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Impulsive Control for a General Hopfield Neural Network with Distributed Delays

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Abstract

In this paper, the problem of global stability for impulsive Hopfield neural networks with distributed delays is considered. The stability criteria are established by employing Lyapunov functions and extended impulsive delayed Halanay inequality. An illustrative example is given to demonstrate the effectiveness of the obtained results.

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1 Introduction

Hopfield neural networks (HNNs) have been extensively studied in past years and found many applications in different areas such as pattern recognition, associative memory, and combinatorial optimization. Such applications heavily depend on the dynamical behaviors. Thus, the analysis of the dynamical behaviors is a necessary step for practical design of neural networks.

One of the most investigated problems in the study of HNNs is the stability of the equilibrium point. If an equilibrium of a neural network is globally stable, it means that the domain of attraction of the equilibrium point is the whole space. This is significant both theoretically and practically. Many researchers have studied the global stability of the HNNs with delays [1–3]. In [4, 5], the authors derived the criterion of global asymptotic stability for the equilibrium point of the HNNs with distributed delays. But they requested the kernel functions k_{ij} satisfy $\displaystyle\int_0^{+\infty} sk_{ij}(s)ds < +\infty, i, j = 1, 2, \dots, n.$

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On the other hand, HNNs are subject to sudden and sharp perturbations instantaneously, which cannot be well described by using pure continuous HNNs or pure discrete HNN models, that is, they do exhibit impulsive effects [6, 7]. These are neural networks with impulsive effects, which can be used as an appropriate description of the phenomena of abrupt qualitative dynamical changes of essentially continuous time systems. Since impulses can affect the dynamical behaviors of the system creating unstable characteristics, it is necessary to investigate impulse effects on the stability of HNNs. In [8], the author studied the exponential stability of a class of impulsive HNN with distributed delays. But the author requested that the distant between any solution of the system and equilibrium point is nonincreasing at impulse times.

The main aim of this paper is to present some new sufficient conditions for global exponential stability and global asymptotic stability of a class of impulsive HNN with distributed delays by means of constructing the extended impulsive delayed Halanay inequality and Lyapunov function methods. The effects of impulses and delays on the solutions are stressed here. The result of exponential stability permits that the distant between the solutions of the system and equilibrium point is obviously increased at impulse times. To derive the criterion of global asymptotic stability for the equilibrium point, it only requests the kernel functions k_{ij} satisfy $\int_0^{+\infty} k_{ij}(s)ds < +\infty, i, j = 1, 2, \dots, n.$

This paper is organized as follows. In Section 2, some notations and definitions are introduced. In Section 3, a criterion on global asymptotic stability is established for impulsive HNN with distributed delays. In Section 4, a criterion on exponential stability is established for impulsive HNN with distributed delays. In the last section, an example is given to illustrate the advantage of the results obtained.

2 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers and $\mathbb{R}^n$ the n-dimensional real space equipped with the norm $|x| = \sum_{i=1}^{n} |x_i|$.

Consider the impulsive HNN model with distributed delays

$$\begin{cases} \dot{x}_i(t) = -a_i x_i(t) + \sum_{j=1}^{n} b_{ij} f_j(x_j(t)) \\ \qquad\qquad + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} k_{ij}(t-s) f_j(x_j(s))ds + I_i, \quad t \neq t_k,\ t \geq t_0, \\ \Delta(x_i(t_k)) = x_i(t_k) - x_i(t_k^-), \qquad\qquad\qquad k \in Z_+,\ i \in \Lambda, \end{cases} \tag{2.1}$$

where $\Lambda = \{1, 2, \dots, n\}$; $n \geq 2$ corresponds to the number of units in a neural network; the impulse times t_k satisfy $0 \leq t_0 < t_1 < \dots < t_k < \dots$, $\lim_{k \to +\infty} t_k = +\infty$; $x_i(t)$

corresponds to the state of the ith unit at time t; f_j is the activation function of the jth unit; a_i is positive constant; $B = (b_{ij})_{n\times n}$, $C = (C_{ij})_{n\times n}$ denote the connection weight matrices; I_i is the constant input.

