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On the Exponential Operator Functions on Time Scales

Alaa E. Hamza
Cairo University
Department of Mathematics
Giza, Egypt
hamzaaeg2003@yahoo.com

Moneer A. Al-Qubaty
Hodeidah University
Department of Mathematics
Hodeidah, Yemen
monirabdo72@yahoo.com

Abstract

In this paper, we investigate some properties of the operator exponential solutions of the initial value problem

$$\begin{aligned} X^{\Delta}(t) &= A(t)X(t), \quad t \in \mathbb{T} \\ X(s) &= I, \end{aligned}$$

where I is the identity operator on a Banach space $\mathbb{X}$, s is an initial point in a time scale $\mathbb{T}$ and $A(t)$ is a bounded linear operator on $\mathbb{X}$, $t \in \mathbb{T}$. Also, we give an example of differentiable 2×2 matrices $A(t)$ and $B(t)$ to show that the commutativity of $e_A(t,s)$ with $(\ominus A)(t)$ (resp. $e_A(t,s)$ with $B(t)$) is an essential condition for the inverse $e_A^{-1}(t,s)$ to equal $e_A(s,t)$ (resp. $e_A(t,s)e_B(t,s)$ to equal $e_{A\oplus B}(t,s)$).

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1 Introduction

A calculus on time scales was initiated by Stefan Hilger (1988) in order to create a theory that can unify discrete and continuous analysis [8], see also [3, 4]. A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of real numbers $\mathbb{R}$. The delta derivative f^{Δ} for a function f on $\mathbb{T}$ was defined in a way such that $f^{\Delta}(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}$ is the usual derivative if $\mathbb{T} = \mathbb{R}$, and $f^{\Delta}(t) = \Delta f(t) = f(t+1) - f(t)$ is the usual forward difference operator if $\mathbb{T} = \mathbb{Z}$ [7].

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Suppose that $\mathbb{T}$ has the topology inherited from the standard topology on $\mathbb{R}$. We define the time scale interval

$$[a, b] = [a, b] \cap \mathbb{T}.$$

Open intervals and open neighborhoods are defined similarly. A set we need to consider is $\mathbb{T}^k$ which is defined as $\mathbb{T}^k = \mathbb{T}\backslash\{M\}$ if $\mathbb{T}$ has a left-scattered maximum M, and $\mathbb{T}^k = \mathbb{T}$ otherwise. Similarly, $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$ if $\mathbb{T}$ has a right-scattered minimum m, and $\mathbb{T}_k = \mathbb{T}$ otherwise. For the terminology used here, we refer the reader to [3, 4].

Throughout this paper, $\mathbb{X}$ is a Banach space and $L(\mathbb{X})$ denotes the space of all bounded linear operators in $\mathbb{X}$. We begin by introducing some definitions which we will need later on.

Definition 1.1. A function $f : \mathbb{T} \to \mathbb{X}$ is said to be regulated if

(i) $\lim\limits_{s\to t^+} f(s)$ exists for all right-dense points t in $\mathbb{T}$,

(ii) $\lim\limits_{s\to t^-} f(s)$ exists for all left-dense points t in $\mathbb{T}$.

Definition 1.2. A function $f : \mathbb{T} \to \mathbb{X}$ is said to be rd-continuous if

(i) f is continuous at every right-dense point t in $\mathbb{T}$ i.e $\lim\limits_{s\to t^+} f(s) = f(t)$,

(ii) $\lim\limits_{s\to t^-} f(s)$ exists at every left-dense point $t \in \mathbb{T}$.

A function $f(t, x) : \mathbb{T} \times \mathbb{X} \to \mathbb{X}$ is called rd-continuous if $f(t, x(t)) : \mathbb{T} \to \mathbb{X}$ is so for every continuous function $x : \mathbb{T} \to \mathbb{X}$.

Definition 1.3. A function $f : \mathbb{T} \to \mathbb{X}$ is called delta differentiable (or simply differentiable) at $t \in \mathbb{T}^k$ provided there exists an α such that for every $\epsilon > 0$ there is a neighborhood U of t with

$$\| f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s) \| \leq \epsilon|\sigma(t) - s| \text{ for all } s \in U.$$

In this case we denote the α by $f^\Delta(t)$; and if f is differentiable at every $t \in \mathbb{T}^k$, then f is said to be differentiable on $\mathbb{T}$ and f^Δ is a new function defined on $\mathbb{T}$. If f is differentiable at $t \in \mathbb{T}^k$, then it is easy to see that

$$f^\Delta(t) = \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0; \\ \lim\limits_{s\to t} \dfrac{f(t) - f(s)}{t - s}, & \text{if } \mu(t) = 0. \end{cases} \tag{1.1}$$

For any time scale $\mathbb{T}$, we define its norm as (see [1, 3, 5])

$$\| \mathbb{T} \| := \sup\{\mu(t) : t \in \mathbb{T}\}.$$

Clearly, $\| \mathbb{T} \| \in [0, +\infty]$, e.g., $\| \mathbb{R} \| = 0$, $\| \mathbb{Z} \| = 1$, if $a, b > 0$, then $\| \cup_{k=1}^{\infty} [k(a+b), k(a+b)+a] \| = b$ and $\| \{n^2 : n \in \mathbb{N}\} \| = +\infty$.

We consider the initial value problem (IVP)

$$X^{\Delta}(t) = A(t)X(t), \qquad X(s) = I, \;\; t \in \mathbb{T}$$

which has a unique solution under certain appropriate conditions. Here $I : \mathbb{X} \to \mathbb{X}$ is the identity operator. This solution is denoted by $e_A(t, s)$. When $A(t) \in M_n(\mathbb{R})$, the family of $n \times n$ real matrices, many properties of $e_A(t, s)$ were proved. See [3, Theorem 5.21]. In this paper, we establish these properties when $A(t) \in L(\mathbb{X}), \; t \in \mathbb{T}$. See Section 3. In Section 4, we give an example of differentiable $n \times n$ matrices $A(t)$ and $B(t)$ to show that the commutativity of $e_A(t, s)$ with $(\ominus A)(t)$ (resp. $e_A(t, s)$ with $B(t)$) is an essential condition for the inverse $e_A^{-1}(t, s)$ to equal $e_A(s, t)$ (resp. $e_A(t, s)e_B(t, s)$ to equal $e_{A \oplus B}(t, s)$).

2 Preliminary Results

Let $\mathbb{Y}$ be a Banach algebra with unit e. See [6]. We need the following theorems to prove the main properties of abstract exponential functions.

