On the Exponential Operator Functions on Time Scales

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Abstract

In this paper, we investigate some properties of the operator exponential solutions of the initial value problem

\[ X^\Delta(t) = A(t)X(t), \quad t \in T \]
\[ X(s) = I, \]

where \( I \) is the identity operator on a Banach space \( X \), \( s \) is an initial point in a time scale \( T \) and \( A(t) \) is a bounded linear operator on \( X \), \( t \in T \). Also, we give an example of differentiable \( 2 \times 2 \) matrices \( A(t) \) and \( B(t) \) to show that the commutativity of \( e_A(t, s) \) with \((\ominus A)(t)\) (resp. \( e_A(t, s) \) with \( B(t) \)) is an essential condition for the inverse \( e_A^{-1}(t, s) \) to equal \( e_A(s, t) \) (resp. \( e_A(t, s)e_B(t, s) \) to equal \( e_{A\oplus B}(t, s) \)).

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1 Introduction

A calculus on time scales was initiated by Stefan Hilger (1988) in order to create a theory that can unify discrete and continuous analysis [8], see also [3,4]. A time scale \( T \) is an arbitrary non-empty closed subset of real numbers \( \mathbb{R} \). The delta derivative \( f^\Delta \) for a function \( f \) on \( T \) was defined in a way such that \( f^\Delta(t) = f'(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} \) is the usual derivative if \( T = \mathbb{R} \), and \( f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t) \) is the usual forward difference operator if \( T = \mathbb{Z} \) [7].
Suppose that $\mathbb{T}$ has the topology inherited from the standard topology on $\mathbb{R}$. We define the time scale interval $[a, b] = [a, b] \cap \mathbb{T}$.

Open intervals and open neighborhoods are defined similarly. A set we need to consider is $\mathbb{T}^k$ which is defined as $\mathbb{T}^k = \mathbb{T}\setminus\{M\}$ if $\mathbb{T}$ has a left-scattered maximum $M$, and $\mathbb{T}^k = \mathbb{T}$ otherwise. Similarly, $\mathbb{T}_k = \mathbb{T}\setminus\{m\}$ if $\mathbb{T}$ has a right-scattered minimum $m$, and $\mathbb{T}_k = \mathbb{T}$ otherwise. For the terminology used here, we refer the reader to [3, 4].

Throughout this paper, $X$ is a Banach space and $L(X)$ denotes the space of all bounded linear operators in $X$. We begin by introducing some definitions which we will need later on.

**Definition 1.1.** A function $f : \mathbb{T} \rightarrow X$ is said to be regulated if

(i) $\lim_{s \rightarrow t^+} f(s)$ exists for all right-dense points $t$ in $\mathbb{T}$,

(ii) $\lim_{s \rightarrow t^-} f(s)$ exists for all left-dense points $t$ in $\mathbb{T}$.

**Definition 1.2.** A function $f : \mathbb{T} \rightarrow X$ is said to be rd-continuous if

(i) $f$ is continuous at every right-dense point $t$ in $\mathbb{T}$ i.e $\lim_{s \rightarrow t^+} f(s) = f(t)$,

(ii) $\lim_{s \rightarrow t^-} f(s)$ exists at every left-dense point $t \in \mathbb{T}$.

A function $f(t, x) : \mathbb{T} \times X \rightarrow X$ is called rd-continuous if $f(t, x(t)) : \mathbb{T} \rightarrow X$ is so for every continuous function $x : \mathbb{T} \rightarrow X$.

**Definition 1.3.** A function $f : \mathbb{T} \rightarrow X$ is called delta differentiable (or simply differentiable) at $t \in \mathbb{T}^k$ provided there exists an $\alpha$ such that for every $\epsilon > 0$ there is a neighborhood $U$ of $t$ with

$$|| f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s) || \leq \epsilon |\sigma(t) - s|$$

for all $s \in U$.

In this case we denote the $\alpha$ by $f^\Delta(t)$; and if $f$ is differentiable at every $t \in \mathbb{T}^k$, then $f$ is said to be differentiable on $\mathbb{T}$ and $f^\Delta$ is a new function defined on $\mathbb{T}$. If $f$ is differentiable at $t \in \mathbb{T}_k$, then it is easy to see that

$$f^\Delta(t) = \begin{cases} 
\frac{f(\sigma(t)) - f(t)}{\mu(t)}, & \text{if } \mu(t) > 0; \\
\lim_{s \rightarrow t} \frac{f(t) - f(s)}{t - s}, & \text{if } \mu(t) = 0.
\end{cases} \quad (1.1)$$

For any time scale $\mathbb{T}$, we define its norm as (see [1, 3, 5])

$$|| \mathbb{T} || := \sup \{ \mu(t) : t \in \mathbb{T} \}.$$
Clearly, $\| T \| \in [0, +\infty]$, e.g., $\| \mathbb{R} \| = 0$, $\| \mathbb{Z} \| = 1$, if $a, b > 0$, then $\| \bigcup_{k=1}^{\infty} [k(a + b), k(a + b) + a] \| = b$ and $\| \{ n^2 : n \in \mathbb{N} \} \| = +\infty$.

We consider the initial value problem (IVP)

$$X^\Delta(t) = A(t)X(t), \quad X(s) = I, \quad t \in \mathbb{T}$$

which has a unique solution under certain appropriate conditions. Here $I : \mathbb{X} \to \mathbb{X}$ is the identity operator. This solution is denoted by $e_A(t, s)$. When $A(t) \in M_n(\mathbb{R})$, the family of $n \times n$ real matrices, many properties of $e_A(t, s)$ were proved. See [3, Theorem 5.21]. In this paper, we establish these properties when $A(t) \in L(\mathbb{X})$, $t \in \mathbb{T}$. See Section 3. In Section 4, we give an example of differentiable $n \times n$ matrices $A(t)$ and $B(t)$ to show that the commutativity of $e_A(t, s)$ with $(\ominus A)(t)$ (resp. $e_A(t, s)$ with $B(t)$) is an essential condition for the inverse $e^{-1}_A(t, s)$ to equal $e_A(s, t)$ (resp. $e_A(t, s)e_B(t, s)$ to equal $e_{A\oplus B}(t, s)$).

## 2 Preliminary Results

Let $\mathbb{Y}$ be a Banach algebra with unit $e$. See [6]. We need the following theorems to prove the main properties of abstract exponential functions.