Define $PC((-\infty,0],\mathbb{R}^n) = \{\psi : (-\infty,0] \to \mathbb{R}^n$ is continuous everywhere except at a finite number of points t_k, at which $\psi(t_k^+)$ and $\psi(t_k^-)$ exist and $\psi(t_k^+) = \psi(t_k)\}$.

Define $PCB(t) = \{\psi \in PC((-\infty,0],\mathbb{R}^n) | \psi$ is bounded$\}$. For $\psi \in PCB(t)$, the norm of ψ is defined by $\|\psi\| = \sup\limits_{-\infty<\theta\le 0} |\psi(\theta)|$.

For any $\sigma \ge t_0$, let $PCB_\delta(\sigma) = \{\psi \in PCB(\sigma) | \|\psi\| \le \delta\}$.

Assume that the system (2.1) is supplemented with initial conditions of the form

$$x_i(\sigma + s) = \phi_i(s), \qquad s \in (-\infty, 0],$$

where $\phi = (\phi_1, \phi_2, \ldots, \phi_n)^T \in PC((-\infty,0],\mathbb{R}^n)$.

In this paper, the impulsive operator is viewed as a perturbation of the equilibrium point x^* of system (2.1) without impulse effects. We assume that the following conditions hold:

(H$_1$) $\Delta(x_i(t_k)) = x_i(t_k) - x_i(t_k^-) = J_{ik}(x_i(t_k^-) - x_i^*)$,$i \in \Lambda, k \in Z_+$, and $J_{ik} = 0$ has exactly one equilibrium point x^*;

(H$_2$) f_i is globally Lipschitz with Lipschitz constant L_i, i.e.,

$$|f_i(u) - f_i(v)| \le L_i|u - v|, \quad \forall u, v \in \mathbb{R},\ i \in \Lambda;$$

(H$_3$) The kernel functions $k_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ are continuous on $\mathbb{R}_+$ with $\displaystyle\int_0^{+\infty} k_{ij}(s)ds = 1$, and there exists $K : \mathbb{R}_+ \to \mathbb{R}_+$ such that $k_{ij}(s) \le K(s), i, j = 1, 2, \ldots, n$.

Definition 2.1. $x^* = (x_1^*, x_2^*, \ldots, x_n^*)^T$ is said to be an equilibrium point of system (2.1), if

$$-a_i x_i^* + \sum_{j=1}^n b_{ij} f_j(x_j^*) + \sum_{j=1}^n c_{ij} \int_{-\infty}^t k_{ij}(t-s) f_j(x_j^*) ds + I_i = 0, \quad i \in \Lambda.$$

Since x^* is an equilibrium point of system (2.1), one can derive from system (2.1) that the transformation $y_i = x_i - x_i^*$, $i \in \Lambda$ transforms such system into the system

$$\begin{cases} \dot{y}_i(t) = -a_i(y_i(t) + x_i^*) + \displaystyle\sum_{j=1}^n b_{ij} f_j(y_j(t) + x_j^*) \\ \qquad\qquad + \displaystyle\sum_{j=1}^n c_{ij} \int_{-\infty}^t k_{ij}(t-s) f_j(y_j(s) + x_j^*) ds + I_i, & t \ne t_k,\ t \ge t_0, \\ \Delta(y_i(t_k)) = y_i(t_k) - y_i(t_k^-), & k \in \mathbb{Z}_+,\ i \in \Lambda, \\ y(\sigma + \theta) = \varphi(\theta), & \theta \in (-\infty, 0], \end{cases} \tag{2.2}$$

where $\varphi(\theta) = x(\sigma + \theta) - x^*$.

Hence, to prove the stability of x^* of system (2.1), it is sufficient to prove the stability of the zero solution of system (2.2).

