Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 7, Number 1, pp. 81–88 (2012)
http://campus.mst.edu/adsa

Fractional Virasoro Algebra

E. H. El Kinani and **M. R. Sidi Ammi**
Moulay Ismaïl University
Faculty of Sciences and Technics
Department of Mathematics
Box. 509, Errachidia, Morocco
sidiammi@ua.pt

Abstract

In this work we introduce a new extension of the Virasoro algebra namely fractional Virasoro algebra. Based on the Riemann–Liouville fractional derivatives we give the representation of the fractional Virasoro operators in terms of the fractional derivatives. The correspondence between q-deformation and fractional calculus is also given.

AMS Subject Classifications: 17B68, 81R10, 26A33, 53D55.
Keywords: Fractional calculus, fractional Virasoro algebra, q-deformation.

1 Introduction

The Virasoro algebra is an infinite-dimensional Lie algebra with many applications in physics, e.g, in conformal field theory, string theory, statistical mechanics and condensed matter physics [1, 11]. The centerless Virasoro L_0 algebra is the complexification of the Lie algebra VectS^1 of (real) vector fields on the circle S^1 and it coincides with the algebra of differential operators defined in $\mathbb{C} - \{0\}$ as

$$L_m = -z^{m+1}\frac{d}{dz}. \tag{1.1}$$

Furthermore the corresponding commutation relations verify the relations

$$[L_m, L_n] = (m-n)L_{m+n}. \tag{1.2}$$

Received November 8, 2010; Accepted March 15, 2011
Communicated by Delfim F. M. Torres

Recall that L_0 admits a unique 1-dimensional central extension $L_\kappa = L_0 \oplus \kappa$ with the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \kappa \frac{m^3 - m}{12}\delta_{m+n,0}, \qquad [L_m, \kappa] = 0,$$

where the parameter κ, called the value of the central charge which characterizes a theory in context of conformal field theory. The generators l_{-1}, l_0, l_1 form a subalgebra isomorphic to algebra $sl(2, \mathbb{R})$.

Here we will introduce a new extension of the Virasoro algebra, which we call fractional Virasoro algebra, and we will give its representation in terms of fractional derivatives. Then the fractional Virasoro algebra is generated by the generators $L_m^{(\alpha)}$, defined for $\alpha, \beta \in [0, 1]$ and $m, n \in \mathbb{Z}$ and satisfy the commutation relations

$$[L_m^{(\alpha)}, L_n^{(\beta)}] = \Lambda(m+1, \alpha)L_{m+n}^{(\alpha)} - \Lambda(n+1, \beta)L_{m+n}^{(\beta)}, \tag{1.3}$$

where $\Lambda(m, \alpha)$ is defined using the gamma function $\Gamma(x)$ in the following ways: For a nonnegative integer ($m \geq 0$), $\Lambda(m, \alpha)$ is defined by

$$\Lambda(m, \alpha) = \frac{\Gamma(1 + m\alpha)}{\Gamma(1 + (m-1)\alpha)},$$

and for negative integer m, $\Lambda(m, \alpha)$ is given by

$$\Lambda(m, \alpha) = \frac{\Gamma(1 - m\alpha)}{\Gamma(1 + (1-m)\alpha)}.$$

When $\alpha = \beta$, the algebra Eq. (1.3) becomes

$$[L_m^{(\alpha)}, L_n^{(\alpha)}] = \left(\Lambda(m+1, \alpha) - \Lambda(n+1, \alpha)\right) L_{m+n}^{(\alpha)}. \tag{1.4}$$

For $\alpha = 1$, and a positive integer m, we get $\Lambda(m, \alpha) = m$. Then the corresponding generators satisfy the classical Virasoro algebra Eq. (1.2)

$$[L_m, L_n] = (m-n)L_{m+n}.$$

Similarly, for a nonpositive integer m, we have $\Lambda(m, \alpha) = \dfrac{1}{1-m}$. In this case the generators $L_m, (m < 0)$ satisfy the commutation relations

$$[L_m, L_n] = \left(\frac{1}{1-m} - \frac{1}{1-n}\right) L_{m+n} = \frac{m-n}{(1-m)(1-n)} L_{m+n}. \tag{1.5}$$

For $n = 0$, we have

$$[L_m, L_0] = \frac{m}{1-m} L_m.$$

Another important case is when $m = 1$ and $n = -1$, we have

$$[L_1, L_{-1}] = \frac{1}{2} L_0.$$

The paper is organized as follows. In Section 2 we give some definitions of fractional calculus. Fractional derivatives used in this paper are the fractional order derivative introduced by Riemann–Liouville. In Section 3, we give a realization of the fractional Virasoro algebra in terms of the Riemann–Liouville fractional derivatives. While Section 4 is devoted to the connection between the q-deformation and fractional calculus.