**Theorem 2.1.** If $A : [a, b] \to \mathbb{Y}$ is regulated, then it is bounded.

**Proof.** Assume towards a contradiction, that there is a sequence $\{ t_n \} \subseteq [a, b]$ such that $\| A(t_n) \| > n \quad \forall n \in \mathbb{N}$. Then $\{ t_n \}$ has a subsequence $\{ t_{n_k} \} \subseteq [a, b]$ which decreases to $s$ or increases to $s$ for some $s \in [a, b]$. Since $s \in \mathbb{T}$ is either a right-dense point or a left-dense point, then $\lim_{k \to \infty} A(t_{n_k})$ exists which contradicts with $\lim_{n \to \infty} \| A(t_n) \| = \infty$. \hfill $\Box$

**Lemma 2.2.** Assume $g, h : \mathbb{T} \to \mathbb{Y}$ such that $\lim_{s \to t} g(s) = T_1$ and $\lim_{s \to t} h(s) = T_2$. Then

$$\lim_{s \to t} g(s) h(s) = T_1 T_2, \quad T_1, T_2 \in \mathbb{Y}. \quad (2.1)$$

**Proof.** We have

$$\| g(s) h(s) - T_1 T_2 \| \leq \| g(s) h(s) - T_1 h(s) \| + \| T_1 h(s) - T_1 T_2 \| \leq \| g(s) - T_1 \| \| h(s) \| + \| T_1 \| \| h(s) - T_2 \|. \quad (2.1)$$

Taking $s \to t$, since $h$ is bounded in a neighborhood of $t$, we get Relation (2.1). \hfill $\Box$

**Theorem 2.3.** Let $A, B : \mathbb{T} \to \mathbb{Y}$ be $\Delta$-differentiable at $t \in \mathbb{T}^k$. Then $AB : \mathbb{T} \to \mathbb{Y}$ is $\Delta$-differentiable at $t \in \mathbb{T}^k$ and

$$(AB)^\Delta(t) = A(\sigma(t))B^\Delta(t) + A^\Delta(t)B(t) = A^\Delta(t)B(\sigma(t)) + A(t)B^\Delta(t). \quad (2.2)$$

Here, $AB(t) = A(t)B(t)$. 


Proof. We have
\[
(AB)^\Delta(t) = \lim_{s \to t, \; s, t \in T} \frac{(AB)(\sigma(t)) - (AB)(s)}{\sigma(t) - s}
\]
\[
= \lim_{s \to t, \; s, t \in T} \frac{A(\sigma(t))B(\sigma(t)) - A(s)B(\sigma(t)) + A(s)B(\sigma(t)) - A(s)B(s)}{\sigma(t) - s}
\]
\[
= \lim_{s \to t, \; s, t \in T} \left[ \frac{A(\sigma(t))B(\sigma(t)) - A(s)B(\sigma(t))}{\sigma(t) - s} + \frac{A(s)B(\sigma(t)) - A(s)B(s)}{\sigma(t) - s} \right]
\]
\[
= \lim_{s \to t, \; s, t \in T} \frac{A(\sigma(t)) - A(s)}{\sigma(t) - s} B(\sigma(t)) + \lim_{s \to t, \; s, t \in T} A(s) \frac{B(\sigma(t)) - B(s)}{\sigma(t) - s}
\]
\[
= A^\Delta(t)B(\sigma(t)) + A(t)B^\Delta(t).
\]
This completes the proof. \(\square\)

Definition 2.4. A function \(F : T \to \mathbb{Y}\) is called an antiderivative of \(f : T \to \mathbb{Y}\) if \(F^\Delta(t) = f(t), \; t \in T^k\). In this case we define an indefinite integral of \(f\) by
\[
\int f(\tau) \Delta \tau = F(t) + C
\]
where \(C\) is an arbitrary constant. We define the Cauchy integral by
\[
\int_s^t f(\tau) \Delta \tau = F(t) - F(s), \; \text{where} \; s, t \in T.
\]

Theorem 2.5 (See [3, Theorems 1.74, 1.75]).
(i) Every rd-continuous \(f : T \to \mathbb{Y}\) has an antiderivative and \(F(t) = \int_s^t f(\tau) \Delta \tau\) is an antiderivative of \(f\), i.e., \(F^\Delta(t) = f(t), \; t \in T^k\).
(ii) If \(f \in C_{rd}\) and \(t \in T^k\), then
\[
\int_s^{\sigma(t)} f(\tau) \Delta \tau = \mu(t)f(t).
\]

We need the following theorem in our study.

Theorem 2.6. Let \(f : T \to \mathbb{Y}\) be rd-continuous. Then
\[
\int_s^t f(\tau) y \Delta \tau = \int_s^t f(\tau) \Delta \tau \; y, \; y \in \mathbb{Y}.
\]
Proof. Let $F(t) = \int_s^t f(\tau) y \Delta \tau$ and $G(t) = \int_s^t f(\tau) y \Delta \tau$. Then $F^\Delta(t) = f(t) y$ and $G^\Delta(t) = f(t) y$. Consequently, $F(t) = G(t) + C$. Hence, $F(s) = G(s) + C$. Thus $C = 0$.

Definition 2.7. Let $A : \mathbb{T} \to \mathbb{Y}$ be rd-continuous at every $t \in \mathbb{T}^k$ such that $A(t)$ has an inverse for all $t \in \mathbb{T}^k$, we define $A^{-1} : \mathbb{T} \to \mathbb{Y}$ by

$$A^{-1}(t) = [A(t)]^{-1}.$$ 

We note that $A^{-1}$ is rd-continuous, since the inverse function on the invertible elements $I(\mathbb{Y}) \subseteq \mathbb{Y}$

$$I(\mathbb{Y}) \to I(\mathbb{Y})$$

$$x \mapsto x^{-1}$$

is continuous.

A mapping $A : \mathbb{T} \to \mathbb{Y}$ is called regressive if $I + \mu(t) A(t)$ is invertible for every $t \in \mathbb{T}$. The class of all regressive mappings is denoted by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{Y}),$$

and the space of rd-continuous, regressive mappings from $\mathbb{T}$ to $\mathbb{Y}$ is denoted by

$$C_{rd} \mathcal{R}(\mathbb{T}, \mathbb{Y}) = \{A : \mathbb{T} \to \mathbb{Y} \mid A \in C_{rd}(\mathbb{T}, \mathbb{Y}) \text{ and } I + \mu(t) A(t) \text{ is invertible for all } t \in \mathbb{T}\}.$$ 

Definition 2.8. Let $A, B \in C_{rd} \mathcal{R}(\mathbb{T}, \mathbb{Y})$. Then we define:

(i) $A \oplus B$ by

$$(A \oplus B)(t) = A(t) + B(t) + \mu(t) A(t) B(t) \text{ for all } t \in \mathbb{T}^k,$$

(ii) $\ominus A$ by

$$(\ominus A)(t) = -[I + \mu(t) A(t)]^{-1} A(t) \text{ for all } t \in \mathbb{T}^k,$$

(iii) $A \ominus B$ by

$$(A \ominus B)(t) = (A \oplus (\ominus B))(t) \text{ for all } t \in \mathbb{T}^k.$$

We can see, if $A \in \mathcal{R}$, then $\ominus A$ satisfies $(A \oplus (\ominus A))(t) = 0$ for all $t \in \mathbb{T}$.

Remark 2.9. Let $A \in C_{rd} \mathcal{R}(\mathbb{T}, \mathbb{Y})$. Then

(i) $(\ominus A)(t) = -[I + \mu(t) A(t)]^{-1} A(t) = -A(t)[I + \mu(t) A(t)]^{-1} \text{ for all } t \in \mathbb{T}^k,$

(ii) $\ominus(\ominus A) = A$. 