Definition 2.2. Assume $y(t) = y(t, \sigma, \varphi)$ to be the solution of (2.2) through (σ, φ), then the zero solution of (2.2) is said to be

(S1) stable, if for any $\sigma \geq t_0$ and $\epsilon > 0$, there exists some $\delta = \delta(\epsilon, \sigma) > 0$ such that $\varphi \in PCB_\delta(\sigma)$ implies $|y(t)| < \epsilon$, $t \geq \sigma$;

(S2) uniformly stable if the δ in (S1) is independent of σ;

(S3) uniformly asymptotically stable if (S2) holds and there exists some $\eta > 0$ such that for any $\epsilon > 0$ there exists some $T = T(\epsilon, \eta) > 0$ such that $\sigma \geq t_0$ and $\varphi \in PCB_\eta(\sigma)$ imply $|y(t)| < \epsilon$, $t \geq \sigma + T$;

(S4) globally asymptotically stable if (S1) holds and for any given initial data $x_\sigma = \phi$, the following inequality holds: $|y(t)| \to 0$ as $t \to +\infty$;

(S5) globally exponentially stable if there exist constants $\lambda > 0$, $M \geq 1$ such that for any $\varphi \in PCB(\sigma)$, $|y(t)| < M\|\varphi\|e^{-\lambda(t-\sigma)}$, $t \geq \sigma$.

3 Global Asymptotic Stability of Equilibrium Point

Lemma 3.1. *Assume that $m : \mathbb{R} \to \mathbb{R}_+$ is continuous, and there exist positive constants r, l satisfy:*

(i) *For all $t \geq \sigma$, $t \neq t_k$, $k \in Z_+$, $D^+m(t) \leq -rm(t) + l \int_{-\infty}^{t} K(t-s)m(s)ds$;*

(ii) *$m(t_k) \leq \gamma_k m(t_k^-)$, where $\gamma_k \geq 1$ and $\prod_{k=1}^{+\infty} \gamma_k =: M < +\infty$;*

(iii) *$-r + Ml\mu < 0$, where $\mu = \int_0^{+\infty} K(s)ds < +\infty$.*

Then

$$m(t) \leq M\widetilde{m}(\sigma), \quad t \geq \sigma, \quad \textit{and} \quad \lim_{t\to+\infty} m(t) = 0,$$

where $\widetilde{m}(\sigma) = \sup_{-\infty<\theta\leq 0} m(\sigma + \theta)$.

Proof. Suppose that $\sigma \in [t_{l-1}, t_l)$. First, we shall prove that

$$m(t) \leq \widetilde{m}(\sigma), \qquad t \in [\sigma, t_l). \tag{3.1}$$

Suppose on the contrary, then there exists some $\bar{t} \in [\sigma, t_l)$ such that

$$m(\bar{t}) = \widetilde{m}(\sigma), \qquad D^+ m(\bar{t}) \geq 0, \qquad m(t) \leq \widetilde{m}(\sigma),\ t \in [\sigma, \bar{t}).$$

Therefore, we get

$$\begin{aligned} D^+ m(\bar{t}) &\leq -rm(\bar{t}) + l \int_{-\infty}^{\bar{t}} K(\bar{t}-s) m(s) ds \\ &\leq -r\widetilde{m}(\sigma) + l\widetilde{m}(\sigma) \int_0^{+\infty} K(s) ds \\ &\leq (-r + l\mu)\widetilde{m}(\sigma) \\ &< 0. \end{aligned}$$

But this contradicts the fact $D^+ m(\bar{t}) \geq 0$. So (3.1) holds for all $t \in [\sigma, t_l)$.

