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q-analogues of Triple Series Reduction Formulas due to Srivastava and Panda with General Terms

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Abstract

We find six double q-analogues of the reduction formulas by Srivastava and Panda for triple series, with general terms. These are then spezialized to summations for q-Kampé de Fériet functions.

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1 Introduction

The history of the theme in this article goes many years back. It deals with equalities between quite general triple and double sums. The coefficients consist of q-shifted factorials, Γ_q-functions and q-binomial coefficients. Srivastava and Panda [10] collected these formulas, previously scattered in the literature into 6 formulas of a slightly different character than we present here (for $q = 1$). In Section 2 we find double q-forms of these, by the 2 q-Vandermonde summation formulas, i.e., 12 q-forms. These are then specialized, in Section 3, to six summation formulas for q-Kampé de Fériet functions. We will use the $\triangle(q; l; \lambda)$ operator, which is explored in detail in [6].

We turn to the first definitions.

Definition 1.1. The power function is defined by $q^a \equiv e^{a \log(q)}$. Let $\delta > 0$ be an arbitrary small number. We will use the following branch of the logarithm: $-\pi + \delta < \mathrm{Im}\,(\log q) \leq \pi + \delta$. This defines a simply connected space in the complex plane.

The variables $a, b, c, \ldots \in \mathbb{C}$ denote certain parameters. The variables i, j, k, l, m, n, p, r will denote natural numbers except for certain cases where it will be clear from

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the context that i will denote the imaginary unit. Let the q-shifted factorial be defined by

$$\langle a;q\rangle_n \equiv \begin{cases} 1, & n=0; \\ \prod_{m=0}^{n-1}(1-q^{a+m}) & n=1,2,\ldots. \end{cases} \tag{1.1}$$

Since products of q-shifted factorials occur so often, to simplify them we shall frequently use the more compact notation

$$\langle a_1,\ldots,a_m;q\rangle_n \equiv \prod_{j=1}^{m}\langle a_j;q\rangle_n. \tag{1.2}$$

The operator

$$\widetilde{}: \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{\pi i}{\log q}. \tag{1.3}$$

By (1.3) it follows that

$$\widetilde{\langle a;q\rangle_n} = \prod_{m=0}^{n-1}(1+q^{a+m}), \tag{1.4}$$

Assume that $(m,l)=1$, i.e., m and l relatively prime. The operator

$$\widetilde{\tfrac{m}{l}} : \frac{\mathbb{C}}{\mathbb{Z}} \mapsto \frac{\mathbb{C}}{\mathbb{Z}}$$

is defined by

$$a \mapsto a + \frac{2\pi i m}{l\log q}. \tag{1.5}$$

We will also need another generalization of the tilde operator.

$$_k\langle\widetilde{a};q\rangle_n \equiv \prod_{m=0}^{n-1}\Big(\sum_{i=0}^{k-1} q^{i(a+m)}\Big). \tag{1.6}$$

This leads to the following q-analogue of [11, p.22, (2)].

Theorem 1.2 (See [4])**.**

$$\langle a;q\rangle_{kn} = \prod_{m=0}^{k-1}\langle\frac{a+m}{k};q\rangle_n \times_k \langle\frac{\widetilde{a+m}}{k};q\rangle_n. \tag{1.7}$$

The following notation will be convenient:

$$\text{QE}(x) \equiv q^x.$$

Definition 1.3.

$$\langle\triangle(q;l;\lambda);q\rangle_n \equiv \prod_{m=0}^{l-1}\langle\frac{\lambda+m}{l};q\rangle_n \times_l \langle\frac{\widetilde{\lambda+m}}{l};q\rangle_n. \tag{1.8}$$

When λ is a vector, we mean the corresponding product of vector elements. When λ is replaced by a sequence of numbers separated by commas, we mean the corresponding product as in the case of q-shifted factorials.