Now let $A \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$ and $A^*(t) : \mathbb{X}^* \to \mathbb{X}^*$, where $\mathbb{X}^*$ is the dual space of $\mathbb{X}$ [9], be the adjoint operator of $A(t) : \mathbb{X} \to \mathbb{X}$, $t \in \mathbb{T}$ which is defined by

$$(A^*(t)f)(x) = f(A(t)x) \text{ for all } f \in \mathbb{X}^* \text{ and } x \in \mathbb{X}.$$ 

The proof of the following lemma is straightforward and will be omitted.

**Lemma 2.10.** The following statements are true:

(i) $((A^*)^\Delta)^\ast = (A^\Delta)^\ast$ holds for any differentiable operator $A$;

(ii) $A^\ast$ is regressive;

(iii) $\ominus A^\ast = (\ominus A)^\ast$;

(iv) $A^\ast \oplus B^\ast = (A \oplus B)^\ast$, where $B \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$.

### 3 The Operator Exponential function $e_A(t, s)$ in Banach Spaces

In the sequel, we assume that $\sup \mathbb{T} = \infty$ and $A \in C_{rd}\mathcal{R}(\mathbb{T}, L(\mathbb{X}))$.

**Definition 3.1.** Let $s \in \mathbb{T}$. A function $X : \mathbb{T} \to L(\mathbb{X})$ that satisfies the operator delta dynamic equation

$$X^\Delta(t) = A(t)X(t), \quad X(s) = I,$$

is called the operator exponential function (corresponding to $A(t)$) initialized at $s$. Its value at $t \in \mathbb{T}$, is denoted by $e_A(t, s)$. Here $X^\Delta(t) = \lim_{s \to t, s \in \mathbb{T}} \frac{X(\sigma(t)) - X(s)}{\sigma(t) - s}$, $I$ is the identity operator on $\mathbb{X}$.

The regressive initial value problem (IVP)

$$x^\Delta(t) = A(t)x(t), \quad x(s) = x_s \in \mathbb{X}$$

has a unique solution $x : \mathbb{T} \to \mathbb{X}$, which is given by $x(t) = e_A(t, s)x_s$ provided

(i) The function $A(t)x(t)$ is rd-continuous for every continuous function $x : \mathbb{T} \to \mathbb{X}$;

(ii) $I + \mu(t)A(t) : \mathbb{X} \to \mathbb{X}$ is invertible for every $t \in \mathbb{T}$;

(iii) $\sup_{t \in \mathbb{T}} \| A(t) \| < \infty$.

See [3, Chapter 8].

**Example 3.2.** Let $A$ be a constant operator.
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(i) If $A$ is bounded and $T = \mathbb{R}$, then $e_A(t, s) = e^{(t-s)A}$, $t, s \in \mathbb{R}$. Here, $e^{tA} = \sum_{n=0}^{\infty} (tA)^n/n!$.

(ii) If $T = h\mathbb{Z}$, then $e_A(t, s) = (I + hA)^{t-s}$, where $t, s \in h\mathbb{Z}$, with $t \geq s$.

(iii) If $T = 2^\mathbb{N}_0$, then $e_A(t, s) = \prod_{k=1}^{\log_2 \frac{t}{s}} (I + \frac{t}{2k}A), t, s \in T$, with $t \geq s$.

Now we establish some fundamental properties of $e_A(t, s)$ which are satisfied when $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, where $M_n(\mathbb{R})$ is the family of all $n \times n$ real matrices [3]. The proof of the following theorem is similar to the case $A(t)$ is an $n \times n$ matrix [3, Theorem 5.21], so it will be omitted.

Theorem 3.3. If $A, B \in C_{rd}\mathcal{R}(\mathbb{T}, L(X))$ and $t, s \in \mathbb{T}$, then the following statements are true

(i) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$;

(ii) $e_A(\sigma(t), s) = [I + \mu(t)A(t)]e_A(t, s)$;

(iii) $e_A(t, s)e_B(t, s) = e_{A \oplus B}(t, s)$ if $e_A(t, s)$ and $B(t)$ commute;

(iv) Assume that we have the following commutativity condition:

(C) $e_A(t, s)$ and $(\ominus A)(t)$ commute, and $e_{\ominus A}(t, s)$ and $A(t)$ commute.

Then $e_{-1}^A(t, s) = e_{\ominus A}(t, s)$.

The following theorem was proved when $A \in C_{rd}\mathcal{R}(\mathbb{T}, M_n(\mathbb{R}))$, see [3, Theorem 5.21]. Now we prove it when $A \in C_{rd}\mathcal{R}(\mathbb{T}, L(X))$.

Theorem 3.4. If $A, B \in C_{rd}\mathcal{R}(\mathbb{T}, L(X))$ and $t, r, s \in \mathbb{T}$, then

(i) Condition (C) implies

\[ [e_{-1}^A(t, s)]^\Delta = -e_{-1}^A(\sigma(t), s)e_A(t, s)e_{-1}^A(t, s) = -[e_{\ominus A}(t, s)]^\sigma A(t) = (\ominus A)(t)e_{\ominus A}(t, s); \]

(ii) $e_{-1}^A(t, s) = e_{\ominus A}^*(t, s)$, provided that condition (C) holds;

(iii) If $a, b, s \in \mathbb{T}$, then we have

(a) $e_{\Delta^A}(t, s) = -e_A(t, \sigma(s))A(s)$,
(b) $e_A(t, \sigma(s)) = e_A(t, s)[I + \mu(s)(\ominus A)(s)]$, 
(c) \[ \int_a^b e_A(t, \sigma(s))A(s)\Delta s = e_A(t, a) - e_A(t, b); \]

(iv) \[ e_A(t, s) = e_A^{-1}(t, s), \text{ provided } e_A(t, s) \text{ commutes with } (\oplus A)(t); \]

(v) \[ e_A(t, s)e_A(s, r) = e_A(t, r), \text{ provided } e_A(t, s) \text{ commutes with } (\ominus A)(t). \]

\textbf{Proof.} (i) In view of \[ e_A(t, s)e_A^{-1}(t, s) = I, \] we obtain \[ [e_A(t, s)e_A^{-1}(t, s)]^\Delta = 0. \] By Theorem 2.3, we get

\[ e_A^\Delta(t, s)e_A^{-1}(t, s) + e_A(\sigma(t), s)(e_A^{-1}(t, s))^\Delta = 0. \]

Then

\[ e_A(\sigma(t), s)(e_A^{-1}(t, s))^\Delta = -e_A^\Delta(t, s)e_A^{-1}(t, s), \]

which implies that

\[ (e_A^{-1}(t, s))^\Delta = -e_A^{-1}(\sigma(t), s)e_A^\Delta(t, s)e_A^{-1}(t, s) \]

\[ = -e_A^{-1}(\sigma(t), s)A(t)e_A(t, s)e_A^{-1}(t, s) \]

\[ = -e_A^{-1}(\sigma(t), s)A(t) \]

\[ = -e_A^{-1}(t, s)[I + \mu(t)A(t)]^{-1}A(t) \]

\[ = e_A^{-1}(t, s)(\ominus A)(t). \]