Theorem 2.1. *If $A : [a, b] \to \mathbb{Y}$ is regulated, then it is bounded.*

Proof. Assume towards a contradiction, that there is a sequence $\{t_n\} \subseteq [a, b]$ such that $\| A(t_n) \| > n \;\; \forall\, n \in \mathbb{N}$. Then $\{t_n\}$ has a subsequence $\{t_{n_k}\}_{k \in \mathbb{N}}$ which decreases to s or increases to s for some $s \in [a, b]$. Since $s \in \mathbb{T}$ is either a right-dense point or a left-dense point, then $\lim\limits_{k \to \infty} A(t_{n_k})$ exists which contradicts with $\lim\limits_{n \to \infty} \| A(t_n) \| = \infty$. □

Lemma 2.2. *Assume $g, h : \mathbb{T} \to \mathbb{Y}$ such that $\lim\limits_{s \to t} g(s) = T_1$ and $\lim\limits_{s \to t} h(s) = T_2$. Then*

$$\lim_{s \to t} g(s)h(s) = T_1 T_2, \quad T_1, T_2 \in \mathbb{Y}. \tag{2.1}$$

Proof. We have

$$\begin{aligned} \| g(s)h(s) - T_1 T_2 \| &\leq \| g(s)h(s) - T_1 h(s) \| + \| T_1 h(s) - T_1 T_2 \| \\ &\leq \| g(s) - T_1 \| \| h(s) \| + \| T_1 \| \| h(s) - T_2 \| . \end{aligned}$$

Taking $s \to t$, since h is bounded in a neighborhood of t, we get Relation (2.1). □

Theorem 2.3. *Let $A, B : \mathbb{T} \to \mathbb{Y}$ be Δ-differentiable at $t \in \mathbb{T}^k$. Then $AB : \mathbb{T} \to \mathbb{Y}$ is Δ-differentiable at $t \in \mathbb{T}^k$ and*

$$(AB)^{\Delta}(t) = A(\sigma(t))B^{\Delta}(t) + A^{\Delta}(t)B(t) = A^{\Delta}(t)B(\sigma(t)) + A(t)B^{\Delta}(t). \tag{2.2}$$

Here, $(AB)(t) = A(t)B(t)$.

Proof. We have

$$\begin{aligned}
(AB)^{\Delta}(t) &= \lim_{s\to t,\, s\in\mathbb{T}} \frac{(AB)(\sigma(t)) - (AB)(s)}{\sigma(t) - s} \\
&= \lim_{s\to t,\, s\in\mathbb{T}} \frac{A(\sigma(t))B(\sigma(t)) - A(s)B(\sigma(t)) + A(s)B(\sigma(t)) - A(s)B(s)}{\sigma(t) - s} \\
&= \lim_{s\to t,\, s\in\mathbb{T}} \left[\frac{A(\sigma(t))B(\sigma(t)) - A(s)B(\sigma(t))}{\sigma(t) - s}\right. \\
&\qquad\qquad \left. + \frac{A(s)B(\sigma(t)) - A(s)B(s)}{\sigma(t) - s}\right] \\
&= \lim_{s\to t,\, s\in\mathbb{T}} \frac{A(\sigma(t)) - A(s)}{\sigma(t) - s} B(\sigma(t)) + \lim_{s\to t,\, s\in\mathbb{T}} A(s) \frac{B(\sigma(t)) - B(s)}{\sigma(t) - s} \\
&= A^{\Delta}(t)B(\sigma(t)) + A(t)B^{\Delta}(t).
\end{aligned}$$

This completes the proof. □

Definition 2.4. A function $F : \mathbb{T} \to \mathbb{Y}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{Y}$ if $F^{\Delta}(t) = f(t),\ t \in \mathbb{T}^k$. In this case we define an indefinite integral of f by

$$\int f(\tau)\Delta\tau = F(t) + C$$

where C is an arbitrary constant. We define the Cauchy integral by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \quad \text{where} \quad s, t \in \mathbb{T}.$$

Theorem 2.5 (See [3, Theorems 1.74, 1.75])**.** (i) *Every rd-continuous $f : \mathbb{T} \to \mathbb{Y}$ has an antiderivative and $F(t) = \int_s^t f(\tau)\Delta\tau$ is an antiderivative of f, i.e., $F^{\Delta}(t) = f(t),\ t \in \mathbb{T}^k$.*

(ii) *If $f \in C_{rd}$ and $t \in \mathbb{T}^k$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

We need the following theorem in our study.

Theorem 2.6. *Let $f : \mathbb{T} \to \mathbb{Y}$ be rd-continuous. Then*

$$\int_s^t f(\tau)\ y\Delta\tau = \int_s^t f(\tau)\Delta\tau\ \ y, \quad y \in \mathbb{Y}. \tag{2.3}$$

Proof. Let $F(t) = \int_s^t f(\tau)\ y\Delta\tau$ and $G(t) = \int_s^t f(\tau)\Delta\tau\ y$. Then $F^\Delta(t) = f(t)\ y$ and $G^\Delta(t) = f(t)y$. Consequently, $F(t) = G(t) + C$. Hence, $F(s) = G(s) + C$. Thus $C = 0$. □

Definition 2.7. Let $A : \mathbb{T} \to \mathbb{Y}$ be rd-continuous at every $t \in \mathbb{T}^k$ such that $A(t)$ has an inverse for all $t \in \mathbb{T}^k$, we define $A_{-1} : \mathbb{T} \to \mathbb{Y}$ by

$$A_{-1}(t) = [A(t)]^{-1}.$$

We note that A_{-1} is rd-continuous, since the inverse function on the invertible elements $I(\mathbb{Y}) \subseteq \mathbb{Y}$

$$\begin{array}{ccc} I(\mathbb{Y}) & \to & I(\mathbb{Y}) \\ x & \longmapsto & x^{-1} \end{array}$$

is continuous.

A mapping $A : \mathbb{T} \to \mathbb{Y}$ is called regressive if $I + \mu(t)A(t)$ is invertible for every $t \in \mathbb{T}$. The class of all regressive mappings is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{Y}),$$

and the space of rd-continuous, regressive mappings from $\mathbb{T}$ to $\mathbb{Y}$ is denoted by

$$\begin{aligned} C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{Y}) \quad = \quad & \{A : \mathbb{T} \to \mathbb{Y} \mid A \in C_{rd}(\mathbb{T}, \mathbb{Y}) \text{ and } I + \mu(t)A(t) \\ & \text{is invertible for all } t \in \mathbb{T}\}. \end{aligned}$$

Definition 2.8. Let A, $B \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{Y})$. Then we define:

(i) $A \oplus B$ by

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t) \ \text{ for all } t \in \mathbb{T}^k,$$

(ii) $\ominus A$ by

$$(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) \ \text{ for all } t \in \mathbb{T}^k,$$

(iii) $A \ominus B$ by

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t) \text{ for all } t \in \mathbb{T}^k.$$

We can see, if $A \in \mathcal{R}$, then $\ominus A$ satisfies $(A \oplus (\ominus A))(t) = 0$ for all $t \in \mathbb{T}$.

Remark 2.9. Let $A \in C_{rd}\mathcal{R}(\mathbb{T}, \mathbb{Y})$. Then

(i) $(\ominus A)(t) = -[I + \mu(t)A(t)]^{-1}A(t) = -A(t)[I + \mu(t)A(t)]^{-1}$ for all $t \in \mathbb{T}^k$,

(ii) $\ominus(\ominus A) = A$.

Now let $A \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$ and $A^*(t) : \mathbb{X}^* \to \mathbb{X}^*$, where $\mathbb{X}^*$ is the dual space of $\mathbb{X}$ [9], be the adjoint operator of $A(t) : \mathbb{X} \to \mathbb{X}, \quad t \in \mathbb{T}$ which is defined by

$$(A^*(t)f)(x) = f(A(t)x) \text{ for all } f \in \mathbb{X}^* \text{ and } x \in \mathbb{X}.$$

The proof of the following lemma is straightforward and will be omitted.