2 Review on the Fractional Order Derivative

In this section we briefly recall some notions of the fractional order derivatives. We refer interested reader to [3, 7, 10]. The theory of fractional calculus has a long mathematical history. It goes back to more than 300 years. It was initiated by Leibniz and l'Hospital as a result of a correspondence which lasted several months in 1695. Nowadays it starts to attract much attention in phenomenological theories for complex systems. It has numerous applications in many areas, essentially in measurement of different phenomena. Recently it was found that many physical, chemical, biological and medical processes are governed by fractional differential equations (FDEs) [5, 8, 9]. Note also that the fractional quantum mechanics and the corresponding stationary fractional Schrödinger's equation is studied by many authors [5, 8, 13].

The definition of fractional order derivative is not unique. Several definitions are introduced by Caputo, Weyl, Riesz and Grünwald [2, 12, 14]. The fractional derivatives used in this paper are of Riemann–Liouville type [6]. Then the fractional derivatives D_x^α is defined by its action on the function $f(x)$ of any real number $\alpha \in [0, 1]$, and for $x > 0$ by

$$D_x^\alpha(x) f(x) = \frac{1}{\Gamma(1-\alpha)} \frac{\partial}{\partial x} \int_0^x (x - \xi)^{-\alpha} f(\xi) d\xi. \tag{2.1}$$

Since the derivative Eq. (2.1) is defined only for $x > 0$, as in [3, 4], to calculate the fractional derivatives for all $x \in \mathbb{R}$, we introduce a new variable $\bar{\chi}$ of x and the corresponding derivative $\bar{D}_\chi^\alpha$ as :

$$\bar{\chi} = \operatorname{sgn}(x)|x|^\alpha \quad \textit{and} \quad \bar{D}_\chi^\alpha = sing(x) D_x^\alpha(|x|).$$

Then, we are now able to calculate the α-fractional order derivative for any real number. For example if $f(\bar{\chi}) = \bar{\chi}^p, p \neq 0$, then the α-fractional derivative of $\bar{\chi}^p$ is given by

$$\bar{D}_\chi^\alpha \bar{\chi}^p = \frac{\Gamma(1 + p\alpha)}{\Gamma(1 + (p-1)\alpha)} \bar{\chi}^{p-1} = \Lambda(p, \alpha) \bar{\chi}^{p-1}, \qquad \alpha > \frac{-1}{p}. \tag{2.2}$$

In general for any real value function $f(\bar{\chi})$, we define its Riemann–Taylor series as

$$f(\bar{\chi}) = |\bar{\chi}|^{\frac{\alpha-1}{\alpha}} \sum_{n=0}^{\infty} a_n \bar{\chi}^n.$$

For $\alpha = 1$, we obtain the classical Riemann–Taylor series of $f(x)$. The fractional scalar product of $f(\bar{\chi})$ and $g(\bar{\chi})$ is defined by the form

$$< f(\bar{\chi}), g(\bar{\chi}) >= \int_{-|\chi|}^{|\chi|} d\bar{u} f^*(\bar{\chi}) g(\bar{\chi}),$$

where the measure $d\bar{u}$ is defined on function of the Riemann–Liouville fractional integral of order α as

$$\int_{-|\chi|}^{|\chi|} f(x) d\bar{u} = \frac{\operatorname{sgn}(x)}{\Gamma(\alpha)} \int_0^x d\xi (x - \xi)^{\alpha - 1} f(\xi),$$

for all $x \in \mathbb{R}$.

3 Realization of Fractional Virasoro Algebra

In this section, we introduce the realization of the fractional Virasoro algebra in terms of the variables $\bar{\chi}$ and the corresponding α-fractional derivatives $\bar{D}_{\chi}^{\alpha}$. Recall that the classical Eq. (1.2) Virasoro algebra coincides with the algebra of differential operators defined in $\mathbb{C} - \{0\}$ by Eq. (1.1). Now, if we introduce the operators

$$L_m^{(\alpha)} = -\bar{\chi}^{m+1} \bar{D}_{\chi}^{\alpha}, \tag{3.1}$$

which is a naturel extension of the differential realization Eq. (1.1) in the fractional case. From Eq. (2.2) and (3.1), the operators $L_m^{(\alpha)}$ act on $\bar{\chi}^p$ as follows :

$$L_m^{(\alpha)} \bar{\chi}^p = -\bar{\chi}^{m+p} \frac{\Gamma(1 + p\alpha)}{\Gamma(1 + (p-1)\alpha)}, \quad m \geq 0, \tag{3.2}$$

and the α-fractional derivative for a nonpositive integer m, is defined by

$$L_m^{(\alpha)} \bar{\chi}^p = -\bar{\chi}^{m+p} \frac{\Gamma(1 - p\alpha)}{\Gamma(1 + (1-p)\alpha)} \quad m < 0. \tag{3.3}$$

