k) = a_1^\tau(k)\big(\Lambda' + A(k) - \mu(k)\big)^{-1}. \tag{7.7}$$

- For some k (e.g., for $k = 0$), assume that we found an eigenvalue $\mu(k)$ which is not an eigenvalue of $\Lambda' + A(k)$, and have defined $t_1^\tau(k)$ by (7.7). Then $\mu(k) + 1$ is an eigenvalue of $A_1(k{+}1)$, and if it is not an eigenvalue of $\Lambda' + A(k{+}1)$, we may choose $\mu(k{+}1) := \mu(k) + 1$ to proceed. Assuming this can be done for all $k \in \mathbb{N}_0$, the number $\mu := \mu(k) + k = -t_1^\tau(k) a_2(k)$ does not depend upon k.

- With $T(k)$ as above, we may use (5.1) to find that
$$\begin{aligned} A(k) &= \begin{bmatrix} 1 & t_1^\tau(k) \\ 0 & I \end{bmatrix} \begin{bmatrix} \mu - k & t_1^\tau(k)(k{+}\Lambda') \\ -\Lambda t_2(k) & T(k) \end{bmatrix} \begin{bmatrix} 1 & -t_1^\tau(k) \\ 0 & I \end{bmatrix}, \\ A(k{+}1) &= \begin{bmatrix} 1 & 0 \\ -t_2(k) & I \end{bmatrix} \begin{bmatrix} \mu - k & -t_1^\tau(k)\Lambda \\ (k{+}1{+}\Lambda')t_2(k) & T(k) \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t_2(k) & I \end{bmatrix}. \end{aligned} \tag{7.8}$$

Observe that in case of $n = 2$ all blocks are one-dimensional, hence commute with one another. In particular, all blocks $A(k)$ vanish, hence (7.4) becomes trivial. The remaining ones of formulas (7.1) – (7.5), switching from upper- to lower-case letters, become equivalent to

$$\begin{aligned} a_2(k) = -\lambda t_2(k), \quad a_1(k+1) = -\lambda t_1(k), \\ a_2(k+1) = (k+1+\lambda')t_2(k) - \lambda t_2(k)^2 t_1(k), \\ t_1(k)(k+\lambda') = a_1(k) - t_1(k)^2 a_2(k). \end{aligned} \tag{7.9}$$

Observing that $\mu = -t_1(k)a_2(k)$ does not depend upon k, we obtain the following simple first order difference equations for the sequences $a_1(k)$, $a_2(k)$:

$$\begin{aligned} \lambda a_2(k+1) = -(k+1+\lambda'-\mu)a_2(k), \\ a_1(k+1)(k+\lambda'-\mu) = -\lambda a_1(k). \end{aligned} \tag{7.10}$$

These simple equations can be solved explicitly and imply

$$a_1(k) = \frac{(-\lambda)^k \Gamma(\lambda'-\mu)}{\Gamma(k+\lambda'-\mu)} a_1(0), \qquad a_2(k) = \frac{\Gamma(1+k+\lambda'-\mu)}{(-\lambda)^k \Gamma(1+\lambda'-\mu)} a_2(0).$$

For higher dimensions of $n \geq 3$, the corresponding identities are nonlinear difference equations and can, therefore, not be solved explicitly. Note, however, that according to results by the author [5] the matrices $A_1(k)$ (compare (6.1)) can, for all cases where the eigenvalue condition at the origin is satisfied, be regarded as locally holomorphic functions of the entries in the Stokes matrix V (which does not depend on k). The blocks $A_{\nu\mu}(k) = A_{\nu\mu}(k; V)$ then give rise to a family of solutions of these nonlinear system of difference equations, depending holomorphically upon $n(n-1)$ parameters!

8 Summary and Outlook

As was said before, the Stokes matrix function very likely cannot be computed in terms of previously known (higher) transcendental functions. Therefore it is of great importance to investigate its properties, such as its holomorphic dependence upon the various entries in the matrices Λ, Λ', and A. On the other hand, it is highly desirable to give a few (relatively) simple properties that characterize $V(\Lambda, \Lambda', A)$ uniquely. One such characterisation has been given in the articles mentioned above by expanding the Stokes matrix function as a power series in the entries of A, with coefficients that can be recursively computed as functions of Λ and Λ'. In a way, it is quite satisfying that this can be done. From looking into the case of $n = 2$, however, one gets the impression that such results may not be very natural, since it roughly speaking corresponds to finding the power series expansion of the reciprocal of the Gamma function. Instead, it is so much more natural to use its functional equation that has been found in this article. This equation can also be viewed as a nonlinear difference equation for the reverse Stokes matrix function. In view of this, it is quite likely that this reverse function may be characterized as the only solution of this difference equation that has a certain asymptotic behaviour as $k \to \infty$. We illustrate this by looking once more at the case of $n = 2$:

- Observe that in the notation used here we have

$$A_1(k) = \begin{bmatrix} -k & a_1(k) \\ a_2(k) & \lambda' \end{bmatrix}.$$

With $\mu = -t_1(k)a_2(k)$ as defined above, we obtain from (7.9), (7.10) that $\mu^2 - \mu(k + \lambda') - a_1(k)a_2(k) = 0$. With

$$\alpha_k + \beta_k = \lambda' + k, \qquad \alpha_k\beta_k = -a_1(k)a_2(k),$$

note that this implies that α_k, β_k are the roots of the same quadratic equation that we found for μ. Therefore we choose $\alpha_k = \mu$ (independent of k), and then $\beta_k = k + \lambda' - \mu$. The Stokes matrix function (which also does not depend on k) then, rewriting the identities (1.4) in the new notation, becomes equal to

$$V = \begin{bmatrix} 0 & v_1 \\ v_2 & 0 \end{bmatrix}, \quad v_2 = \frac{2\pi i a_2(k)\lambda^{\lambda'+k}\mathrm{e}^{-i\pi(\lambda'+k)}}{\Gamma(1+\mu)\Gamma(1+k+\lambda'-\mu)},$$

$$v_1 = \frac{2\pi i a_1(k)\lambda^{-\lambda'-k}\mathrm{e}^{2\pi i\lambda'}}{\Gamma(1-\mu)\Gamma(1-k-\lambda'+\mu)}.$$

From these identities we can trivially observe that the entries v_1, v_2 can be obtained from the asymptotic behaviour of the sequences $a_1(k)$, $a_2(k)$, which in turn are determined by the difference equations (7.9), with given initial values $a_1(0)$, $a_2(0)$. While in this article we have found the higher dimensional analogues of (7.9), it is not entirely clear but very likely that even for $n \geq 3$ the (nonlinear) difference equations (7.1) – (7.5), or more precisely the asymptotic behaviour of their solutions, determine the Stokes matrix function. This, however, is left for future research.

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