Thus

\[ (e_A^{-1}(t, s))^\Delta = e_\ominus A(t, s)(\ominus A)(t). \] (3.3)

Finally, since

\[ e_\ominus A(\sigma(t), s) = e_\ominus A(t, s) + \mu(t)e_\ominus A(t, s), \]

then

\[ [e_\ominus A(t, s)]^\sigma A(t) = e_\ominus A(t, s)A(t) + \mu(t)e_\ominus A(t, s)A(t) \]

\[ = e_\ominus A(t, s)A(t) + \mu(t)(\ominus A)(t)e_\ominus A(t, s)A(t) \]

\[ = [I + \mu(t)(\ominus A)(t)]e_\ominus A(t, s)A(t) \]

\[ = [I - \mu(t)A(t)(I + \mu(t)A(t))^{-1}]e_\ominus A(t, s)A(t) \]

\[ = [I + \mu(t)A(t)]^{-1}e_\ominus A(t, s)A(t) \]

\[ = [I + \mu(t)A(t)]^{-1}A(t)e_\ominus A(t, s)A(t) \]

\[ = -(\ominus A)(t)e_\ominus A(t, s) \]

\[ = -e_\ominus A(t, s) \]

\[ = -(e_A^{-1}(t, s))^\Delta \]

\[ = -e_\ominus A(t, s)(\ominus A)(t). \text{ (by Equation (3.3))} \]
(ii) Put $X(t) = (e^{-1}_A(t, s))^*$. Then from (i) and Lemma 2.10, we have

\[
X^\Delta(t) = ((e^{-1}_A(t, s))^*)^\Delta = ((e^{-1}_A(t, s))^*)^* \\
= -[e^{-1}_A(\sigma(t), s)e^\Delta_A(t, s)e^{-1}_A(t, s)]^* \\
= -[e^{-1}_A(t, s)(I + \mu(t)A(t))^{-1}A(t)e_A(t, s)e^{-1}_A(t, s)]^* \\
= [e^{-1}_A(t, s)(\ominus A)(t)]^* \\
= (\ominus A^*)(t)(e^{-1}_A(t, s))^* \\
= (\ominus A^*)(t)X(t),
\]

and

\[
X(s) = (e^{-1}_A(s, s))^* = (I^{-1})^* = I.
\]

Hence, $X(t)$ solves the IVP

\[
X^\Delta(t) = (\ominus A^*)(t)X(t), \quad X(s) = I
\]

which has exactly the solution

\[
X(t) = e_{\ominus A^*}(t, s).
\]

Thus

\[
e^{-1}_A(t, s) = e_{\ominus A^*}^*(t, s).
\]

(iii) Let $a, b, s \in \mathbb{T}$.

(a) By differentiating the integral equation

\[
e_A(t, s) = I + \int_s^t A(\tau)e_A(\tau, s)\Delta\tau \tag{3.4}
\]

corresponding the IVP (3.1) with respect to $s$, we get two cases:

(1) First case $\mu(s) > 0$. We have, from Relations (1.1),

\[
e^\Delta_A^*(t, s) = \frac{e_A(t, \sigma(s)) - e_A(t, s)}{\mu(s)}. \tag{3.5}
\]
Equation (3.4) and Theorem 2.5 imply that
\[
e^A_{\Delta s}(t, s) = \frac{1}{\mu(s)} \left[ I + \int_{\sigma(s)}^t A(\tau)e_A(\tau, \sigma(s))d\tau \right. \\
- I - \int_s^t A(\tau)e_A(\tau, s)d\tau \\
= \frac{1}{\mu(s)} \left[ \int_s^t A(\tau)(e_A(\tau, \sigma(s)) - e_A(\tau, s))d\tau \\
- \int_s^\sigma A(\tau)e_A(\tau, \sigma(s))d\tau \right] \\
= \int_s^t A(\tau)e^A_{\Delta s}(\tau, s)d\tau - \frac{1}{\mu(s)} \int_s^{\sigma(s)} A(\tau)e_A(\tau, \sigma(s))d\tau,
\]
and consequently,
\[
e^A_{\Delta s}(t, s) = \int_s^t A(\tau)e^A_{\Delta s}(\tau, s)d\tau + \frac{1}{\mu(s)} \int_s^{\sigma(s)} A(\tau)e_A(\tau, \sigma(s))d\tau. \tag{3.6}
\]
From Equations (3.4) and (3.6), we get
\[
e^A_{\Delta s}(t, s) = \frac{1}{\mu(s)}[e_A(s, \sigma(s)) - I] + \int_s^t A(\tau)e^A_{\Delta s}(\tau, s)d\tau. \tag{3.7}
\]
Thus
\[
\Psi(t, s) = e^A_{\Delta s}(t, s)
\]
is a solution of the integral equation
\[
\Psi(t, s) = U(s) + \int_{\sigma(s)}^t A(\tau)\Psi(\tau, s)d\tau \tag{3.8}
\]
with
\[
U(s) = \frac{1}{\mu(s)}[e_A(s, \sigma(s)) - I]. \tag{3.9}
\]
Multiplying Equation (3.4) by \(\frac{1}{\mu(s)}[e_A(s, \sigma(s)) - I]\) and applying Theorem 2.6, we get
\[
\frac{1}{\mu(s)}e_A(t, s)[e_A(s, \sigma(s)) - I] \\
= \frac{1}{\mu(s)}[e_A(s, \sigma(s)) - I] + \frac{1}{\mu(s)} \int_s^t A(\tau)e_A(\tau, s)[e_A(s, \sigma(s)) - I]d\tau.
\]
This implies that
\[ \Psi(t, s) = e_A(t, s)[e_A(s, \sigma(s)) - I] \] (3.10)
is a solution of Equation (3.8) with \( U(s) = \frac{1}{\mu(s)}[e_A(s, \sigma(s)) - I] \). By the uniqueness of solution of Equation (3.8), we deduce that,
\[ e^\Delta_A(t, s) = \frac{1}{\mu(s)} e_A(t, s)[e_A(s, \sigma(s)) - I]. \] (3.11)

By Theorem 2.5 part (ii), Equation (3.6) yields
\[ e^\Delta_A(t, s) = -A(s)e_A(s, \sigma(s)) + \int_s^t A(\tau)e^\Delta_A(\tau, s) \Delta \tau. \] (3.12)

Multiplying Equation (3.4) by \(-A(s)e_A(s, \sigma(s))\) and applying Theorem 2.6, we obtain
\[ [-e_A(t, s)A(s)e_A(s, \sigma(s))] = -A(s)e_A(s, \sigma(s)) + \int_s^t A(\tau)[e_A(\tau, s)A(s)e_A(s, \sigma(s))] \Delta \tau. \] (3.13)

From the uniqueness of solutions of (3.8) with \( U(s) = -A(s)e_A(s, \sigma(s)) \), we conclude that
\[ e^\Delta_A(t, s) = -e_A(t, s)A(s)e_A(s, \sigma(s)). \] (3.14)