Definition 1.4. Let

$$(a),(b),(g_i),(h_i),(a'),(b'),(g'_i),(h'_i)$$

have dimensions

$$A,B,G_i,H_i,A',B',G'_i,H'_i.$$

Let

$$1+B+B'+H_i+H'_i-A-A'-G_i-G'_i\geq 0, i=1,\dots,n.$$

Then the generalized q-Kampé de Fériet function is defined by

$$\Phi^{A+A':G_1+G'_1;\dots;G_n+G'_n}_{B+B':H_1+H'_1;\dots;H_n+H'_n}\left[\begin{array}{c}(\hat{a}):(\hat{g_1});\dots;(\hat{g_n})\\(\hat{b}):(\hat{h_1});\dots;(\hat{h_n})\end{array}|\vec{q};\vec{x}||\begin{array}{c}(a'):(g'_1);\dots;(g'_n)\\(b'):(h'_1);\dots;(h'_n)\end{array}\right]\equiv$$
$$\sum_{\vec{m}}\frac{\langle(\hat{a});q_0\rangle_m(a')(q_0,m)\prod_{j=1}^n(\langle(\hat{g_j});q_j\rangle_{m_j}((g'_j)(q_j,m_j)x_j^{m_j})}{\langle(\hat{b});q_0\rangle_m(b')(q_0,m)\prod_{j=1}^n(\langle(\hat{h_j});q_j\rangle_{m_j}(h'_j)(q_j,m_j)\langle1;q_j\rangle_{m_j})}\times$$
$$(-1)^{\sum_{j=1}^n m_j(1+H_j+H'_j-G_j-G'_j+B+B'-A-A')}\times$$
$$\mathrm{QE}\left((B+B'-A-A')\binom{m}{2},q_0\right)\prod_{j=1}^n\mathrm{QE}\left((1+H_j+H'_j-G_j-G'_j)\binom{m_j}{2},q_j\right). \tag{1.9}$$

We assume that no factors in the denominator are zero. We assume that

$$(a')(q_0,m),(g'_j)(q_j,m_j),(b')(q_0,m),(h'_j)(q_j,m_j)$$

contain factors of the form $\langle a(\hat{k});q\rangle_k$, $(s;q)_k$, $(s(k);q)_k$ or $\mathrm{QE}\left(f(\vec{m})\right)$.

Definition 1.5. Let the Gauss q-binomial coefficient be defined by

$$\binom{n}{k}_q \equiv \frac{\langle1;q\rangle_n}{\langle1;q\rangle_k\langle1;q\rangle_{n-k}},\ k=0,1,\dots,n. \tag{1.10}$$

Let the Γ_q-function be defined in the unit disk $0<|q|<1$ by

$$\Gamma_q(x)\equiv\frac{\langle1;q\rangle_\infty}{\langle x;q\rangle_\infty}(1-q)^{1-x}. \tag{1.11}$$

2 Certain q-summation Formulas

Srivastava [12] and Panda & Srivastava [10] have systematically collected and generalized a number of related summation formulas known from the literature. Our task in this section is to find symmetric q-analogues of these formulas, which always occur in pairs. In certain exceptional cases the convergence in the formulas is not so good, we then replace the equality sign by the sign for formal equality, $\cong$.

We assume throughout that $M = km + ln$ and $\{C(m,n)\}_{m,n=0}^{\infty}$ is a sequence of bounded complex numbers.

Theorem 2.1 (Compare [10, (4) p. 244] and [12, (9) p. 28])**.**

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta+r;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}q^{\binom{r}{2}-rM}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{-NM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}. \tag{2.1}$$

Proof. We have

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta+r;q\rangle_M\frac{x^m y^n q^{\binom{r}{2}-rM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{r(-\alpha+\beta-M+N)}=$$
$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$
$$q^{r(-\alpha+\beta-M+N)}=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-\alpha;q\rangle_N}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{-NM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}. \tag{2.2}$$

The proof is complete. $\square$

Theorem 2.2 (Compare [10, (4) p. 244]).

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta+r;q\rangle_M\frac{x^m y^n q^{\binom{r}{2}+r(\alpha-\beta-N+1)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}. \tag{2.3}$$

Proof. We have

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta+r;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$
$$\mathrm{QE}\left(\binom{r}{2}+r(\alpha-\beta-N+1)\right)=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}$$
$$\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r=$$
$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\langle 1-\beta;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r=$$
$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-\alpha;q\rangle_N}q^{N(1-\beta+M)}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}. \tag{2.4}$$

The proof is complete. □

Theorem 2.3 (Compare [10, (5) p. 244] and [12, (8) p. 28]).