By using (ii) we get that

$$m(t_l) \leq \gamma_l \widetilde{m}(\sigma).$$

Next, we shall prove that

$$m(t) \leq \gamma_l \widetilde{m}(\sigma), \qquad t \in [t_l, t_{l+1}). \tag{3.2}$$

Suppose on the contrary, then there exists some $\hat{t} \in [t_l, t_{l+1})$ such that

$$m(\hat{t}) = \gamma_l \widetilde{m}(\sigma), \qquad D^+ m(\hat{t}) \geq 0, \qquad m(t) \leq \gamma_l \widetilde{m}(\sigma),\ t \in [t_l, \hat{t}).$$

Therefore, we get

$$\begin{aligned} D^+ m(\hat{t}) &\leq -rm(\hat{t}) + l \int_{-\infty}^{\hat{t}} K(\hat{t}-s) m(s) ds \\ &\leq -r\gamma_l \widetilde{m}(\sigma) + l\gamma_l \widetilde{m}(\sigma) \int_0^{+\infty} K(s) ds \\ &\leq (-r + l\mu)\gamma_l \widetilde{m}(\sigma) \\ &< 0. \end{aligned}$$

But this contradicts the fact $D^+ m(\hat{t}) \geq 0$. So (3.2) holds for all $t \in [t_l, t_{l+1})$.

By induction hypothesis, we may prove, in general, that

$$m(t) \leq \prod_{\sigma \leq t_i \leq t} \gamma_i \widetilde{m}(\sigma), \qquad t \geq \sigma,$$

i.e.,

$$m(t) \leq M\widetilde{m}(\sigma), \qquad t \geq \sigma.$$

Next, we shall show that

$$\lim_{t\to+\infty} m(t) = 0.$$

Note that $m(t)$ is bounded, so

$$\limsup_{t\to+\infty} m(t) = \alpha \qquad i \in \Lambda$$

exist. It is clear that $\alpha \geq 0$. We shall only need to prove that $\alpha = 0$.

Suppose on the contrary, then $\alpha > 0$. According to definition of superior limit, for sufficient small $\epsilon > 0$, there exits $T_1 > \sigma$ such that

$$m(t) < \alpha + \epsilon, \qquad t \geq T_1.$$

Since $\int_0^{+\infty} K(s)ds = \mu$, for the above $\epsilon > 0$, there exists $T_2 > 0$ such that

$$\int_t^{+\infty} K(s)ds < \epsilon, \quad t \geq T_2.$$

Hence, when $t \geq T = T_1 + T_2$, we obtain

$$\begin{aligned}
D^+m(t) &\leq -rm(t) + l\int_{-\infty}^{t} K(t-s)m(s)ds \\
&= -rm(t) + l\int_{-\infty}^{t-T} K(t-s)m(s)ds + l\int_{t-T}^{t} K(t-s)m(s)ds \\
&\leq -rm(t) + lM\widetilde{m}(\sigma)\int_{-\infty}^{t-T} K(t-s)ds + l(\alpha+\epsilon)\int_{t-T}^{t} K(t-s)ds \\
&= -rm(t) + lM\widetilde{m}(\sigma)\int_{T}^{+\infty} K(s)ds + l(\alpha+\epsilon)\int_{0}^{T} K(s)ds \\
&\leq -rm(t) + \epsilon lM\widetilde{m}(\sigma) + l\mu(\alpha+\epsilon)
\end{aligned}$$

By using assumption (ii) and Impulsive differential inequality (see [9]), we obtain that

$$\begin{aligned}
m(t) &\leq m(T)\prod_{T<t_k<t}\gamma_k e^{-r(t-T)} + \int_T^t \prod_{s<t_k<t}\gamma_k e^{-r(t-s)}[\epsilon lM\widetilde{m}(\sigma) + l(\alpha+\epsilon)]ds \\
&= m(T)Me^{-r(t-T)} + \frac{M}{r}(1-e^{-r(t-T)})[\epsilon lM\widetilde{m}(\sigma) + l(\alpha+\epsilon)] \\
&\leq m(T)Me^{-r(t-T)} + \frac{M}{r}[\epsilon lM\widetilde{m}(\sigma) + l\mu(\alpha+\epsilon)].
\end{aligned}$$