We have
\[ e_A(t, \sigma(s)) = e_A(t, s) + \mu(s)e^\Delta_A(t, s) = e_A(t, s) - \mu(s)e_A(t, s)A(s)e_A(s, \sigma(s)), \]
so that
\[ e_A(t, \sigma(s)) = e_A(t, s)[I - \mu(s)A(s)e_A(s, \sigma(s))]. \] (3.15)

Put \( t = s \), in Equation (3.15) to get
\[ e_A(s, \sigma(s)) = [I - \mu(s)A(s)e_A(s, \sigma(s))], \] (3.16)
consequently,
\[ [I + \mu(s)A(s)]e_A(s, \sigma(s)) = I. \] (3.17)

Therefore,
\[ e_A(s, \sigma(s)) = [I + \mu(s)A(s)]^{-1}. \] (3.18)

By Equations (3.15) and (3.16), we get
\[ e_A(t, \sigma(s)) = e_A(t, s)e_A(s, \sigma(s)). \] (3.19)
In view of Equation (3.14), $A(s)$ and $e_A(s, \sigma(s))$ commute, we have
\[ e_A^\Delta(t, s) = -e_A(t, s) e_A(s, \sigma(s)) A(s). \] (3.20)

It follows, from Equations (3.19) and (3.20), that
\[ e_A^\Delta(t, s) = -e_A(t, \sigma(s)) A(s). \] (3.21)

(2) Second case $\mu(s) = 0$. We have from Equation (3.4)
\[
\frac{e_A(t, s) - e_A(t, s_0)}{s - s_0} = \frac{-1}{s - s_0} \int_{s_0}^{s} A(\tau) e_A(\tau, s_0) \Delta \tau \\
- \int_t^s A(\tau) \left[ \frac{e_A(\tau, s) - e_A(\tau, s_0)}{s - s_0} \right] \Delta \tau.
\]

By taking the limit to both sides as $s \to s_0$, we obtain
\[ e_A^\Delta(t, s) = -A(s_0) - \int_t^{s_0} A(\tau) e_A^\Delta(\tau, s_0) \Delta \tau. \] (3.22)

Then
\[ e_A^\Delta(t, s) = -A(s) - \int_t^{s} A(\tau) e_A^\Delta(\tau, s) \Delta \tau. \] (3.23)

Again by Equation (3.4), multiplying by $-A(s)$ and using Theorem 2.6, we get
\[ -e_A(t, s) A(s) = -A(s) + \int_t^{s} A(\tau) e_A(\tau, s) A(s) \Delta \tau. \] (3.24)

By the uniqueness of solution of the equation
\[ \psi(t, s) = U(s) + \int_t^{s} A(\tau) \psi(\tau, s) \Delta \tau \quad \text{with} \quad U(s) = -A(s), \] (3.25)
we conclude that
\[ e_A^\Delta(t, s) = -e_A(t, s) A(s). \] (3.26)

(b) From Equations (3.15) and (3.18), we have
\[
e_A(t, \sigma(s)) = e_A(t, s) [I - \mu(s) A(s) e_A(s, \sigma(s))] \\
= e_A(t, s) [I - \mu(s) A(s) [I + \mu(s) A(s)^{-1}]] \\
= e_A(t, s) [I + \mu(s) (\ominus A(s))].
\]

(c) From part (a), we obtain
\[
\int_a^b e_A(t, \sigma(s)) A(s) \Delta s = -\int_a^b [e_A(t, s)]^\Delta \Delta s \\
= -e_A(t, s) \big|_a^b \\
= e_A(t, a) - e_A(t, b), \quad \text{for all} \quad a, b \in \mathbb{T}.
\]
(iv) Let $X(t) = e_A(t, s)e_A(s, t)$, and suppose that $e_A(t, s)$ and $e_A(s, t)$ commute with $(\ominus A)(t)$. Then

$$X^\Delta(t) = e_A^\Delta(t, s)e_A(s, t) + e_A^\Delta(s, t)e_A^\Delta(t, s)$$

$$= A(t)e_A(t, s)e_A(s, t) + [I + \mu(t)A(t)]e_A(t, s)\{-e_A(s, \sigma(t))A(t)\}$$

$$= A(t)e_A(t, s)e_A(s, t) + [I + \mu(t)A(t)]e_A(t, s)\{-e_A(s, t)$$

$$[I + \mu(t)(\ominus A)(t)]A(t)\}$$

$$= A(t)e_A(t, s)e_A(s, t) + [I + \mu(t)A(t)]e_A(t, s)\{-e_A(s, t)$$

$$[I - \mu(t)A(t)(I + \mu(t)A(t))^{-1}]A(t)\}$$

$$= A(t)e_A(t, s)e_A(s, t) + [I + \mu(t)A(t)]e_A(t, s)\{-e_A(s, t)$$

$$[I + \mu(t)A(t)]^{-1}A(t)\}$$

$$= A(t)e_A(t, s)e_A(s, t) + [I + \mu(t)A(t)]\{A(t) + [\ominus A](t) + \mu(t)A(t)\ominus A(t))e_A(t, s)e_A(s, t)$$

$$= [A(t) + (\ominus A)(t) + \mu(t)A(t)\ominus A(t))e_A(t, s)e_A(s, t)$$

$$= [A(t)\ominus A(t)]e_A(t, s)e_A(s, t)$$

$$= 0,$$

and

$$X(s) = e_A(s, s)e_A(s, s) = I \cdot I = I.$$  

So $X(t)$ solves the IVP

$$X^\Delta(t) = 0, \quad X(s) = I$$

which has exactly one solution according to uniqueness solution theorem, and thereby, we have

$$e_A(t, s)e_A(s, t) = I,$$

and consequently,

$$e_A(t, s) = e_A^{-1}(s, t).$$

(v) Let $X(t) = e_A(t, s)e_A(s, r)$. Then

$$X^\Delta(t) = e_A^\Delta(t, s)e_A(s, r)$$

$$= A(t)e_A(t, s)e_A(s, r)$$

$$= A(t)X(t), \quad t \in \mathbb{T},$$

and

$$X(r) = e_A(r, s)e_A(s, r) = I.$$  

Then $X(t)$ solves the dynamic equation

$$X^\Delta(t) = A(t)X(t), \quad X(r) = I,$$
which has exactly one solution
\[ X(t) = e_A(t, r). \]
It follows that
\[ e_A(t, s)e_A(s, r) = e_A(t, r). \]
The proof is complete.

We note that by (3.18) and part (b) we have
\[ e_A(s, \sigma(s)) = [I + \mu(s)A(s)]^{-1} = [I + \mu(s)(\oplus A)(s)]. \]

4 Remarks on the Exponential Matrix Functions

In this section we give an example of differentiable \( n \times n \)-matrices \( A(t) \) and \( B(t) \) to show that the commutativity of \( e_A(t, s) \) with \( \ominus A(t) \) (resp. \( e_A(t, s) \) with \( B(t) \)) is an essential condition for the inverse \( e_A^{-1}(t, s) \) to equal \( e_A(s, t) \) (resp. \( e_A(t, s)e_B(t, s) \) to equal \( e_A \oplus B(t, s) \)).