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M}\frac{x^m y^n q^{\binom{r}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{1}{\langle 1-\alpha+N;q\rangle_M}. \tag{2.5}$$

Proof. We have

$$
\begin{aligned}
&LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M}\frac{x^m y^n q^{\binom{r}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_{r+M}}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{r(-\alpha+\beta+N)}=\\
&\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle 1-\beta;q\rangle_M}{\langle 1-\alpha;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}\\
&q^{r(-\alpha+\beta+N)}=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta;q\rangle_N}{\langle 1-\alpha;q\rangle_M\langle 1-\alpha+M;q\rangle_N}=\\
&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{1}{\langle 1-\alpha+N;q\rangle_M}.
\end{aligned}
\tag{2.6}
$$

The proof is complete. □

Theorem 2.4 (Compare [10, (5) p. 244])**.**

$$
\begin{aligned}
&\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&\mathrm{QE}\left(\binom{r}{2}+r(\alpha-\beta-N+1)\right)=\\
&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{N(1-\beta+M)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1-\alpha+N;q\rangle_M}.
\end{aligned}
\tag{2.7}
$$

Proof. We have

$$
\begin{aligned}
&LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle 1-\beta+r;q\rangle_M}{\langle 1-\alpha+r;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&\mathrm{QE}\left(\binom{r}{2}+r(\alpha-\beta-N+1)\right)=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_{r+M}}{\langle 1-\alpha;q\rangle_{r+M}}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle 1-\beta;q\rangle_M}{\langle 1-\alpha;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&\sum_{r=0}^{N}\frac{\langle 1-\beta+M;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&\frac{\langle -\alpha+\beta;q\rangle_N q^{N(1-\beta+M)}}{\langle 1-\alpha;q\rangle_M\langle 1+\alpha+M;q\rangle_N}=\\
&\frac{\langle \beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle 1-\beta;q\rangle_M q^{N(1-\beta+M)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle 1-\alpha+N;q\rangle_M}.
\end{aligned}
\tag{2.8}
$$

The proof is complete. □

Theorem 2.5 (Compare [10, (6) p. 244] and [12, (10) p. 29])**.**

$$
\begin{aligned}
&\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle \alpha-r;q\rangle_M}{\langle \beta-r;q\rangle_M}\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\mathrm{QE}\left(\binom{r}{2}\right)\\
&=\frac{\langle \beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n q^{NM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle \alpha-N;q\rangle_M}{\langle \beta;q\rangle_M}.
\end{aligned}
\tag{2.9}
$$

Proof. We have

$$
\begin{aligned}
&LHS = \sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle\alpha-r;q\rangle_M}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&= \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\alpha;q\rangle_{M-r}}{\langle\beta;q\rangle_{M-r}}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{rN} =\\
&\frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta-M;q\rangle_r}{\langle 1-\alpha-M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}\\
&q^{r(-\alpha+\beta+N)} = \frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta;q\rangle_N}{\langle 1-\alpha-M;q\rangle_N} =\\
&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^nq^{NM}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle\alpha-N;q\rangle_M}{\langle\beta;q\rangle_M}.
\end{aligned}
\tag{2.10}
$$

The proof is complete. ☐

Theorem 2.6 (Compare [10, (6) p. 244])**.**

$$
\begin{aligned}
&\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle\alpha-r;q\rangle_M}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}+r(\alpha-\beta-N+1)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&= \frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^nq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle\alpha-N;q\rangle_M}{\langle\beta;q\rangle_M}.
\end{aligned}
\tag{2.11}
$$

Proof. We have

$$
\begin{aligned}
&LHS = \sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\frac{\langle\alpha-r;q\rangle_M}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}+r(\alpha-\beta-N+1)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\
&= \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\alpha;q\rangle_{M-r}}{\langle\beta;q\rangle_{M-r}}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{r(\alpha-\beta+1)} =\\
&\frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta-M;q\rangle_r}{\langle 1-\alpha-M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^r\\
&= \frac{\langle\alpha;q\rangle_M}{\langle\beta;q\rangle_M}\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta;q\rangle_Nq^{N(1-\beta-M)}}{\langle 1-\alpha-M;q\rangle_N} =\\
&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle\alpha-N;q\rangle_Mq^{N(1-\beta)}}{\langle\beta;q\rangle_M}.
\end{aligned}
\tag{2.12}
$$

The proof is complete. □

Theorem 2.7 (Compare [10, (7) p. 244] and [12, (11) p. 29]).

$$\begin{aligned}&\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle\beta-r;q\rangle_M}\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}q^{\binom{r}{2}}\cong\\&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^nq^{\binom{r}{2}}}{\langle\beta;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle-\alpha+\beta+N;q\rangle_M}{\langle-\alpha+\beta;q\rangle_M}.\end{aligned}\tag{2.13}$$