Let $t \to +\infty$, $\epsilon \to 0$, we have

$$\alpha \leq \frac{M}{r}l\mu\alpha.$$

Since $\alpha > 0$, we get

$$-r + Ml\mu \geq 0.$$

This contradicts to assumption (iii). So the proof is complete. $\square$

Theorem 3.2. *Assume that the following conditions hold:*

(i) $|u + J_{ik}(u)| \le d_k|u|,\ i \in \Lambda, k \in Z_+$, *with* $d_k > 1$, $M := \prod_{k=1}^{+\infty} d_k < +\infty$*;*

(ii) $\min_{1\le i\le n}\left\{a_i - L_i\sum_{j=1}^{n}|b_{ji}|\right\} > M\mu \max_{1\le i\le n}\left\{L_i\sum_{j=1}^{n}|c_{ji}|\right\}$, *where*

$$\mu := \int_0^{+\infty} K(s)ds < +\infty.$$

Then the equilibrium point of system (2.1) *is globally asymptotically stable.*

Proof. We only need to prove that the zero solution of system (2.2) is globally asymptotically stable. For any $\sigma \ge t_0, \varphi \in PCB(\sigma)$, let $y(t) = y(t, \sigma, \varphi)$ be a solution of (2.2) through (σ, φ).

We choose

$$V(t) = |y(t)| = \sum_{i=1}^{n}|y_i(t)|.$$

Then for $t = t_k, k = 1, 2, \ldots$, from condition (i), we obtain

$$\begin{aligned} V(t_k) &= \sum_{i=1}^{n}|y_i(t_k)| = \sum_{i=1}^{n}|x_i(t_k) - x_i^*| \\ &= \sum_{i=1}^{n}|x_i(t_k^-) - x_i^* + J_{ik}(x_i(t_k^-) - x_i^*)| \\ &\le d_k\sum_{i=1}^{n}|x_i(t_k^-) - x_i^*| = d_kV(t_k^-). \end{aligned}$$

Let $t \ge \sigma$ and $t \ne t_k, k = 1, 2, \ldots$, then for the upper right-hand derivative $D^+V(t)$ of V with respect to system (2.2) we get

$$\begin{aligned} D^+V(t) = & \sum_{i=1}^{n}\operatorname{sgn}(y_i(t))\Bigg[-a_i(y_i(t) + x_i^*) + \sum_{j=1}^{n}b_{ij}f_j(y_j(t) + x_j^*) \\ & + \sum_{j=1}^{n}c_{ij}\int_{-\infty}^{t}k_{ij}(t-s)f_j(y_j(s) + x_j^*)ds + I_i\Bigg]. \end{aligned}$$

Since x^* is the equilibrium of (2.1), then, from condition (H_2), we obtain

$$D^+V(t) \le \sum_{i=1}^{n}\Bigg[-a_i|y_i(t)| + \sum_{j=1}^{n}|b_{ij}|L_j|y_j(t)|$$

$$
\begin{aligned}
& \quad + \sum_{j=1}^{n} |c_{ij}| L_j \int_{-\infty}^{t} k_{ij}(t-s)|y_j(s)|ds \Bigg] \\
&= -\sum_{i=1}^{n} \left[a_i - L_i \sum_{j=1}^{n} |b_{ji}| \right] |y_i(t)| \\
& \quad + \sum_{i=1}^{n} L_i \sum_{j=1}^{n} |c_{ji}| \int_{-\infty}^{t} K(t-s)|y_i(s)|ds \\
&\leq -rV(t) + l \int_{-\infty}^{t} K(t-s)V(s)ds
\end{aligned}
$$

where

$$
r = \min_{1 \leq i \leq n} \left\{ a_i - L_i \sum_{j=1}^{n} |b_{ji}| \right\}, \qquad l = \max_{1 \leq i \leq n} \left\{ L_i \sum_{j=1}^{n} |c_{ji}| \right\}.
$$

Combining with (i) and (ii), from Lemma 3.1, we obtain

$$
V(t) \leq M\widetilde{V}(\sigma), \qquad t \geq \sigma, \tag{3.3}
$$

$$
\lim_{t \to +\infty} V(t) = 0. \tag{3.4}
$$

For any $\epsilon > 0$, we choose $\delta = \delta(\epsilon) > 0$ such that $\delta < \dfrac{\epsilon}{M}$. Then, for any $\sigma \geq t_0$, $\varphi \in PCB_\delta(\sigma)$, we obtain from (3.3)

$$
|y(t)| = V(t) \leq M\widetilde{V}(\sigma) = M\|\varphi\| < \epsilon, \qquad t \geq \sigma.
$$

So the zero solution of system (2.2) is uniformly stable.