**Definition 4.1** (See [3]). Let \( A(t) = (a_{ij}(t)) \) be an \( n \times n \)-matrix-valued function on \( T \). We say that \( A \)

(i) is rd-continuous on \( T \) if each \( a_{ij} \) is rd-continuous on \( T \), \( 1 \leq i, j \leq n \), \( n \in \mathbb{N} \) and the class of all such rd-continuous \( n \times n \)-matrix-valued functions on \( T \) is denoted by \( C_{rd}(T, \mathbb{R}^{n \times n}) \);

(ii) is differentiable on \( T \) provided each \( a_{ij}, 1 \leq i, j \leq n \) is differentiable on \( T \), and in this case we put \( A^\Delta = (a^\Delta_{ij})_{1 \leq i, j \leq n} \);

(iii) is regressive (with respect to \( T \)) provided \( I + \mu(t)A(t) \) is invertible for all \( t \in \mathbb{T}^k \), where \( I \) is the identity \( n \times n \)-matrix. The class of all such regressive \( n \times n \)-matrix-valued functions on \( T \) is denoted by \( R(T, \mathbb{R}^{n \times n}) \).

If \( A(t) \equiv A \) is a constant matrix which is regressive, then \( A \) commutes with \( e_A(t, \tau) \), \( \tau \in \mathbb{T} \), see [3, Corollary 5.26]. For example, suppose \( a, b \in \mathbb{R} \) are distinct numbers as in text by Zafer [10, 11] and the matrix
\[
A = \begin{bmatrix}
a & 0 & 1 \\
0 & a & 0 \\
0 & 0 & b \\
\end{bmatrix}.
\]
Consider $T = \mathbb{R}$ and $\tau = 0$, we can easily see that

$$e_A(t, 0) = e^{At} = \begin{bmatrix} e^{at} & 0 & \frac{e^{at} - e^{bt}}{a - b} \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{bt} \end{bmatrix},$$

and

$$Ae^{At} = \begin{bmatrix} ae^{at} & 0 & \frac{ae^{at} - be^{bt}}{a - b} \\ 0 & ae^{at} & 0 \\ 0 & 0 & be^{bt} \end{bmatrix} = e^{At}A. \quad (4.1)$$

**Example 4.2.** Now we give an $2 \times 2$-matrix $A(t)$ of differentiable functions such that

$$(\ominus A)(t)e_A(t, 0) \neq e_A(t, 0)(\ominus A)(t), \quad \text{for every } t \neq 0, \ t \in T, \quad (4.2)$$

and

$$e^{-1}_A(t, 0) \neq e_A(0, t), \quad t \neq 0, \quad (4.3)$$

where $e_A(t, 0)$ is the solution of the regressive linear dynamic equation

$$X^\Delta(t) = A(t)X(t), \quad X(0) = I. \quad (4.4)$$

Let $A(t) = \begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix}$. The solution of Equation (4.4) is given by

$$X(t) = e_A(t, 0). \quad (4.5)$$

It was shown that $e_A(t, \tau)$ can be given by

$$e_A(t, \tau) = I + \int_{\tau}^{t} A(s_1)\Delta s_1 + \int_{\tau}^{t} A(s_1) \int_{\tau}^{s_1} A(s_2)\Delta s_2 \Delta s_1$$

$$+ \ldots + \int_{\tau}^{t} A(s_1) \int_{\tau}^{s_1} A(s_2) \ldots \int_{\tau}^{s_i-1} A(s_i) \Delta s_i \ldots \Delta s_1 + \ldots, \quad (4.6)$$

see [3, 5], where the integration of matrices should be defined entry-by-entry. Thus the $i, j$-entries of $\int_{\tau}^{t} A(s)\Delta s$ is $\left[ \int_{\tau}^{t} a_{ij}(s)\Delta s \right]$. Using formula (4.6), we conclude that

$$e_A(t, 0)$$

$$= \begin{bmatrix} 1 + \int_{0}^{t} s_1 \Delta s_1 + \int_{0}^{s_1} s_2 \Delta s_2 \Delta s_1 + \ldots & \int_{0}^{t} \Delta s_1 + \int_{0}^{s_1} \int_{0}^{s_2} \Delta s_2 \Delta s_1 + \ldots \\ \end{bmatrix} \quad (4.7)$$
and
\[
e_{A}(0, t) = \left[ 1 + \int_{t}^{0} s_{1} \Delta s_{1} + \int_{t}^{0} s_{1} \int_{0}^{s_{1}} s_{2} \Delta s_{2} \Delta s_{1} + \ldots + \int_{t}^{0} \Delta s_{1} + \int_{t}^{0} s_{1} \int_{0}^{s_{1}} \Delta s_{2} \Delta s_{1} + \ldots \right].
\]

(4.8)

Now, assume that \( T = \mathbb{R} \). Simple calculations show that
\[
e_{A}(t, 0) = \left[ e^{\frac{t^{2}}{2}} \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} 0 \right], \quad t \in \mathbb{R}.
\]

(4.9)

and
\[
e_{A}(0, t) = \left[ e^{-\frac{t^{2}}{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n}t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} 0 \right], \quad t \in \mathbb{R}.
\]

(4.10)

Using Relation (4.9), we obtain
\[
A(t)e_{A}(t, 0) = \left[ te^{\frac{t^{2}}{2}} 1 + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} 0 \right],
\]

(4.11)

and
\[
e_{A}(t, 0)A(t) = \left[ te^{\frac{t^{2}}{2}} e^{\frac{t^{2}}{2}} 0 \right].
\]

(4.12)

It follows, from Relations (4.11) and (4.12), that
\[
A(t)e_{A}(t, 0) \neq e_{A}(t, 0)A(t), \quad \text{for every } t \neq 0, t \in \mathbb{R}.
\]

Since
\[
det e_{A}(t, 0) = e^{t^{2}/2} \neq 0 \forall t \in \mathbb{R},
\]

(4.13)

then the inverse of \( e_{A}(t, 0) \) exists. By Relation (4.9), the inverse of \( e_{A}(t, 0) \) is given by
\[
e_{A}^{-1}(t, 0) = e^{-t^{2}/2} \left[ 1 - \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} e^{\frac{t^{2}}{2}} \right] 0
\]
\[
= \left[ e^{-t^{2}/2} - e^{-t^{2}/2} \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n}(2i-1)} 1 \right].
\]

(4.14)
which satisfies
\[ e_A^{-1}(t, 0)e_A(t, 0) = e_A(t, 0)e_A^{-1}(t, 0) = I. \]