Then from (3.2) and (3.3) and the commutativity property of the Riemann–Liouville fractional integral on the space $C_\mu, \mu \in \mathbb{R}$ (see [7]); the space of a real function $f(x), x > 0$ such that $f(x) = x^p f_1(x), (p > \mu)$ where $f_1(x) \in C[0, \infty)$, namely for $f(x) \in C_\mu, \mu \geq -1$, $D^\alpha D^\beta f(x) = D^\beta D^\alpha f(x)$, it is not difficult to see that the generators $L_m^{(\alpha)}$ satisfy the fractional Virasoro algebra introduced in Eq. (1.3)

$$[L_m^{(\alpha)}, L_n^{(\beta)}] = \Lambda(m+1, \alpha) L_{m+n}^{(\alpha)} - \Lambda(n+1, \beta) L_{m+n}^{(\beta)}.$$

For $\alpha = \beta$, the above relation becomes

$$[L_m^{(\alpha)}, L_n^{(\alpha)}] = (\Lambda(m+1, \alpha) - \Lambda(n+1, \alpha)) L_{m+n}^{(\alpha)}.$$

For $\alpha = 1$, and m, n positive integers we have $\Lambda(m+1, \alpha) = m+1$ and $\Lambda(n+1, \alpha) = n+1$. Then we obtain the classical Virasoro algebra Eq. (1.2). For $\alpha = 1$, and m, n nonpositive integer numbers, we have $\Lambda(m, \alpha) = \dfrac{1}{1-m}$ and $\Lambda(n, \alpha) = \dfrac{1}{1-n}$. Hence we obtain the commutation relations given in Eq. (1.5).

4 q-Deformation and Fractional Calculus

In this section, we examine the connection between the α-fractional derivative and the q-deformation theory. We will see that the α-fractional derivative is interpreted as α-number like it is introduced on the language of the q-deformed Lie algebras [4]. First recall that the q-deformed Lie algebras, called also quantum groups are the extended (or deformation) of the usual Lie algebras. Principal elements of the q-deformation are the q-numbers and the q-derivatives which are defined for any number x and any function $f(x)$ in the form

$$[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}} \qquad and \qquad D_x^q f(x) = \frac{f(qx) - f(q^{-1}x)}{(q - q^{-1})x}. \tag{4.1}$$

The limits $\lim_{q\to 1}[x]_q = 1$ and $\lim_{q\to 1} D_x^q f(x) = df(x)$. For a given $f(x) = x^n$, we get

$$D_x^q x^n = [n]_q x^{n-1}. \tag{4.2}$$

Here we will extend the construction of Eq. (1.1) to the q-deformed case. We then introduce the generators $L_m^q = -x^{m+1} D_x^q$. It follows from Eq. (4.1) that

$$[L_m^q, L_n^q] = ([m+1]_q - [n+1]_q) L_{m+n}^q.$$

By letting q tends to 1, we obtain the algebra Eq. (1.2).

A simple example of the q-deformed algebra is given by the q-deformed oscillator algebra. This latter is generated by the creation $b^\dagger$, the annihilation b and the number's operators N satisfying the commutation relations

$$[N, b^\dagger] = b^\dagger, \qquad [N, b] = -b \qquad and \qquad bb^\dagger - q^{\pm} b^\dagger b = q^{\pm N}. \tag{4.3}$$

From Eq. (4.3), we can also write $b^\dagger b$ and $bb^\dagger$ in terms of the q-number N as

$$b^\dagger b = [N]_q, \qquad and \qquad bb^\dagger = [N+1]_q.$$

The Fock space is defined by application of the creation $b^\dagger$ at the vacuum state $|0>$ which satisfies the condition $b|0>= 0$. Then, we have

$$b^\dagger|n>= \sqrt{[n+1]_q}|n+1>,$$

$$b|n>= \sqrt{[n]_q}|n-1>b \quad and \quad N|n>= n|n> .$$

Hence, the corresponding q-deformed Hamiltonian and its eigenvalues on the basis $|n>$ are given by

$$H_q = \frac{\hbar\omega}{2}(bb^\dagger + b^\dagger b) \qquad and \qquad E_q(n) = \frac{\hbar\omega}{2}([n]_q + [n+1]_q).$$