Lemma 2.10. *The following statements are true:*

(i) $(A^*)^\Delta = (A^\Delta)^*$ *holds for any differentiable operator* A;

(ii) A^* *is regressive;*

(iii) $\ominus A^* = (\ominus A)^*$;

(iv) $A^* \oplus B^* = (A \oplus B)^*$, *where* $B \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$.

3 The Operator Exponential function $e_A(t, s)$ in Banach Spaces

In the sequel, we assume that $\sup \mathbb{T} = \infty$ and $A \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$.

Definition 3.1. Let $s \in \mathbb{T}$. A function $X : \mathbb{T} \to L(\mathbb{X})$ that satisfies the operator delta dynamic equation

$$X^\Delta(t) = A(t)X(t), \quad X(s) = I, \tag{3.1}$$

is called the operator exponential function (corresponding to $A(t)$) initialized at s. Its value at $t \in \mathbb{T}$, is denoted by $e_A(t, s)$. Here $X^\Delta(t) = \lim\limits_{s \to t,\, s \in \mathbb{T}} \dfrac{X(\sigma(t)) - X(s)}{\sigma(t) - s}$, I is the identity operator on $\mathbb{X}$.

The regressive initial value problem (IVP)

$$x^\Delta(t) = A(t)x(t), \quad x(s) = x_s \in \mathbb{X} \tag{3.2}$$

has a unique solution $x : \mathbb{T} \to \mathbb{X}$, which is given by $x(t) = e_A(t, s)x_s$ provided

(i) The function $A(t)x(t)$ is rd-continuous for every continuous function $x : \mathbb{T} \to \mathbb{X}$;

(ii) $I + \mu(t)A(t) : \mathbb{X} \to \mathbb{X}$ is invertible for every $t \in \mathbb{T}$;

(iii) $\sup\limits_{t \in \mathbb{T}} \| A(t) \| < \infty$.

See [3, Chapter 8].

Example 3.2. Let A be a constant operator.

(i) If A is bounded and $\mathbb{T} = \mathbb{R}$, then $e_A(t,s) = e^{(t-s)A}, t, s \in \mathbb{R}$. Here, $e^{tA} = \sum_{n=0}^{\infty}(tA)^n/n!$.

(ii) If $\mathbb{T} = h\mathbb{Z}$, then $e_A(t,s) = (I + hA)^{\frac{t-s}{h}}$, where $t, s \in h\mathbb{Z}$, with $t \geq s$.

(iii) If $\mathbb{T} = 2^{\mathbb{N}_0}$, then $e_A(t,s) = \prod_{k=1}^{\log_2 \frac{t}{s}} (I + \frac{t}{2^k}A), t, s \in \mathbb{T}$, with $t \geq s$.

Now we establish some fundamental properties of $e_A(t,s)$ which are satisfied when $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, where $M_n(\mathbb{R})$ is the family of all $n \times n$ real matrices [3]. The proof of the following theorem is similar to the case $A(t)$ is $n \times n$ matrix [3, Theorem 5.21], so it will be omitted.

Theorem 3.3. *If A, $B \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$ and $t, s \in \mathbb{T}$, then the following statements are true*

(i) $e_0(t,s) \equiv I$ *and* $e_A(t,t) \equiv I$;

(ii) $e_A(\sigma(t), s) = [I + \mu(t)A(t)]e_A(t,s)$;

(iii) $e_A(t,s)e_B(t,s) = e_{A\oplus B}(t,s)$ *if* $e_A(t,s)$ *and* $B(t)$ *commute;*

(iv) *Assume that we have the following commutativity condition:*

(C) $e_A(t,s)$ *and* $(\ominus A)(t)$ *commute, and* $e_{\ominus A}(t,s)$ *and* $A(t)$ *commute.*

Then $e_A^{-1}(t,s) = e_{\ominus A}(t,s)$.

The following theorem was proved when $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, see [3, Theorem 5.21]. Now we prove it when $A \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$.

Theorem 3.4. *If A, $B \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$ and $t, r, s \in \mathbb{T}$, then*

(i) *Condition* (C) *implies*

$$\begin{aligned}[e_A^{-1}(t,s)]^{\Delta} &= -e_A^{-1}(\sigma(t),s)e_A^{\Delta}(t,s)e_A^{-1}(t,s) = -[e_{\ominus A}(t,s)]^{\sigma}A(t) \\ &= (\ominus A)(t)e_{\ominus A}(t,s);\end{aligned}$$

(ii) $e_A^{-1}(t,s) = e^*_{\ominus A^*}(t,s)$, *provided that condition* (C) *holds;*

(iii) *If* $a, b, s \in \mathbb{T}$, *then we have*

(a) $e_A^{\Delta_s}(t,s) = -e_A(t,\sigma(s))A(s)$,

(b) $e_A(t,\sigma(s)) = e_A(t,s)[I + \mu(s)(\ominus A)(s)]$,

(c) $\displaystyle\int_a^b e_A(t,\sigma(s))A(s)\Delta s = e_A(t,a) - e_A(t,b)$;

(iv) $e_A(t,s) = e_A^{-1}(s,t)$, *provided* $e_A(t,s)$ *commutes with* $(\ominus A)(t)$;

(v) $e_A(t,s)e_A(s,r) = e_A(t,r)$, *provided* $e_A(t,s)$ *commutes with* $(\ominus A)(t)$.

Proof.(i) In view of $e_A(t,s)e_A^{-1}(t,s) = I$, we obtain $[e_A(t,s)e_A^{-1}(t,s)]^\Delta = 0$. By Theorem 2.3, we get

$$e_A^\Delta(t,s)e_A^{-1}(t,s) + e_A(\sigma(t),s)(e_A^{-1}(t,s))^\Delta = 0.$$

Then

$$e_A(\sigma(t),s)(e_A^{-1}(t,s))^\Delta = -e_A^\Delta(t,s)e_A^{-1}(t,s).$$

which implies that

$$\begin{aligned}
(e_A^{-1}(t,s))^\Delta &= -e_A^{-1}(\sigma(t),s)e_A^\Delta(t,s)e_A^{-1}(t,s)\\
&= -e_A^{-1}(\sigma(t),s)A(t)e_A(t,s)e_A^{-1}(t,s))\\
&= -e_A^{-1}(\sigma(t),s)A(t)\\
&= -e_A^{-1}(t,s)[I+\mu(t)A(t)]^{-1}A(t)\\
&= e_A^{-1}(t,s)(\ominus A)(t).
\end{aligned}$$