It can be shown that
\[ e^{-t^2/2} \sum_{n=1}^{\infty} \prod_{i=1}^{n} (2i - 1) \frac{t^{2n-1}}{2^{n}} = \sum_{n=1}^{\infty} \frac{(-1)^{3n-1} t^{2n-1}}{2^{n} (2n - 1)}, \]

with the convention \( \prod_{i=1}^{0} a_i = 1 \). Consequently, Relation (4.14) yields
\[ e_A^{-1}(t, 0) = \begin{bmatrix} e^{-t^2/2} & -\sum_{n=1}^{\infty} \frac{(-1)^{3n-1} t^{2n-1}}{\prod_{i=1}^{n} (2i - 1)} \\ 0 & 1 \end{bmatrix}. \quad (4.15) \]

Now we show that the condition: \( e_A(t, 0) \) commutes with \( (\otimes A)(t) \) is essential for \( e_A^{-1}(t, 0) \) to equal \( e_A(0, t) \). One can see that \( (\otimes A)(t) = -A(t) \), \( t \in \mathbb{R} \). Hence
\[ (\otimes A)(t) = \begin{bmatrix} -t & -1 \\ 0 & 0 \end{bmatrix}. \quad (4.16) \]

Then
\[ e_{\otimes A}(t, 0) = \begin{bmatrix} e^{-t^2/2} & \sum_{n=1}^{\infty} \frac{(-1)^{n} t^{2n-1}}{\prod_{i=1}^{n} (2i - 1)} \\ 0 & 1 \end{bmatrix}. \quad (4.17) \]

Relations (4.10), (4.15) and (4.17) imply that
\[ e_A(0, t) = e_{\otimes A}(t, 0) \neq e_A^{-1}(t, 0), \quad \forall t \in \mathbb{R} \setminus \{0\}. \quad (4.18) \]

We have
\[ e_{\otimes A}(t, 0)A(t) = \begin{bmatrix} te^{-t^2/2} & e^{-t^2} \\ 0 & 0 \end{bmatrix}, \quad (4.19) \]

and
\[ A(t)e_{\otimes A}(t, 0) = \begin{bmatrix} te^{-t^2/2} & 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n} t^{2n}}{\prod_{i=1}^{n} (2i - 1)} \\ 0 & 0 \end{bmatrix}. \quad (4.20) \]

Then from Equations (4.19) and (4.20), we obtain
\[ e_{\otimes A}(t, 0)A(t) \neq A(t)e_{\otimes A}(t, 0), \quad t \neq 0. \quad (4.21) \]
Similarly one can see that $e_A(t, 0)$ and $(\ominus A)(t)$ do not commute. Indeed,

$$e_A(t, 0)(\ominus A)(t) = \begin{bmatrix} -te^{-\frac{t^2}{2}} & -e^{-\frac{t^2}{2}} \\ 0 & 0 \end{bmatrix},$$  \hspace{1cm} (4.22)

and

$$(\ominus A)(t)e_A(t, 0) = \begin{bmatrix} -e^{-\frac{t^2}{2}} & -\sum_{n=1}^{\infty} \frac{(-1)^n t^{2n}}{\prod_{i=1}^{n}(2i-1)} - 1 \\ 0 & 0 \end{bmatrix}.$$  \hspace{1cm} (4.23)

Therefore

$$e_A(t, 0)(\ominus A)(t) \neq (\ominus A)(t)e_A(t, 0), \hspace{0.5cm} t \neq 0.$$  \hspace{1cm} (4.24)

We can verify Theorem 3.4 (ii) as follows. Since $A(t) \in M_n(\mathbb{R})$, then the conjugate transpose of $A(t)$ is

$$A^*(t) = \begin{bmatrix} t & 0 \\ 1 & 0 \end{bmatrix}.$$  

Using Definition 2.8 part (ii) and Lemma 2.10 part (iii) to obtain $(\ominus A^*)(t) = -A^*(t)$. Consequently

$$(\ominus A^*)(t) = \begin{bmatrix} -t & 0 \\ -1 & 0 \end{bmatrix}.$$
Hence, \( e_{\ominus A^*}(t, 0) = e_{-A^*}(t, 0) \). Thus

\[
\begin{align*}
e_{\ominus A^*}(t, 0) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left[ -\int_0^t s_1 ds_1 \right] 0 + \left[ \int_0^t s_1 \int_0^{s_1} s_2 ds_1 \Delta s_1 \right] 0 \\
&+ \left[ -\int_0^t s_1 \int_0^{s_1} s_2 \int_0^{s_2} s_3 ds_3 ds_1 \right] 0 \\
&+ \left[ \int_0^t s_1 \int_0^{s_1} s_2 \int_0^{s_2} \Delta s_3 ds_3 \right] 0 \\
&+ \left[ \int_0^t s_1 \int_0^{s_1} s_2 \int_0^{s_2} \int_0^{s_3} s_4 ds_4 ds_3 ds_1 \right] 0 + \ldots \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \left[ -\frac{t^2}{2} \right] 0 + \left[ -\frac{t^4}{2.4} \right] 0 + \left[ -\frac{t^6}{2.4.6} \right] 0 \\
&+ \left[ \frac{t^8}{2.4.6.8} \right] 0 + \ldots \\
&= \begin{bmatrix} 1 - \frac{1}{1!} \left( \frac{t^2}{2} \right) + \frac{1}{2!} \left( \frac{t^2}{2} \right)^2 - \frac{1}{3!} \left( \frac{t^2}{2} \right)^3 + \ldots & 0 \\ -t + \frac{t^3}{2.3} - \frac{t^5}{2.4.5} + \frac{t^7}{2.4.6.7} - \ldots & 1 \end{bmatrix} \\
&= \begin{bmatrix} e^{-\frac{t^2}{2}} & 0 \\ -\sum_{n=1}^{\infty} \frac{(-1)^{3n-1} t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} & 1 \end{bmatrix}.
\end{align*}
\]

Consequently, we conclude that

\[
e_{A^*}^*(t, 0) = \begin{bmatrix} e^{-\frac{t^2}{2}} & -\sum_{n=1}^{\infty} \frac{(-1)^{3n-1} t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} \\ 0 & 1 \end{bmatrix}.
\]
Hence, by Equations (4.15) and (4.25), we obtain
\[ e^{-t}A(t,0) = e^{-t}A(t,0), \quad \forall \ t \in \mathbb{R}. \] (4.26)

**Example 4.3.** Now, we show that the condition: \( e_A(t,0) \) commutes with \( B(t) \), in Theorem 3.3 part (iii), is essential for \( e_{A\oplus B}(t,0) \) to equal \( e_A(t,0)e_B(t,0) \). Let \( A(t) \) and \( B(t) \) be the regressive \( 2 \times 2 \)-matrices of differentiable functions defined by
\[ A(t) = \begin{bmatrix} t & 1 \\ 0 & 0 \end{bmatrix}, \quad B(t) = \begin{bmatrix} -t & 0 \\ 1 & 0 \end{bmatrix}. \] (4.27)

Consequently, we conclude that
\[ e_B(t,0) = \begin{bmatrix} e^{-t/2} & 0 \\ \sum_{n=1}^{\infty} \frac{(-1)^{3n-1}t^{2n-1}}{(\prod_{i=1}^{n-1} 2i)(2n-1)} & 1 \end{bmatrix}. \] (4.28)