Proof. We have

$$\begin{aligned}&LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle\beta-r;q\rangle_M}\frac{x^my^nq^{\binom{r}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\\&=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle-N;q\rangle_r}{\langle\alpha-r;q\rangle_r\langle 1;q\rangle_r}\frac{q^{rN}}{\langle\beta;q\rangle_{M-r}}=\\&\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)\langle\beta;q\rangle_M}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta-M;q\rangle_r}{\langle 1-\alpha;q\rangle_r}\frac{\langle-N;q\rangle_r}{\langle 1;q\rangle_r}\\&q^{r(-\alpha+\beta+M+N)}=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle\beta;q\rangle_M}\frac{\langle-\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha;q\rangle_N}=\\&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n}{\langle\beta;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle-\alpha+\beta+N;q\rangle_M}{\langle-\alpha+\beta;q\rangle_M}.\end{aligned}\tag{2.14}$$

The proof is complete. □

Remark 2.8. The convergence in formula (2.13) is not so good. However, this formula works well in a number of special cases. One example is

$$C(m,n)=1,q=.85,\alpha=5.3,\beta=5.543,N=4,x=.2,y=.176.\tag{2.15}$$

In this case the difference LHS-RHS in (2.13) is 4.09273×10^{-12} for $0\leq m\leq 60,\ 0\leq n\leq 60$.

Theorem 2.9 (Compare [10, (7) p. 244]).

$$\begin{aligned}&\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle\beta-r;q\rangle_M}\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}q^{\binom{r}{2}+r(1+\alpha-\beta-M-N)}=\\&\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^nq^{N(1-\beta-M)}}{\langle\beta;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle-\alpha+\beta+N;q\rangle_M}{\langle-\alpha+\beta;q\rangle_M}.\end{aligned}\tag{2.16}$$

Proof. We have

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)}{\langle\beta-r;q\rangle_M}\frac{x^m y^n q^{r(1+\alpha-\beta-M-N}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}q^{\binom{r}{2}+1-2M}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle -N;q\rangle_r}{\langle\alpha-r;q\rangle_r\langle 1;q\rangle_r}\frac{q^{r(1+\alpha-\beta-M)}}{\langle\beta;q\rangle_{M-r}}=$$
$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)\langle\beta;q\rangle_M}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta-M;q\rangle_r}{\langle 1-\alpha;q\rangle_r}\frac{\langle -N;q\rangle_r q^r}{\langle 1;q\rangle_r}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^m y^n q^{N(1-\beta-M)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n\langle\beta;q\rangle_M}\frac{\langle -\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha;q\rangle_N}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^m y^n q^{N(1-\beta-M)}}{\langle\beta;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle -\alpha+\beta;q\rangle_M}.\tag{2.17}$$

The proof is complete. □

Theorem 2.10 (Compare [10, (8) p. 244] and [12, (12) p. 29])**.**

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha-r;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\mathrm{QE}\left(\binom{r}{2}\right)=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n\langle\alpha-N;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.\tag{2.18}$$

Proof. We have

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha-r;q\rangle_M\frac{x^m y^n q^{\binom{r}{2}}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\beta-r;q\rangle_r\langle\alpha;q\rangle_{M-r}\langle -N;q\rangle_r}{\langle 1;q\rangle_r}q^{rN}=$$
$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha;q\rangle_M\frac{x^m y^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_r q^{r(-\alpha+\beta-M+N)}}{\langle 1-\alpha-M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^m y^n\langle\alpha;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-M-\alpha;q\rangle_N}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^m y^n\langle\alpha-N;q\rangle_M}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}.\tag{2.19}$$

The proof is complete. □

Theorem 2.11 (Compare [10, (8) p. 244]).