Thus

$$(e_A^{-1}(t,s))^\Delta = e_{\ominus A}(t,s)(\ominus A)(t). \tag{3.3}$$

Finally, since

$$e_{\ominus A}(\sigma(t),s) = e_{\ominus A}(t,s) + \mu(t)e_{\ominus A}^\Delta(t,s),$$

then

$$\begin{aligned}
[e_{\ominus A}(t,s)]^\sigma A(t) &= e_{\ominus A}(t,s)A(t) + \mu(t)e_{\ominus A}^\Delta(t,s)A(t)\\
&= e_{\ominus A}(t,s)A(t) + \mu(t)(\ominus A)(t)e_{\ominus A}(t,s)A(t)\\
&= [I+\mu(t)(\ominus A)(t)]e_{\ominus A}(t,s)A(t)\\
&= [I-\mu(t)A(t)(I+\mu(t)A(t))^{-1}]e_{\ominus A}(t,s)A(t)\\
&= [I+\mu(t)A(t)]^{-1}e_{\ominus A}(t,s)A(t)\\
&= [I+\mu(t)A(t)]^{-1}A(t)e_{\ominus A}(t,s)\\
&= -(\ominus A)(t)e_{\ominus A}(t,s)\\
&= -e_{\ominus A}^\Delta(t,s)\\
&= -(e_A^{-1}(t,s))^\Delta\\
&= -e_{\ominus A}(t,s)(\ominus A)(t).\text{(by Equation (3.3))}
\end{aligned}$$

(ii) Put $X(t) = (e_A^{-1}(t,s))^*$. Then from (i) and Lemma 2.10, we have

$$\begin{aligned}
X^{\Delta}(t) &= ((e_A^{-1}(t,s))^*)^{\Delta} \\
&= ((e_A^{-1}(t,s))^{\Delta})^* \\
&= -[e_A^{-1}(\sigma(t),s)e_A^{\Delta}(t,s)e_A^{-1}(t,s)]^* \\
&= -[e_A^{-1}(t,s)(I+\mu(t)A(t))^{-1}A(t)e_A(t,s)e_A^{-1}(t,s)]^* \\
&= [e_A^{-1}(t,s)(\ominus A)(t)]^* \\
&= (\ominus A^*)(t)(e_A^{-1}(t,s))^* \\
&= (\ominus A^*)(t)X(t),
\end{aligned}$$

and

$$X(s) = (e_A^{-1}(s,s))^* = (I^{-1})^* = I.$$

Hence, $X(t)$ solves the IVP

$$X^{\Delta}(t) = (\ominus A^*)(t)X(t), \quad X(s) = I$$

which has exactly the solution

$$X(t) = e_{\ominus A^*}(t,s).$$

Thus

$$e_A^{-1}(t,s) = e^*_{\ominus A^*}(t,s).$$

(iii) Let a, b, $s \in \mathbb{T}$.

(a) By differentiating the integral equation

$$e_A(t,s) = I + \int_s^t A(\tau)e_A(\tau,s)\Delta\tau \tag{3.4}$$

corresponding the IVP (3.1) with respect to s, we get two cases:
(1) First case $\mu(s) > 0$. We have, from Relations (1.1),

$$e_A^{\Delta_s}(t,s) = \frac{e_A(t,\sigma(s)) - e_A(t,s)}{\mu(s)}. \tag{3.5}$$

Equation (3.4) and Theorem 2.5 imply that

$$\begin{aligned}
e_A^{\Delta_s}(t,s) &= \frac{1}{\mu(s)}\Bigg[I+\int_{\sigma(s)}^{t} A(\tau)e_A(\tau,\sigma(s))\Delta\tau \\
&\qquad -I-\int_{s}^{t} A(\tau)e_A(\tau,s)\Delta\tau\Bigg] \\
&= \frac{1}{\mu(s)}\Bigg[\int_{s}^{t} A(\tau)e_A(\tau,\sigma(s))\Delta\tau+\int_{\sigma(s)}^{s} A(\tau)e_A(\tau,\sigma(s))\Delta\tau \\
&\qquad -\int_{s}^{t} A(\tau)e_A(\tau,s)\Delta\tau\Bigg] \\
&= \frac{1}{\mu(s)}\Bigg[\int_{s}^{t} A(\tau)(e_A(\tau,\sigma(s))-e_A(\tau,s))\Delta\tau \\
&\qquad -\int_{s}^{\sigma(s)} A(\tau)e_A(\tau,\sigma(s))\Delta\tau\Bigg] \\
&= \int_{s}^{t} A(\tau)e_A^{\Delta_s}(\tau,s)\Delta\tau-\frac{1}{\mu(s)}\int_{s}^{\sigma(s)} A(\tau)e_A(\tau,\sigma(s))\Delta\tau,
\end{aligned}$$

and consequently,

$$e_A^{\Delta_s}(t,s)=\int_{s}^{t} A(\tau)e_A^{\Delta_s}(\tau,s)\Delta\tau+\frac{1}{\mu(s)}\int_{\sigma(s)}^{s} A(\tau)e_A(\tau,\sigma(s))\Delta\tau. \tag{3.6}$$

From Equations (3.4) and (3.6), we get

$$e_A^{\Delta_s}(t,s)=\frac{1}{\mu(s)}[e_A(s,\sigma(s))-I]+\int_{s}^{t} A(\tau)e_A^{\Delta_s}(\tau,s)\Delta\tau. \tag{3.7}$$

Thus

$$\Psi(t,s)=e_A^{\Delta_s}(t,s)$$

is a solution of the integral equation

$$\Psi(t,s)=U(s)+\int_{s}^{t} A(\tau)\Psi(\tau,s)\Delta\tau \tag{3.8}$$

with

$$U(s)=\frac{1}{\mu(s)}[e_A(s,\sigma(s))-I]. \tag{3.9}$$

Multiplying Equation (3.4) by $\frac{1}{\mu(s)}[e_A(s,\sigma(s))-I]$ and applying Theorem 2.6, we get

$$\begin{aligned}
&\frac{1}{\mu(s)}e_A(t,s)[e_A(s,\sigma(s))-I] \\
&=\frac{1}{\mu(s)}[e_A(s,\sigma(s))-I]+\frac{1}{\mu(s)}\int_{s}^{t} A(\tau)e_A(\tau,s)[e_A(s,\sigma(s))-I]\Delta\tau.
\end{aligned}$$

This implies that

$$\Psi(t,s) = e_A(t,s)[e_A(s,\sigma(s)) - I] \tag{3.10}$$

is a solution of Equation (3.8) with $U(s) = \dfrac{1}{\mu(s)}[e_A(s,\sigma(s)) - I]$. By the uniqueness of solution of Equation (3.8), we deduce that,

$$e_A^{\Delta_s}(t,s) = \frac{1}{\mu(s)} e_A(t,s)[e_A(s,\sigma(s)) - I]. \tag{3.11}$$

By Theorem 2.5 part (ii), Equation (3.6) yields

$$e_A^{\Delta_s}(t,s) = -A(s)e_A(s,\sigma(s)) + \int_s^t A(\tau)e_A^{\Delta_s}(\tau,s)\Delta\tau. \tag{3.12}$$

Multiplying Equation (3.4) by $-A(s)e_A(s,\sigma(s))$ and applying Theorem 2.6, we obtain

$$\begin{aligned}&[-e_A(t,s)A(s)e_A(s,\sigma(s))]\\ &\qquad = -A(s)e_A(s,\sigma(s)) + \int_s^t A(\tau)[e_A(\tau,s)A(s)e_A(s,\sigma(s))]\Delta\tau.\end{aligned} \tag{3.13}$$