From (3.4), we obtain, for any $\varphi \in PCB(\sigma)$,

$$
\lim_{t \to +\infty} |y(t)| = \lim_{t \to +\infty} V(t) = 0.
$$

So the zero solution of (2.2) is globally asymptotically stable, i.e., the equilibrium point system (2.1) is globally asymptotically stable. □

4 Global Exponential Stability of Equilibrium Point

In this section, we assume that the following condition holds:

(H_4) there exists positive constant λ_0 such that

$$
\int_{0}^{+\infty} K(s)e^{\lambda_0 s}ds =: \delta < +\infty.
$$

Lemma 4.1. *Assume that $m : \mathbb{R} \to \mathbb{R}_+$ is continuous, and there exist positive constants r, l satisfy:*

(i) *For all $t \geq \sigma$, $t \neq t_k, k \in Z_+$, $D^+m(t) \leq -rm(t) + l \int_{-\infty}^{t} K(t-s)m(s)ds$;*

(ii) *$m(t_k) \leq \gamma_k m(t_k^-)$, where $\gamma_k \geq 1$ and $\prod_{k=1}^{+\infty} \gamma_k =: M < +\infty$;*

(iii) *$-r + l\delta < 0$.*

Then there exists $\lambda \in (0, \lambda_0]$ such that

$$m(t) \leq M\widetilde{m}(\sigma)e^{-\lambda(t-\sigma)}, \;\; t \geq \sigma,$$

where $\widetilde{m}(\sigma) = \sup_{-\infty<\theta\leq 0} m(\sigma+\theta)$.

Proof. From (iii), there exists $\lambda \in (0, \lambda_0]$ such that

$$-r + \lambda + l\delta < 0.$$

Suppose that $\sigma \in [t_{l-1}, t_l)$. Let

$$L(t) = \begin{cases} m(t)e^{\lambda(t-\sigma)}, & t \geq \sigma, \\ m(t), & -\infty < t \leq \sigma. \end{cases}$$

Thus we only need to prove

$$L(t) \leq M\widetilde{m}(\sigma), \qquad t \geq \sigma. \tag{4.1}$$

First, we shall prove that

$$L(t) \leq \widetilde{m}(\sigma), \qquad t \in [\sigma, t_l).$$

Suppose on the contrary, then there exists some $\bar{t} \in [\sigma, t_l)$ such that

$$L(\bar{t}) = \widetilde{m}(\sigma), \qquad D^+L(\bar{t}) \geq 0, \qquad L(t) \leq \widetilde{m}(\sigma), \; t \in [\sigma, \bar{t}).$$

Therefore, we get

$$\begin{aligned} D^+L(\bar{t}) &= D^+m(\bar{t})e^{\lambda(\bar{t}-\sigma)} + \lambda m(\bar{t})e^{\lambda(\bar{t}-\sigma)} \\ &\leq e^{\lambda(\bar{t}-\sigma)} \left[-rm(\bar{t}) + l \int_{-\infty}^{\bar{t}} K(\bar{t}-s)m(s)ds \right] + \lambda L(\bar{t}) \\ &\leq (-r+\lambda)L(\bar{t}) + l \int_{0}^{+\infty} K(s)e^{\lambda_0 s}L(\bar{t}-s)ds \\ &\leq (-r+\lambda+l\delta)\widetilde{m}(\sigma) \\ &< 0. \end{aligned}$$

But this contradicts the fact $D^+L(\bar{t}) \geq 0$. So (4.1) holds for all $t \in [\sigma, t_l)$.