Thus
\[ e_A(t,0)B(t) = \begin{bmatrix} -te^{t/2} + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n} (2i-1)} & 0 \\ 1 & 0 \end{bmatrix}, \] (4.29)

and
\[ B(t)e_A(t,0) = \begin{bmatrix} -te^{t/2} - \sum_{n=1}^{\infty} \frac{t^{2n}}{\prod_{i=1}^{n} (2i-1)} & \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n} (2i-1)} \\ e^{t/2} & \sum_{n=1}^{\infty} \frac{t^{2n}}{\prod_{i=1}^{n} (2i-1)} \end{bmatrix}, \] (4.30)

from which we have
\[ e_A(t,0)B(t) \neq B(t)e_A(t,0) \quad \forall \ t \in \mathbb{R}\{0\}. \] (4.31)

By using Relations (4.9) and (4.28), we get
\[ e_A(t,0)e_B(t,0) = \begin{bmatrix} 1 + \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n} (2i-1)} \sum_{n=1}^{\infty} \frac{(-1)^{3n-1}t^{2n-1}}{\prod_{i=1}^{n-1} 2i(2n-1)} & \sum_{n=1}^{\infty} \frac{t^{2n-1}}{\prod_{i=1}^{n} (2i-1)} \\ \sum_{n=1}^{\infty} \frac{(-1)^{3n-1}t^{2n-1}}{\prod_{i=1}^{n-1} 2i(2n-1)} & 1 \end{bmatrix}. \] (4.32)
Since \( \mu(t) = 0, t \in \mathbb{R} \), we have
\[
(A \oplus B)(t) = A(t) + B(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]

So,
\[
e_{A \oplus B}(t, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ds_1 + \int_0^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ds_2 ds_1
\]
\[
+ \int_0^t \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ds_3 ds_2 ds_1 + \ldots
\]
\[
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix} + \begin{bmatrix} \frac{t^2}{2} & 0 \\ 0 & \frac{t^2}{2} \end{bmatrix} + \begin{bmatrix} 0 & \frac{t^3}{2.3} \\ \frac{t^3}{2.3} & 0 \end{bmatrix} + \begin{bmatrix} \frac{t^4}{2.3.4} & 0 \\ 0 & \frac{t^4}{2.3.4} \end{bmatrix}
\]
\[
+ \begin{bmatrix} 0 & \frac{t^5}{2.3.4.5} \\ \frac{t^5}{2.3.4.5} & 0 \end{bmatrix} + \ldots
\]
\[
= \begin{bmatrix} 1 + \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} + \frac{t^6}{1.2.3.4.5.6} + \ldots & t + \frac{t^3}{1.2.3} + \frac{t^5}{1.2.3.4.5} + \ldots \\
\frac{t^3}{1.2.3} & \frac{t^5}{1.2.3.4.5} + \frac{t^7}{1.2.3.4.5.6.7} + \ldots & 1 + \frac{t^2}{1.2} + \frac{t^4}{1.2.3.4} + \ldots \end{bmatrix}.
\]

Consequently, we conclude that
\[
e_{A \oplus B}(t, 0) = \left[ \sum_{n=1}^{\infty} \frac{t^{2n-2}}{(2n-2)!} \right] \left[ \sum_{n=1}^{\infty} \frac{t^{2n-1}}{(2n-1)!} \right] \left[ \sum_{n=1}^{\infty} \frac{t^{2n-2}}{(2n-2)!} \right]. \tag{4.33}
\]

Then
\[
e_{A \oplus B}(t, 0) \neq e_A(t, 0) e_B(t, 0) \quad \forall t \in \mathbb{R} \setminus \{0\}. \tag{4.34}
\]
Finally, we verify Theorem 3.3 (iii) as follows:

**Example 4.4.** In this example we give two $2 \times 2$-matrices $C(t)$ and $D(t)$ such that the commutativity condition mentioned in Theorem 3.3 (iii) is satisfied and $e_{C \oplus D}(t, 0) = e_C(t, 0)e_D(t, 0)$. Take the matrices $C(t)$ and $D(t)$ as follows:

\[
C(t) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix},
\]

and

\[
D(t) = \begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix}.
\]

Consequently, we conclude that

\[
e_C(t, 0) = \begin{bmatrix} e^{t^2} & 0 \\ 0 & 1 \end{bmatrix},
\]

and

\[
e_D(t, 0) = \begin{bmatrix} e^{-t^2} & 0 \\ 0 & e^{-t^2} \end{bmatrix}.
\]

We have

\[
e_C(t, 0)e_D(t, 0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t^2} \end{bmatrix},
\]

and

\[
e_D(t, 0)e_C(t, 0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t^2} \end{bmatrix}.
\]

By Relations (4.36), (4.37), (4.35) and (4.38) we obtain

\[
 D(t)e_C(t, 0) = \begin{bmatrix} -t & 0 \\ 0 & -t \end{bmatrix} \begin{bmatrix} e^{t^2} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t^2} \end{bmatrix} = e_C(t, 0)D(t),
\]

and

\[
 C(t)e_D(t, 0) = \begin{bmatrix} t & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e^{-t^2} & 0 \\ 0 & e^{-t^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t^2} \end{bmatrix} = e_D(t, 0)C(t).
\]

Since $(C \oplus D)(t) = C(t) + D(t) = \begin{bmatrix} 0 & 0 \\ 0 & -t \end{bmatrix}$, we have

\[
e_{C \oplus D}(t, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -t \end{bmatrix} - \int_0^t ds_1 + \int_0^t \left[ \begin{bmatrix} 0 & 0 \\ 0 & -s_1 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & -s_2 \end{bmatrix} ds_2 ds_1 ight. \\
+ \left. \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & -s_1 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & -s_2 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 0 \\ 0 & -s_3 \end{bmatrix} ds_3 ds_2 ds_1 \right. \\
+ \left. \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & -s_1 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & -s_2 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 0 \\ 0 & -s_3 \end{bmatrix} ds_3 ds_2 ds_1 \right.
\]
\[
+ \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & -s_1 \end{bmatrix} \int_0^{s_1} \begin{bmatrix} 0 & 0 \\ 0 & -s_2 \end{bmatrix} \int_0^{s_2} \begin{bmatrix} 0 & 0 \\ 0 & -s_3 \end{bmatrix} \int_0^{s_3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} ds_4 ds_3 ds_2 ds_1 + \ldots \\
= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{t^2}{2} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{t^4}{2.4} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -\frac{t^6}{2.4.6} \end{bmatrix} + \ldots \\
= \begin{bmatrix} 1 & 0 \\ 0 & 1 - \frac{t^2}{2} + \frac{t^4}{2.4} - \frac{t^6}{2.4.6} + \ldots \end{bmatrix}.
\]

Thus
\[
e_{C \oplus D}(t, 0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\frac{t^2}{2}} \end{bmatrix}. \quad (4.43)
\]

Relations (4.39) and (4.43) yield
\[
e_{C \oplus D}(t, 0) = e_C(t, 0)e_D(t, 0) \quad \forall \ t \in \mathbb{R}. \quad (4.44)
\]

References