From the uniqueness of solutions of (3.8) with $U(s) = -A(s)e_A(s,\sigma(s))$, we conclude that

$$e_A^{\Delta_s}(t,s) = -e_A(t,s)A(s)e_A(s,\sigma(s)). \tag{3.14}$$

We have

$$\begin{aligned} e_A(t,\sigma(s)) &= e_A(t,s) + \mu(s)e_A^{\Delta_s}(t,s)\\ &= e_A(t,s) - \mu(s)e_A(t,s)A(s)e_A(s,\sigma(s)),\end{aligned}$$

so that

$$e_A(t,\sigma(s)) = e_A(t,s)[I - \mu(s)A(s)e_A(s,\sigma(s))]. \tag{3.15}$$

Put $t = s$, in Equation (3.15) to get

$$e_A(s,\sigma(s)) = [I - \mu(s)A(s)e_A(s,\sigma(s))], \tag{3.16}$$

consequently,

$$[I + \mu(s)A(s)]e_A(s,\sigma(s)) = I. \tag{3.17}$$

Therefore,

$$e_A(s,\sigma(s)) = [I + \mu(s)A(s)]^{-1}. \tag{3.18}$$

By Equations (3.15) and (3.16), we get

$$e_A(t,\sigma(s)) = e_A(t,s)e_A(s,\sigma(s)). \tag{3.19}$$

In view of Equation (3.14), $A(s)$ and $e_A(s,\sigma(s))$ commute, we have

$$e_A^{\Delta_s}(t,s) = -e_A(t,s)e_A(s,\sigma(s))A(s). \tag{3.20}$$

It follows, from Equations (3.19) and (3.20), that

$$e_A^{\Delta_s}(t,s) = -e_A(t,\sigma(s))A(s). \tag{3.21}$$

(2) Second case $\mu(s) = 0$. We have from Equation (3.4)

$$\begin{aligned}\frac{e_A(t,s) - e_A(t,s_0)}{s - s_0} &= \frac{-1}{s - s_0}\int_{s_0}^{s} A(\tau)e_A(\tau,s_0)\Delta\tau \\ &\quad - \int_t^s A(\tau)\left[\frac{e_A(\tau,s) - e_A(\tau,s_0)}{s - s_0}\right]\Delta\tau.\end{aligned}$$

By taking the limit to both sides as $s \to s_0$, we obtain

$$e_A^{\Delta_s}(t,s) = -A(s_0) - \int_t^{s_0} A(\tau)e_A^{\Delta_s}(\tau,s_0)\Delta\tau. \tag{3.22}$$

Then

$$e_A^{\Delta_s}(t,s) = -A(s) - \int_t^{s} A(\tau)e_A^{\Delta_s}(\tau,s)\Delta\tau. \tag{3.23}$$

Again by Equation (3.4), multiplying by $-A(s)$ and using Theorem 2.6, we get

$$-e_A(t,s)A(s) = -A(s) + \int_t^s A(\tau)e_A(\tau,s)A(s)\Delta\tau. \tag{3.24}$$

By the uniqueness of solution of the equation

$$\psi(t,s) = U(s) + \int_t^s A(\tau)\psi(\tau,s)\Delta\tau \quad \text{with} \quad U(s) = -A(s), \tag{3.25}$$

we conclude that

$$e_A^{\Delta_s}(t,s) = -e_A(t,s)A(s). \tag{3.26}$$

(b) From Equations (3.15) and (3.18), we have

$$\begin{aligned}e_A(t,\sigma(s)) &= e_A(t,s)[I - \mu(s)A(s)e_A(s,\sigma(s))] \\ &= e_A(t,s)[I - \mu(s)A(s)[I + \mu(s)A(s)]^{-1}] \\ &= e_A(t,s)[I + \mu(s)(\ominus A)(s)].\end{aligned}$$

(c) From part (a), we obtain

$$\begin{aligned}\int_a^b e_A(t,\sigma(s))A(s)\Delta s &= -\int_a^b [e_A(t,s)]^{\Delta}\Delta s \\ &= -e_A(t,s)\,\Big|_a^b \\ &= e_A(t,a) - e_A(t,b), \quad \text{for all} \quad a,b \in \mathbb{T}.\end{aligned}$$

(iv) Let $X(t) = e_A(t,s)e_A(s,t)$, and suppose that $e_A(t,s)$ and $e_A(s,t)$ commute with $(\ominus A)(t)$. Then

$$\begin{aligned}
X^{\Delta}(t) &= e_A^{\Delta}(t,s)e_A(s,t) + e_A^{\sigma}(t,s)e_A^{\Delta}(s,t) \\
&= A(t)e_A(t,s)e_A(s,t) + [I+\mu(t)A(t)]e_A(t,s)\{-e_A(s,\sigma(t))A(t)\} \\
&= A(t)e_A(t,s)e_A(s,t) + [I+\mu(t)A(t)]e_A(t,s)\{-e_A(s,t) \\
&\qquad\qquad [I+\mu(t)(\ominus A)(t)]A(t)\} \\
&= A(t)e_A(t,s)e_A(s,t) + [I+\mu(t)A(t)]e_A(t,s)\{-e_A(s,t) \\
&\qquad\qquad [I-\mu(t)A(t)(I+\mu(t)A(t))^{-1}]A(t)\} \\
&= A(t)e_A(t,s)e_A(s,t) + [I+\mu(t)A(t)]e_A(t,s)\{-e_A(s,t) \\
&\qquad\qquad [I+\mu(t)A(t)]^{-1}A(t)\} \\
&= A(t)e_A(t,s)e_A(s,t) + [I+\mu(t)A(t)]e_A(t,s)e_A(s,t)(\ominus A)(t) \\
&= A(t)e_A(t,s)e_A(s,t) + [I+\mu(t)A(t)](\ominus A)(t)e_A(t,s)e_A(s,t) \\
&= \{A(t) + [I+\mu(t)A(t)](\ominus A)(t)\}e_A(t,s)e_A(s,t) \\
&= [A(t) + (\ominus A)(t) + \mu(t)A(t)(\ominus A)(t)]e_A(t,s)e_A(s,t) \\
&= [A(t) \oplus (\ominus A)(t)]e_A(t,s)e_A(s,t) \\
&= [A(t) \ominus A(t)]e_A(t,s)e_A(s,t) \\
&= 0,
\end{aligned}$$

and

$$X(s) = e_A(s,s)e_A(s,s) = I \cdot I = I.$$

So $X(t)$ solves the IVP

$$X^{\Delta}(t) = 0, \quad X(s) = I$$

which has exactly one solution according to uniqueness solution theorem, and thereby, we have

$$e_A(t,s)e_A(s,t) = I,$$

and consequently,

$$e_A(t,s) = e_A^{-1}(s,t).$$

(v) Let $X(t) = e_A(t,s)e_A(s,r)$. Then

$$\begin{aligned}
X^{\Delta}(t) &= e_A^{\Delta}(t,s)e_A(s,r) \\
&= A(t)e_A(t,s)e_A(s,r) \\
&= A(t)X(t), \quad t \in \mathbb{T},
\end{aligned}$$

and

$$X(r) = e_A(r,s)e_A(s,r) = I.$$

Then $X(t)$ solves the dynamic equation

$$X^{\Delta}(t) = A(t)X(t), \quad X(r) = I,$$

which has exactly one solution

$$X(t) = e_A(t, r).$$

It follows that

$$e_A(t, s)e_A(s, r) = e_A(t, r).$$

The proof is complete.