In order to establish the connection between the q-deformed algebra and the α-fractional derivatives, setting the state $|n>$ as $\bar{\chi}^n, n > 0$. Then from Eq. (2.2), we have

$$\bar{D}^{\alpha}_{\chi}\bar{\chi}^n = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\bar{\chi}^{n-1}.$$

It yields that

$$\bar{D}^{\alpha}_{\chi}|n>= [n]_{\alpha}|n-1>,$$

where the α-number is defined in terms of the function gamma as

$$[n]_{\alpha} = \frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}.$$

So the α-fractional derivative can be interpreted as α-deformation in the q-deformation language ($D^{\alpha}_{x} \to [n]_{\alpha}$) Eq. (4.2).

$$D^{\alpha}_{x}x^n = [n]_{\alpha}x^{n-1},$$

and obviously we have the limit

$$\lim_{\alpha\to 1}[n]_{\alpha} = n.$$

Due to this connection, the fractional Virasoro algebra takes the form

$$[L^{(\alpha)}_{m}, L^{(\beta)}_{n}] = ([m]_{\alpha}L^{(\alpha)}_{m} - [n]_{\beta}L^{(\beta)}_{m}).$$

For $\alpha = \beta$, we obtain

$$[L^{(\alpha)}_{m}, L^{(\alpha)}_{n}] = ([m]_{\alpha} - [n]_{\alpha})L^{(\alpha)}_{m+n},$$

and for $\alpha = 1 (q = 1)$ we have

$$[L^{(1)}_{m}, L^{(1)}_{n}] = ([m]_{1} - [n]_{1})L^{(1)}_{m+n} = [L_m, L_n] = (m-n)L_{m+n}.$$

5 Conclusion

In this work an extension of Virasoro algebra namely fractional Virasoro algebra is introduced. Its realization in terms of the fractional derivative is given. The connection between the q-deformation algebra and the fractional calculus is also investigated. Then, we have seen that via this correspondence the α-fractional derivative may be interpreted as α-deformation in the language of the deformation defined by the gamma function. Finally, we claim that one can extend the oscillator's representation of the elements $L^{(\alpha)}_{m}$ for the fractional Virasoro algebra by introducing the fractional oscillators algebra.

Acknowledgments

Work supported by the project "New Explorations in Control Theory Through Advanced Research (NECTAR)" cofinanced by the Fundacão para Ciência e a technologia (FCT), Portugal, and the Centre National de la Recherche Scientifique et Technique (CNRST), Morocco.

References

[1] A. A. Belavin, A.M. Polyakov, A. B. Zamoloochikov, Infinite conformal symmetry in two-dimensional quantum field theory, Nucl. Phys. **B 241** (1984), 333–380.

[2] M. Caputo, Linear models of dissipation whose Q is almost frequency independent, Part II, J. Roy. Astral. Soc. **13** (1967), 529–539.

[3] R. Herrmann, The Fractional Symmetric rigid Rotor, J. Phys. G. Nucl. Part. Phys. **34** (2007), 607–625.

[4] R. Herrmann, Common aspects of the q-deformed Lie algebras and fractional calculus, Physica **A** 389 (2010), 4613–4622.

[5] N. Laskin, Fractional Schrödinger equation, Phys. Rev. E **66** (2002), 1–7.

[6] J. Liouvillle, Sur le calcul des différentielles à indices quelconques, J. Ecole Polytech. **13** (1832), 71–162.

[7] K. Miller, B. Ross, An introduction to fractional calculus and fractional differentials equations, Wiley, New York, 1993.

[8] M. Naber, Time Fractional Schrödinger equation, J. Math. Phys. 45 (2004), 3339–3354.

[9] T. F. Nonnenmacher, G. A. Losa, E. R. Weibl (Eds), Fractals in biology and medicine, Vol. II, Birkhäuser, Basel, 1993.

[10] I. Podlubny, Fractional differential equations, Academic Press, New York, 1999.

[11] A.M.Polyakov, Conformal field of critical fluctuations, JETP Lett. **12** (1970), 381–383.

[12] M. Riesz, L'integral de Riemann-Liouville et le problème de Cauchy, Acta Mathematica, **81** (1949), 1–223.

[13] I. Sami Muslih, Om P. Agrawal and D. Baleanu, A Fractional Schrödinger equation and its solution, Int. J. Theor. Phys. Vol. 49, N. 8 (2010), 1746–1752.

[14] H. Weyl, Vierteljahresschrift der naturforschenden Gesellschaft Zürich, **62** (1917), 296–302.