By using (ii) we get that

$$L(t_l) \leq \gamma_l \widetilde{m}(\sigma).$$

Next, we shall prove that

$$L(t) \leq \gamma_l \widetilde{m}(\sigma), \qquad t \in [t_l, t_{l+1}).$$

Suppose on the contrary, then there exists some $\hat{t} \in [t_l, t_{l+1})$ such that

$$L(\hat{t}) = \gamma_l \widetilde{m}(\sigma), \qquad D^+L(\hat{t}) \geq 0, \qquad L(t) \leq \gamma_l \widetilde{m}(\sigma),\ t \in [t_l, \hat{t}).$$

Therefore, we get

$$\begin{aligned} D^+L(\hat{t}) &= D^m(\hat{t}) e^{\lambda(\hat{t}-\sigma)} + \lambda m(\hat{t}) e^{\lambda(\hat{t}-\sigma)} \\ &\leq e^{\lambda(\hat{t}-\sigma)} \left[-rm(\hat{t}) + l \int_{-\infty}^{\hat{t}} K(\hat{t}-s) m(s) ds \right] + \lambda L(\hat{t}) \\ &\leq (-r+\lambda) L(\hat{t}) + l \int_0^{+\infty} K(s) e^{\lambda_0 s} L(\hat{t}-s) ds \\ &\leq (-r+\lambda+l\delta) \gamma_l \widetilde{m}(\sigma) \\ &< 0. \end{aligned}$$

But this contradicts the fact $D^+m(\hat{t}) \geq 0$. So (4.1) holds for all $t \in [t_l, t_{l+1})$.

By induction hypothesis, we may prove, in general, that

$$L(t) \leq \prod_{\sigma \leq t_i \leq t} \gamma_i \widetilde{m}(\sigma), \qquad t \geq \sigma,$$

i.e.,

$$m(t) \leq M \widetilde{m}(\sigma) e^{-\lambda(t-\sigma)}, \qquad t \geq \sigma.$$

The proof is complete. □

Theorem 4.2. *Assume that the following conditions hold:*

(i) $|u + J_{ik}(u)| \leq d_k |u|,\ i \in \Lambda, k \in Z_+$, *with* $d_k > 1, M := \prod_{k=1}^{+\infty} d_k < +\infty$*;*

(ii) $\min\limits_{1 \leq i \leq n} \left\{ a_i - L_i \sum_{j=1}^{n} |b_{ji}| \right\} > \delta \max\limits_{1 \leq i \leq n} \left\{ L_i \sum_{j=1}^{n} |c_{ji}| \right\}.$

Then the equilibrium point of system (2.1) *is globally exponentially stable.*

Proof. We only need to prove that the zero solution of system (2.2) is globally exponentially stable. For any $\sigma \geq t_0$, $\varphi \in PCB(\sigma)$, let $y(t) = y(t, \sigma, \varphi)$ be a solution of (2.2) through (σ, φ).

We choose

$$V(t) = |y(t)| = \sum_{i=1}^{n} |y_i(t)|.$$

Then for $t = t_k, k = 1, 2, \ldots$, from condition (i), we obtain

$$\begin{aligned} V(t_k) &= \sum_{i=1}^{n} |y_i(t_k)| = \sum_{i=1}^{n} |x_i(t_k) - x_i^*| = \sum_{i=1}^{n} |x_i(t_k^-) - x_i^* + J_{ik}(x_i(t_k^-) - x_i^*)| \\ &\leq d_k \sum_{i=1}^{n} |x_i(t_k^-) - x_i^*| = d_k V(t_k^-). \end{aligned}$$

Let $t \geq \sigma$ and $t \neq t_k, k = 1, 2, \ldots$, then for the upper right-hand derivative $D^+V(t)$ of V with respect to system (2.2) we get

$$\begin{aligned} D^+V(t) = & \sum_{i=1}^{n} \mathrm{sgn}(y_i(t)) \left[-a_i(y_i(t) + x_i^*) + \sum_{j=1}^{n} b_{ij} f_j(y_j(t) + x_j^*) \right. \\ & \left. + \sum_{j=1}^{n} c_{ij} \int_{-\infty}^{t} k_{ij}(t-s) f_j(y_j(s) + x_j^*) ds + I_i \right]. \end{aligned}$$