□

We note that by (3.18) and part (b) we have

$$e_A(s, \sigma(s)) = [I + \mu(s)A(s)]^{-1} = [I + \mu(s)(\ominus A)(s)].$$

4 Remarks on the Exponential Matrix Functions

In this section we give an example of differentiable $n \times n$-matrices $A(t)$ and $B(t)$ to show that the commutativity of $e_A(t, s)$ with $(\ominus A)(t)$ (resp. $e_A(t, s)$ with $B(t)$) is an essential condition for the inverse $e_A^{-1}(t, s)$ to equal $e_A(s, t)$ (resp. $e_A(t, s)e_B(t, s)$ to equal $e_{A\oplus B}(t, s)$).

Definition 4.1 (See [3]). Let $A(t) = (a_{ij}(t))$ be an $n \times n$-matrix-valued function on $\mathbb{T}$. We say that A

(i) is rd-continuous on $\mathbb{T}$ if each a_{ij} is rd-continuous on $\mathbb{T},\ 1 \leq i, j \leq n,\ n \in \mathbb{N}$ and the class of all such rd-continuous $n \times n$-matrix-valued functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$;

(ii) is differentiable on $\mathbb{T}$ provided each $a_{ij},\ 1 \leq i, j \leq n$ is differentiable on $\mathbb{T}$, and in this case we put $A^\Delta = (a_{ij}^\Delta)_{1\leq i,j\leq n}$;

(iii) is regressive (with respect to $\mathbb{T}$) provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^k$, where I is the identity $n \times n$-matrix. The class of all such regressive $n \times n$-matrix-valued functions on $\mathbb{T}$ is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

If $A(t) \equiv A$ is a constant matrix which is regressive, then A commutes with $e_A(t, \tau)$, $\tau \in \mathbb{T}$, see [3, Corollary 5.26]. For example, suppose $a, b \in \mathcal{R}$ are distinct numbers as in text by Zafer [10, 11] and the matrix

$$A = \begin{bmatrix} a & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & b \end{bmatrix}.$$

Consider $\mathbb{T} = \mathbb{R}$ and $\tau = 0$, we can easily see that

$$e_A(t,0) = e^{At} = \begin{bmatrix} e^{at} & 0 & \dfrac{e^{at} - e^{bt}}{a-b} \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{bt} \end{bmatrix},$$

and

$$Ae^{At} = \begin{bmatrix} ae^{at} & 0 & \dfrac{ae^{at} - be^{bt}}{a-b} \\ 0 & ae^{at} & 0 \\ 0 & 0 & be^{bt} \end{bmatrix} = e^{At}A. \tag{4.1}$$

Example 4.2. Now we give an 2×2-matrix $A(t)$ of differentiable functions such that

$$(\ominus A)(t)e_A(t,0) \neq e_A(t,0)(\ominus A)(t), \quad \text{for every} \;\; t \neq 0, \; t \in \mathbb{T}, \tag{4.2}$$

and

$$e_A^{-1}(t,0) \neq e_A(0,t), \quad t \neq 0, \tag{4.3}$$

where $e_A(t,0)$ is the solution of the regressive linear dynamic equation

$$X^{\Delta}(t) = A(t)X(t), \quad X(0) = I. \tag{4.4}$$

Let $A(t) = \begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix}$. The solution of Equation (4.4) is given by

$$X(t) = e_A(t,0). \tag{4.5}$$

It was shown that $e_A(t,\tau)$ can be given by

$$\begin{aligned} e_A(t,\tau) = I + \int_\tau^t A(s_1)\Delta s_1 + \int_\tau^t A(s_1) \int_\tau^{s_1} A(s_2)\Delta s_2 \Delta s_1 \\ + \ldots + \int_\tau^t A(s_1) \int_\tau^{s_1} A(s_2) \ldots \int_\tau^{s_{i-1}} A(s_i)\Delta s_i \ldots \Delta s_1 + \ldots, \end{aligned} \tag{4.6}$$

see [3, 5], where the integration of matrices should be defined entry-by-entry. Thus the i,j-entries of $\displaystyle\int_\tau^t A(s)\Delta s$ is $\left[\displaystyle\int_\tau^t a_{ij}(s)\Delta s\right]$. Using formula (4.6), we conclude that

$$\begin{aligned} & e_A(t,0) \\ &= \begin{bmatrix} 1 + \displaystyle\int_0^t s_1 \Delta s_1 + \int_0^t s_1 \int_0^{s_1} s_2 \Delta s_2 \Delta s_1 + \ldots & \displaystyle\int_0^t \Delta s_1 + \int_0^t s_1 \int_0^{s_1} \Delta s_2 \Delta s_1 + \ldots \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{4.7}$$

and

$$e_A(0,t) = \begin{bmatrix} 1+\int_t^0 s_1\Delta s_1+\int_t^0 s_1\int_0^{s_1} s_2\Delta s_2\Delta s_1+\dots & \int_t^0 \Delta s_1+\int_t^0 s_1\int_0^{s_1} \Delta s_2\Delta s_1+\dots \\ 0 & 1 \end{bmatrix}. \tag{4.8}$$

Now, assume that $\mathbb{T}=\mathbb{R}$. Simple calculations show that

$$e_A(t,0) = \begin{bmatrix} e^{\frac{t^2}{2}} & \sum_{n=1}^{\infty}\frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \\ 0 & 1 \end{bmatrix}, \quad t\in\mathbb{R}. \tag{4.9}$$

and

$$e_A(0,t) = \begin{bmatrix} e^{\frac{-t^2}{2}} & \sum_{n=1}^{\infty}\frac{(-1)^n t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \\ 0 & 1 \end{bmatrix}, \quad t\in\mathbb{R}. \tag{4.10}$$

Using Relation (4.9), we obtain

$$A(t)e_A(t,0) = \begin{bmatrix} te^{\frac{t^2}{2}} & 1+\sum_{n=1}^{\infty}\frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \\ 0 & 0 \end{bmatrix}, \tag{4.11}$$

and

$$e_A(t,0)A(t) = \begin{bmatrix} te^{\frac{t^2}{2}} & e^{\frac{t^2}{2}} \\ 0 & 0 \end{bmatrix}. \tag{4.12}$$

It follows, from Relations (4.11) and (4.12), that

$$A(t)e_A(t,0) \neq e_A(t,0)A(t), \quad \text{for every } \ t\neq 0,\ t\in\mathbb{R}.$$