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha-r;q\rangle_M\frac{x^my^nq^{\binom{r}{2}+r(1+\alpha-\beta+M-N)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\cong$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n\langle 1-\beta;q\rangle_Mq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}. \tag{2.20}$$

Proof. We have

$$LHS=\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha-r;q\rangle_M\frac{x^my^nq^{\binom{r}{2}+r(1+\alpha-\beta+M-N)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle\beta-r;q\rangle_r\langle\alpha;q\rangle_{M-r}\langle -N;q\rangle_rq^{r(1+\alpha-\beta+M)}}{\langle 1;q\rangle_r}=$$
$$\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\langle\alpha;q\rangle_M\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\sum_{r=0}^{N}\frac{\langle 1-\beta;q\rangle_rq^r}{\langle 1-\alpha-M;q\rangle_r}\frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r}$$
$$=\frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^n\langle\alpha;q\rangle_Mq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta-M;q\rangle_N}{\langle 1-M-\alpha;q\rangle_N}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}\frac{x^my^n\langle\alpha-N;q\rangle_Mq^{N(1-\beta)}}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle 1+\alpha-\beta;q\rangle_M}{\langle 1+\alpha-\beta-N;q\rangle_M}. \tag{2.21}$$

The proof is complete. □

Remark 2.12. The convergence in formula (2.20) is not so good. However, this formula works somehow in a number of special cases. One example is

$$C(m,n)=1,q=.99,\alpha=4.3,\beta=5.543,N=4,x=.1,y=.076. \tag{2.22}$$

In this case the difference LHS-RHS in (2.20) is 0.0000127104 for $0\leq m\leq 20,\ 0\leq n\leq 20$.

Theorem 2.13 (Compare [10, (9) p. 244] and [12, (13) p. 29]).

$$\sum_{r=0}^{N}(-1)^r\binom{N}{r}_q\frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)}\sum_{m,n=0}^{\infty}\frac{C(m,n)x^my^nq^{\binom{r}{2}+rM}}{\langle 1-\alpha+r;q\rangle_M\langle 1;q\rangle_m\langle 1;q\rangle_n}=$$
$$\frac{\langle\beta-\alpha;q\rangle_N\Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N\Gamma_q(\beta)}\sum_{m,n=0}^{\infty}C(m,n)\frac{x^my^n}{\langle 1;q\rangle_m\langle 1;q\rangle_n}\frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}. \tag{2.23}$$

Proof. We have

$$
\begin{aligned}
LHS = {} & \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n q^{\binom{r}{2}+rM}}{\langle 1-\alpha+r;q\rangle_M \langle 1;q\rangle_m \langle 1;q\rangle_n} \\
= {} & \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha;q\rangle_{r+M}} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r} q^{r(-\alpha+\beta+M+N)} \\
= {} & \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)}{\langle 1-\alpha;q\rangle_M} \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r} \\
& q^{r(-\alpha+\beta+M+N)} = \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)}{\langle 1-\alpha;q\rangle_M} \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha+M;q\rangle_N} = \\
& \frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}.
\end{aligned}
\tag{2.24}
$$

The proof is complete. □

Theorem 2.14 (Compare [10, (9) p. 244])**.**

$$
\begin{aligned}
& \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n q^{\binom{r}{2}+r(\alpha-\beta+1-N)}}{\langle 1-\alpha+r;q\rangle_M \langle 1;q\rangle_m \langle 1;q\rangle_n} = \\
& \frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n q^{NM}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+N;q\rangle_M q^{N(1-\beta)}}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}.
\end{aligned}
\tag{2.25}
$$

Proof. We have

$$
\begin{aligned}
LHS = {} & \sum_{r=0}^{N}(-1)^r \binom{N}{r}_q \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n q^{\binom{r}{2}+r(\alpha-\beta+1-N)}}{\langle 1-\alpha+r;q\rangle_M \langle 1;q\rangle_m \langle 1;q\rangle_n} \\
= {} & \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha;q\rangle_{r+M}} \frac{\langle -N;q\rangle_r}{\langle 1;q\rangle_r} q^r \\
= {} & \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)}{\langle 1-\alpha;q\rangle_M} \frac{x^m y^n}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \sum_{r=0}^{N} \frac{\langle 1-\beta;q\rangle_r}{\langle 1-\alpha+M;q\rangle_r} \frac{\langle -N;q\rangle_r q^r}{\langle 1;q\rangle_r} \\
= {} & \frac{\Gamma_q(\alpha)}{\Gamma_q(\beta)} \sum_{m,n=0}^{\infty} \frac{C(m,n)}{\langle 1-\alpha;q\rangle_M} \frac{x^m y^n q^{N(1-\beta)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+M;q\rangle_N}{\langle 1-\alpha+M;q\rangle_N} = \\
& \frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)} \sum_{m,n=0}^{\infty} C(m,n) \frac{x^m y^n q^{N(1-\beta)}}{\langle 1;q\rangle_m \langle 1;q\rangle_n} \frac{\langle -\alpha+\beta+N;q\rangle_M}{\langle 1-\alpha+N,-\alpha+\beta;q\rangle_M}.
\end{aligned}
\tag{2.26}
$$