Since x^* is the equilibrium of (2.1), then, from condition (H_2), we obtain

$$\begin{aligned} D^+V(t) &\leq \sum_{i=1}^{n} \left[-a_i |y_i(t)| + \sum_{j=1}^{n} |b_{ij}| L_j |y_j(t)| \right. \\ & \quad \left. + \sum_{j=1}^{n} |c_{ij}| L_j \int_{-\infty}^{t} k_{ij}(t-s) |y_j(s)| ds \right] \\ &= -\sum_{i=1}^{n} \left[a_i - L_i \sum_{j=1}^{n} |b_{ji}| \right] |y_i(t)| \\ & \quad + \sum_{i=1}^{n} L_i \sum_{j=1}^{n} |c_{ji}| \int_{-\infty}^{t} K(t-s) |y_i(s)| ds \\ &\leq -rV(t) + l \int_{-\infty}^{t} K(t-s) V(s) ds \end{aligned}$$

where

$$r = \min_{1 \leq i \leq n} \left\{ a_i - L_i \sum_{j=1}^{n} |b_{ji}| \right\}, \qquad l = \max_{1 \leq i \leq n} \left\{ L_i \sum_{j=1}^{n} |c_{ji}| \right\}.$$

Combining with (i) and (ii), from Lemma 4.1, we obtain

$$V(t) \le M\widetilde{V}(\sigma)e^{-\lambda(t-\sigma)}, \qquad t \ge \sigma.$$

i.e.,

$$|y(t)| \le M\|\varphi\|e^{-\lambda(t-\sigma)}, \qquad t \ge \sigma,$$

which implies that the zero solution of system (2.2) is globally exponentially stable. Hence, the equilibrium point of system (2.1) is globally exponentially stable. □

5 Illustrative Example

Consider the impulsive neural network of type (2.1), where $n = 2$, $\sigma = 0$, $I_1 = 2.9$, $I_2 = 0.9$, $a_1 = 4$, $a_2 = 5$, $f_i(x_i) = \frac{1}{2}(|x_i + 1| - |x_i - 1|)$, $k_{ij}(s) = 2e^{-2s}, i, j = 1, 2$;

$$(b_{ij})_{2\times 2} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 0.3 & 0.4 \\ 0.7 & 0.5 \end{pmatrix};$$

$$(c_{ij})_{2\times 2} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} = \begin{pmatrix} 0.5 & 0.2 \\ 0.5 & 0.3 \end{pmatrix};$$

with impulsive perturbations of the form

$$\begin{cases} x_1(t_k) = x_1(t_k^-) + \dfrac{10}{k^2}(x_1(t_k^-) - x_1^*), & k = 1, 2, \ldots \\ x_2(t_k) = x_2(t_k^-) + \dfrac{10}{k^2}(x_2(t_k^-) - x_2^*), & k = 1, 2, \ldots. \end{cases}$$

It is easy to verify that the condition of Theorem 4.2 is satisfied for $\lambda_0 = 1$, $L_1 = L_2 = 1$. We also have that

$$d_k = 1 + \frac{10}{k^2} \qquad \text{and} \qquad \prod_{k=1}^{+\infty}\left(1 + \frac{10}{k^2}\right) < +\infty.$$

According to Theorem 4.2, the unique equilibrium of the system

$$x^* = (x_1^*, x_2^*)^T = (1, 0.5)^T$$

is globally exponentially stable.

6 Conclusion

In this paper, by means of constructing the extended impulsive delayed Halanay inequality, the criterion of global exponential stability and global asymptotic stability are established for impulsive HNN with distributed delays. It also shows that the stability properties of the neural networks can still be guaranteed under the influence of the appropriate impulsive perturbations.

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