Since

$$\det e_A(t,0) = e^{t^2/2} \neq 0 \ \forall\, t\in\mathbb{R}, \tag{4.13}$$

then the inverse of $e_A(t,0)$ exists. By Relation (4.9), the inverse of $e_A(t,0)$ is given by

$$\begin{aligned} e_A^{-1}(t,0) &= e^{-t^2/2}\begin{bmatrix} 1 & -\sum_{n=1}^{\infty}\frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \\ 0 & e^{\frac{t^2}{2}} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t^2/2} & -e^{-t^2/2}\sum_{n=1}^{\infty}\frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{4.14}$$

which satisfies

$$e_A^{-1}(t,0)e_A(t,0) = e_A(t,0)e_A^{-1}(t,0) = I.$$

It can be shown that

$$e^{-t^2/2}\sum_{n=1}^{\infty}\frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} = \sum_{n=1}^{\infty}\frac{(-1)^{3n-1}t^{2n-1}}{(\prod_{i=1}^{n-1}2i)(2n-1)},$$

with the convention $\prod_{i=1}^{0} a_i = 1$. Consequently, Relation (4.14) yields

$$e_A^{-1}(t,0) = \begin{bmatrix} e^{-t^2/2} & -\sum_{n=1}^{\infty}\frac{(-1)^{3n-1}t^{2n-1}}{(\prod_{i=1}^{n-1}2i)(2n-1)} \\ 0 & 1 \end{bmatrix} \tag{4.15}$$

Now we show that the condition: $e_A(t,0)$ commutes with $(\ominus A)(t)$ is essential for $e_A^{-1}(t,0)$ to equal $e_A(0,t)$. One can see that $(\ominus A)(t) = -A(t),\ t \in \mathbb{R}$. Hence

$$(\ominus A)(t) = \begin{bmatrix} -t & -1 \\ 0 & 0 \end{bmatrix} \tag{4.16}$$

Then

$$e_{\ominus A}(t,0) = \begin{bmatrix} e^{-t^2/2} & \sum_{n=1}^{\infty}\frac{(-1)^{n}t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \\ 0 & 1 \end{bmatrix}. \tag{4.17}$$

Relations (4.10), (4.15) and (4.17) imply that

$$e_A(0,t) = e_{\ominus A}(t,0) \neq e_A^{-1}(t,0), \quad \forall\, t \in \mathbb{R}\backslash\{0\}. \tag{4.18}$$

We have

$$e_{\ominus A}(t,0)A(t) = \begin{bmatrix} t\, e^{\frac{-t^2}{2}} & e^{\frac{-t^2}{2}} \\ 0 & 0 \end{bmatrix}, \tag{4.19}$$

and

$$A(t)e_{\ominus A}(t,0) = \begin{bmatrix} te^{\frac{-t^2}{2}} & 1+\sum_{n=1}^{\infty}\frac{(-1)^{n}\, t^{2n}}{\prod_{i=1}^{n}(2i-1)} \\ 0 & 0 \end{bmatrix}. \tag{4.20}$$

Then from Equations (4.19) and (4.20), we obtain

$$e_{\ominus A}(t,0)A(t) \neq A(t)e_{\ominus A}(t,0), \quad t \neq 0. \tag{4.21}$$

Similarly one can see that $e_A(t,0)$ and $(\ominus A)(t)$ do not commute. Indeed,

$$e_A(t,0)(\ominus A)(t) = \begin{bmatrix} -te^{\frac{-t^2}{2}} & -e^{\frac{-t^2}{2}} \\ 0 & 0 \end{bmatrix}, \tag{4.22}$$

and

$$(\ominus A)(t)e_A(t,0) = \begin{bmatrix} -te^{\frac{-t^2}{2}} & -\sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{\prod_{i=1}^{n}(2i-1)} - 1 \\ 0 & 0 \end{bmatrix}. \tag{4.23}$$

Therefore

$$e_A(t,0)(\ominus A)(t) \neq (\ominus A)(t)e_A(t,0), \quad t \neq 0. \tag{4.24}$$

We can verify Theorem 3.4 (ii) as follows. Since $A(t) \in M_n(\mathbb{R})$, then the conjugate transpose of $A(t)$ is

$$A^*(t) = \begin{bmatrix} t & 0 \\ 1 & 0 \end{bmatrix}.$$

Using Definition 2.8 part (ii) and Lemma 2.10 part (iii) to obtain $(\ominus A^*)(t) = -A^*(t)$. Consequently

$$(\ominus A^*)(t) = \begin{bmatrix} -t & 0 \\ -1 & 0 \end{bmatrix}.$$

Hence, $e_{\ominus A^*}(t,0) = e_{-A^*}(t,0)$. Thus

$$
\begin{aligned}
e_{\ominus A^*}(t,0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -\int_0^t s_1 ds_1 & 0 \\ -\int_0^t ds_1 & 0 \end{bmatrix} + \begin{bmatrix} \int_0^t s_1 \int_0^{s_1} s_2 ds_2 \Delta s_1 & 0 \\ \int_0^t s_1 \int_0^{s_1} ds_2 \Delta s_1 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} -\int_0^t s_1 \int_0^{s_1} s_2 \int_0^{s_2} s_3 ds_3 ds_2 ds_1 & 0 \\ -\int_0^t s_1 \int_0^{s_1} s_2 \int_0^{s_2} \Delta s_3 ds_2 ds_1 & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \int_0^t s_1 \int_0^{s_1} s_2 \int_0^{s_2} s_3 \int_0^{s_3} s_4 ds_4 ds_3 ds_2 ds_1 & 0 \\ \int_0^t s_1 \int_0^{s_1} s_2 \int_0^{s_2} s_3 \int_0^{s_3} ds_4 ds_3 ds_2 ds_1 & 0 \end{bmatrix} + \dots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -\frac{t^2}{2} & 0 \\ -t & 0 \end{bmatrix} + \begin{bmatrix} \frac{t^4}{2.4} & 0 \\ \frac{t^3}{2.3} & 0 \end{bmatrix} + \begin{bmatrix} -\frac{t^6}{2.4.6} & 0 \\ -\frac{t^5}{2.4.5} & 0 \end{bmatrix} \\
&\quad + \begin{bmatrix} \frac{t^8}{2.4.6.8} & 0 \\ \frac{t^7}{2.4.6.7} & 0 \end{bmatrix} + \dots \\
&= \begin{bmatrix} 1 - \frac{1}{1!}(\frac{t^2}{2}) + \frac{1}{2!}(\frac{t^2}{2})^2 - \frac{1}{3!}(\frac{t^2}{2})^3 + \dots & 0 \\ -t + \frac{t^3}{2.3} - \frac{t^5}{2.4.5} + \frac{t^7}{2.4.6.7} - \dots & 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t^2}{2}} & 0 \\ -\sum_{n=1}^{\infty} \frac{(-1)^{3n-1} t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} & 1 \end{bmatrix}.
\end{aligned}
$$

Consequently, we conclude that

$$
e^*_{A^*}(t,0) = \begin{bmatrix} e^{-\frac{t^2}{2}} & -\sum_{n=1}^{\infty} \frac{(-1)^{3n-1} t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} \\ 0 & 1 \end{bmatrix}. \tag{4.25}
$$