The proof is complete. □

3 Specializations

Specializing the previous formulas leads to the following formulas for two variables, where we have put

$$\theta(N;\alpha,\beta;r;q) \equiv (-1)^r \binom{N}{r}_q q^{\binom{r}{2}} \frac{\Gamma_q(\alpha-r)}{\Gamma_q(\beta-r)};\ \omega(N;\alpha,\beta;q) \equiv \frac{\langle \beta-\alpha;q\rangle_N \Gamma_q(\alpha)}{\langle 1-\alpha;q\rangle_N \Gamma_q(\beta)}. \tag{3.1}$$

We have assumed that $B = B' = G = G'$ in all formulas. The conditions for A and E are given separately in each case.

Theorem 3.1 (Compare [10, (16) p. 246]). *We assume that* $A + 2l = E$.

$$\sum_{r=0}^{N} \theta(N;\alpha,\beta;r;q) \Phi_{E:G}^{A+2l:B} \left[\begin{matrix} \triangle(q;l;1-\beta+r),(a):(b);(b') \\ (e):(g);(g') \end{matrix} \middle| q; xq^{l(N-r)}, yq^{l(N-r)} \right] =$$
$$\omega(N;\alpha,\beta;q) \Phi_{E+2l:G}^{A+4l:B} \left[\begin{matrix} \triangle(q;l;1-\beta,1+\alpha-\beta),(a):(b);(b') \\ \triangle(q;l;1+\alpha-\beta-N),(e):(g);(g') \end{matrix} \middle| q; x, y \right]. \tag{3.2}$$

Proof. Put

$$C(m,n) = \frac{\langle (a);q\rangle_{m+n} \langle (b);q\rangle_m \langle (b');q\rangle_n}{\langle (e);q\rangle_{m+n} \langle (g);q\rangle_m \langle (g');q\rangle_n} q^{Nl(m+n)} \tag{3.3}$$

in (2.1). We have assumed that $k = l$. □

In the following proofs we use the value

$$C(m,n) = \frac{\langle (a);q\rangle_{m+n} \langle (b);q\rangle_m \langle (b');q\rangle_n}{\langle (e);q\rangle_{m+n} \langle (g);q\rangle_m \langle (g');q\rangle_n} \tag{3.4}$$

and assume that $k = l$.

Theorem 3.2 (Compare [10, (19) p. 247]). *We assume that* $A = E$.

$$\sum_{r=0}^{N} \theta(N;\alpha,\beta;r;q) \Phi_{E+2l:G}^{A+2l:B} \left[\begin{matrix} \triangle(q;l;1-\beta+r),(a):(b);(b') \\ \triangle(q;l;1-\alpha+r),(e):(g);(g') \end{matrix} \middle| q; x, y \right] =$$
$$\omega(N;\alpha,\beta;q) \Phi_{E+2l:G}^{A+2l:B} \left[\begin{matrix} \triangle(q;l;1-\beta),(a):(b);(b') \\ \triangle(q;l;1-\alpha+N),(e):(g);(g') \end{matrix} \middle| q; x, y \right]. \tag{3.5}$$

Proof. Use (2.5). □

Theorem 3.3 (Compare [10, (20) p. 247]). *We assume that* $A = E$.

$$\sum_{r=0}^{N} \theta(N;\alpha,\beta;r;q) \Phi_{E+2l:G}^{A+2l:B} \left[\begin{matrix} \triangle(q;l;\alpha-r),(a):(b);(b') \\ \triangle(q;l;\beta-r),(e):(g);(g') \end{matrix} \middle| q; x, y \right] =$$
$$\omega(N;\alpha,\beta;q) \Phi_{E+2l:G}^{A+2l:B} \left[\begin{matrix} \triangle(q;l;\alpha-N),(a):(b);(b') \\ \triangle(q;l;\beta),(e):(g);(g') \end{matrix} \middle| q; xq^{Nl}, yq^{Nl} \right]. \tag{3.6}$$