Hence, by Equations (4.15) and (4.25), we obtain

$$e_A^{-1}(t,0) = e_{A^*}^*(t,0), \quad \forall\, t \in \mathbb{R}. \tag{4.26}$$

Example 4.3. Now, we show that the condition: $e_A(t,0)$ commutes with $B(t)$, in Theorem 3.3 part (iii), is essential for $e_{A\oplus B}(t,0)$ to equal $e_A(t,0)e_B(t,0)$. Let $A(t)$ and $B(t)$ be the regressive 2×2-matrices of differentiable functions defined by

$$A(t) = \begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} -t & 0 \\ 1 & 0 \end{bmatrix}. \tag{4.27}$$

Consequently, we conclude that

$$e_B(t,0) = \begin{bmatrix} e^{\frac{-t^2}{2}} & 0 \\ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{3n-1}t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} & 1 \end{bmatrix}. \tag{4.28}$$

Thus

$$e_A(t,0)B(t) = \begin{bmatrix} -te^{\frac{t^2}{2}} + \displaystyle\sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} & 0 \\ 1 & 0 \end{bmatrix}, \tag{4.29}$$

and

$$B(t)e_A(t,0) = \begin{bmatrix} -te^{\frac{t^2}{2}} & -\displaystyle\sum_{n=1}^{\infty} \frac{t^{2n}}{\prod_{i=1}^{n}(2i-1)} \\ e^{\frac{t^2}{2}} & \displaystyle\sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \end{bmatrix}, \tag{4.30}$$

from which we have

$$e_A(t,0)B(t) \neq B(t)e_A(t,0) \;\; \forall\, t \in \mathbb{R}\backslash\{0\}. \tag{4.31}$$

By using Relations (4.9) and (4.28), we get

$$\begin{aligned} &e_A(t,0)e_B(t,0) \\ &= \begin{bmatrix} 1 + \displaystyle\sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \sum_{n=1}^{\infty} \frac{(-1)^{3n-1}t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} & \displaystyle\sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} \\ \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{3n-1}t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} & 1 \end{bmatrix}. \end{aligned} \tag{4.32}$$

Since $\mu(t) = 0, t \in \mathbb{R}$, we have

$$\begin{aligned}(A \oplus B)(t) &= A(t) + B(t) \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.\end{aligned}$$

So,

$$\begin{aligned}e_{A\oplus B}(t,0) = &\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \int_0^t ds_1 \\ \int_0^t ds_1 & 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ds_2 ds_1 \\ &+ \int_0^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ds_3 ds_2 ds_1 \\ &+ \int_0^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_3} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ds_4 ds_3 ds_2 ds_1 + \dots\end{aligned}$$

$$\begin{aligned}= &\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2} & 0 \\ 0 & \frac{t^2}{2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{t^3}{2.3} \\ \frac{t^3}{2.3} & 0 \end{bmatrix} + \begin{bmatrix} \frac{t^4}{2.3.4} & 0 \\ 0 & \frac{t^4}{2.3.4} \end{bmatrix} \\ &+ \begin{bmatrix} 0 & \frac{t^5}{2.3.4.5} \\ \frac{t^5}{2.3.4.5} & 0 \end{bmatrix} + \dots \\ = &\begin{bmatrix} 1 + \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} + \frac{t^6}{1.2.3.4.5.6} + \dots & t + \frac{t^3}{1.2.3} + \frac{t^5}{1.2.3.4.5} + \dots \\ t + \frac{t^3}{1.2.3} + \frac{t^5}{1.2.3.4.5} + \frac{t^7}{1.2.3.4.5.6.7} + \dots & 1 + \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} + \dots \end{bmatrix}.\end{aligned}$$

Consequently, we conclude that

$$e_{A\oplus B}(t,0) = \begin{bmatrix} \sum_{n=1}^{\infty} \frac{t^{2n-2}}{(2n-2)!} & \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \\ \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} & \sum_{n=1}^{\infty} \frac{t^{2n-2}}{(2n-2)!} \end{bmatrix}. \tag{4.33}$$

Then

$$e_{A\oplus B}(t,0) \neq e_A(t,0) e_B(t,0) \ \ \forall\, t \in \mathbb{R}\backslash\{0\}. \tag{4.34}$$

Finally, we verify Theorem 3.3 (iii) as follows:

Example 4.4. In this example we give two 2×2-matrices $C(t)$ and $D(t)$ such that the commutativity condition mentioned in Theorem 3.3 (iii) is satisfied and $e_{C\oplus D}(t,0) = e_C(t,0)e_D(t,0)$. Take the matrices $C(t)$ and $D(t)$ as follows:

$$C(t) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix}, \tag{4.35}$$

and

$$D(t) = \begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix}. \tag{4.36}$$

Consequently, we conclude that

$$e_C(t,0) = \begin{bmatrix} e^{\frac{t^2}{2}} & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.37}$$

and

$$e_D(t,0) = \begin{bmatrix} e^{-\frac{t^2}{2}} & 0 \\ 0 & e^{\frac{-t^2}{2}} \end{bmatrix}. \tag{4.38}$$

We have

$$e_C(t,0)e_D(t,0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{-t^2}{2}} \end{bmatrix}, \tag{4.39}$$

and

$$e_D(t,0)e_C(t,0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{-t^2}{2}} \end{bmatrix}. \tag{4.40}$$

By Relations (4.36), (4.37), (4.35) and (4.38) we obtain

$$D(t)e_C(t,0) = \begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix} \begin{bmatrix} e^{\frac{t^2}{2}} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{-t^2}{2}} \end{bmatrix} = e_C(t,0)D(t), \tag{4.41}$$

and

$$C(t)e_D(t,0) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-\frac{t^2}{2}} & 0 \\ 0 & e^{\frac{-t^2}{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\frac{-t^2}{2}} \end{bmatrix} = e_D(t,0)C(t). \tag{4.42}$$

Since $(C \oplus D)(t) = C(t) + D(t) = \begin{bmatrix} 0 & 0 \\ 0 & -t \end{bmatrix}$, we have

$$\begin{aligned} & e_{C\oplus D}(t,0) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\int_0^t ds_1 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & -s_1 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & -s_2 \end{bmatrix} ds_2 ds_1 \\ &\quad + \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & -s_1 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & -s_2 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 0 \\ 0 & -s_3 \end{bmatrix} ds_3 ds_2 ds_1 \end{aligned}$$

$$+\int_0^t \begin{bmatrix} 0 & 0 \\ 0 & -s_1 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & -s_2 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 0 \\ 0 & -s_3 \end{bmatrix}$$
$$\int_0^{s_3} \begin{bmatrix} 0 & 0 \\ 0 & -s_4 \end{bmatrix} ds_4 ds_3 ds_2 ds_1 + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{t^2}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{t^4}{2.4} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{t^6}{2.4.6} \end{bmatrix} + \dots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 - \frac{t^2}{2} + \frac{t^4}{2.4} - \frac{t^6}{2.4.6} + \dots \end{bmatrix}.$$

Thus

$$e_{C\oplus D}(t,0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{t^2}{2}} \end{bmatrix}. \tag{4.43}$$

Relations (4.39) and (4.43) yield

$$e_{C\oplus D}(t,0) = e_C(t,0)e_D(t,0) \ \ \forall\, t \in \mathbb{R}. \tag{4.44}$$

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