Proof. Use (2.9). □

Theorem 3.4 (Compare [10, (21) p. 247]). *We assume that $A = E + 2l$.*

$$\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)q^{r(1+\alpha-\beta-N)}\Phi_{E+2l:G}^{A:B}\left[\begin{array}{c}(a):(b);(b')\\ \triangle(q;l;\beta-r),(e):(g);(g')\end{array}|q;xq^{-rl},yq^{-rl}\right]$$
$$=\omega(N;\alpha,\beta;q)q^{N(1-\beta)}\Phi_{E+4l:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;\beta-\alpha+N),(a):(b);(b')\\ \triangle(q;l;\beta,-\alpha+\beta),(e):(g);(g')\end{array}|q;xq^{-Nl},yq^{-Nl}\right]. \tag{3.7}$$

Proof. Use (2.16). □

Theorem 3.5 (Compare [10, (22) p. 248]). *We assume that $A + 2l = E$.*

$$\begin{aligned}&\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)\Phi_{E:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;\alpha-r),(a):(b);(b')\\ (e):(g);(g')\end{array}|q;x,y\right]=\\ &\omega(N;\alpha,\beta;q)\Phi_{E+2l:G}^{A+4l:B}\left[\begin{array}{c}\triangle(q;l;\alpha-N,1+\alpha-\beta),(a):(b);(b')\\ \triangle(q;l;1+\alpha-\beta-N),(e):(g);(g')\end{array}|q;x,y\right].\end{aligned} \tag{3.8}$$

Proof. Use (2.18). □

Remark 3.6. In [9, (3.1) p. 439] Kandu tried to derive a similar formula. However, in this article the definitions are insufficient.

Remark 3.7. Formula (3.8) is a q-analogue of [8].

Theorem 3.8 (Compare [10, (23) p. 248]). *We assume that $A = E + 2l$.*

$$\begin{aligned}&\sum_{r=0}^{N}\theta(N;\alpha,\beta;r;q)\Phi_{E+2l:G}^{A:B}\left[\begin{array}{c}(a):(b);(b')\\ \triangle(q;l;1-\alpha+r),(e):(g);(g')\end{array}|q;xq^{rl},yq^{rl}\right]=\\ &\omega(N;\alpha,\beta;q)\Phi_{E+4l:G}^{A+2l:B}\left[\begin{array}{c}\triangle(q;l;-\alpha+\beta+N),(a):(b);(b')\\ \triangle(q;l;1-\alpha+N,-\alpha+\beta),(e):(g);(g')\end{array}|q;x,y\right].\end{aligned} \tag{3.9}$$

Proof. Use (2.23). □

4 Conclusion

We expect that these formulas will be of greatest value when looking for q-analogues of reductions for triple q-series. This is an investigation which has only started, and hopefully will continue in the next years. The connection with Γ_q-functions is quite interesting, as is manifested in the book [7].

References

[1] T. Ernst, A method for q-calculus. *J. nonlinear Math. Physics* **10** No.4 (2003), 487-525.

[2] T. Ernst, Some results for q-functions of many variables. *Rendiconti di Padova* **112** (2004), 199-235.

[3] T. Ernst, A renaissance for a q-umbral calculus. Proceedings of the International Conference Munich, Germany 25 - 30 July 2005. World Scientific, (2007).

[4] T. Ernst, Some new formulas involving Γ_q functions. Rendiconti di Padova **118** (2007), 159–188.

[5] T. Ernst, The different tongues of q-calculus, *Proceedings of Estonian Academy of Sciences* **57** no. 2 (2008), 81-99.

[6] T. Ernst, q-analogues of general reduction formulas by Buschman and Srivastava, together with the important $\triangle(l;\lambda)$ operator reminding of MacRobert *Demonstratio Mathematica* **XLIV** (2011), 285-296.

[7] T. Ernst, *A comprehensive treatment of q-calculus*, Birkhäuser, 2012.

[8] K.C. Gupta and A. Srivastava, Certain results involving Kampé de Fériet's function. *Indian J. Math.* **15** (1973), 99–102.

[9] D. Kandu, Certain expansions involving basic hypergeometric functions of two variables. *Indian J. Pure Appl. Math.* **18**, no. 5, (1987), 438–441.

[10] R. Panda and H.M.Srivastava, Some recursion formulas associated with multiple hypergeometric functions. *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.* 1975 **23**, no. 3 (1975), 243–250.

[11] E.D. Rainville, *Special functions*. Reprint of 1960 first edition. (Chelsea Publishing Co., Bronx, N.Y., 1971).

[12] H. M.Srivastava, Certain summation formulas involving generalized hypergeometric functions. *Comment. Math. Univ. St. Paul.* **21**, (1972/73) 25–34. **58**, 241–245.