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Existence of Solutions for Nonlocal Boundary Value Problems of Nonlinear Fractional Differential Equations

Zhoujin Cui, **Pinneng Yu** and **Zisen Mao**
PLA University of Science and Technology
Institute of Science
Jiangsu Nanjing 211101, China
cuizhoujin@126.com
neng11@sohu.com
maozisen@126.com

Abstract

This paper investigates the existence of solutions of the nonlinear fractional differential equation

$$\begin{cases} {}^cD^\alpha u(t) + f(t, u(t), {}^cD^\beta u(t)) = 0, \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\ u(0) = u'(0) = u''(0) = 0, u(1) = u(\xi), \ 0 < \xi < 1, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $\beta > 0$, $\alpha - \beta \geq 1$. The peculiarity of this equation is that the nonlinear term depends on the fractional derivative of the unknown function, compared with the available results in literature. The equation is firstly converted to an equivalent integral equation of Fredholm type, then results for the existence of its solution are derived by means of Schauder's fixed-point theorem. An example is given for demonstration.

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1 Introduction

The subject of multipoint nonlocal boundary value problems (BVP), initiated by Il'in and Moiseev [1], has been addressed by many authors. Gupta [2] studied certain three-point boundary value problems for nonlinear ordinary differential equations. Since then,

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more general nonlinear multi-point BVPs have been studied by several authors. We refer the readers to [3–9] for some references along this line. The multipoint boundary conditions appear in certain problems of thermodynamics, elasticity, and wave propagation, see [5] and the references therein. For example, the vibrations of a guy wire of a uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem (see [10]); many problems in the theory of elastic stability can be handled by the method of multi-point problems (see [11]). The multipoint boundary conditions may be understood in the sense that the controllers at the end points dissipate or add energy according to censors located at intermediate positions.

For example, Ma in [6] considers the fourth-order boundary value problem

$$\begin{cases} u'''' = f(t, u, u''), \quad 0 < t < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases} \tag{1.1}$$

Under some assumptions of f, the results on the existence of positive solutions are obtained. Problem (1.1) describes the deformation of an elastic beam whose both ends simply supported, see Gupta [2].

In some real world problems, fractional-order models are found to be more adequate than integer-order models. The last two decades have witnessed a great progress in fractional calculus and fractional-order dynamical systems. It has been found that fractional calculus is a mathematical tool that works adequately for anomalous social and physical systems with nonlocal, frequency and history-dependent properties, and for intermediate states such as soft materials, which are neither ideal solid nor ideal fluid (see [12–25]). Differential equations with fractional-order derivatives/integrals are called fractional differential equations. Some basic theory for the initial value problems of fractional differential equations has been discussed by Lakshmikantham [15], El–Sayed et al. [20] and S. Zhang [26] etc. Moreover, there are some works that deal with the existence and multiplicity of solutions to nonlinear fractional differential equations by using a fixed-point theorem or the topological degree theory.

As for fractional BVPs, Bai [7] discussed the nonlinear problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + a(t)f(t, u(t), u''(t)) = 0, \ 0 < t < 1, \ 3 < \alpha \leq 4, \\ u(0) = u'(0) = u''(0) = u''(1) = 0, \end{cases} \tag{1.2}$$

where D^{α}_{0+} is the standard Riemann–Liouville fractional derivative. The author used a new fixed-point theorem due to Bai and Ge on cone expansion and compression to show the existence of triple positive solutions.

In [8], Liang et al. studied the nonlinear fractional boundary value problem

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \quad 3 < \alpha \leq 4, \\ u(0) = u'(0) = u''(0) = u''(1) = 0. \end{cases} \tag{1.3}$$

By means of lower and upper solution method and fixed-point theorems, some results on the existence of positive solutions are obtained for the above fractional boundary value problems.

In [9], B. Ahmad et al. studied the nonlinear fractional differential equation with nonlocal boundary value

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \quad 0 < t < 1, \\ x(0) = x'(0) = x''(0) = \cdots = x^{m-2}(0) = 0, x(1) = ax(\eta), \end{cases} \tag{1.4}$$

in which $q \in (m-1, m]$, $m \in N$, $m \geq 2$. Existence results are based on the contraction mapping principle and Krasnoselskii's fixed-point theorem.

Motivated by all the works above, the purpose of this paper is to establish existence results to the nonlinear fractional differential equation with nonlocal boundary conditions

$$\begin{cases} {}^cD^\alpha u(t) + f(t, u(t), {}^cD^\beta u(t)) = 0, \;\; 0 < t < 1, \;\; 3 < \alpha \leq 4, \\ u(0) = u'(0) = u''(0) = 0, u(1) = u(\xi), \end{cases} \tag{1.5}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $\beta > 0$, $\alpha - \beta \geq 1$, $\xi \in (0, 1)$, $f : [0, 1] \times X \times X \to X$ is continuous. The peculiarity of this equation is that the nonlinear term depends on the fractional derivative of the unknown function, compared with the available results in literature. To the best of our knowledge, no one has studied the existence of solutions for nonlinear fractional boundary value problems (1.5).

We remark that the Caputo fractional derivative is more suitable than the usual Riemann–Liouville derivative for the applications in several engineering problems due to the fact that it has better relations with the Laplace transform and because the differentiation appears inside instead than outside, the integral, so to alleviate the effects of noise and numerical differentiation (see [13, 18, 28]).

The rest of the paper is organized as follows. In Section 2, we shall present some lemmas in order to prove our main results. In analogy with boundary value problem for differential equations of integer order, we firstly derive the corresponding Green's function, named as fractional Green's function. The proof of our main result is given in Section 3. In Section 4, we will give an example to illustrate our main result.

2 Preliminaries and Lemmas

For the readers' convenience, definitions of fractional integral/derivative and some preliminary results are given in this section.

Definition 2.1 (See [16, 21])**.** The fractional integral of order $q > 0$ of a function $x : (0, +\infty) \to \mathbb{R}$ is given by

$$I^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \tag{2.1}$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2.2 (See [16, 21]). The Caputo derivative of order $s > 0$ of a continuous function $\omega : (0, +\infty) \to \mathbb{R}$ is defined to be

$$ {}^cD^s\omega(t) = \frac{1}{\Gamma(n-s)} \int_0^t \frac{\omega^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{2.2}$$

where $n = [q] + 1$, provided that the right side is pointwise defined on $(0, +\infty)$.

Lemma 2.3 (See [16]). (1) *If $x \in L(0, 1)$, $\rho > \sigma > 0$, then*

$$ {}^cD^\sigma I^\rho x(t) = I^{\rho-\sigma}x(t), \quad I^\rho I^\sigma x(t) = I^{\rho+\sigma}x(t).$$

(2) *If $\rho > 0, \lambda > 0$, then* $D^\rho t^{\lambda-1} = \dfrac{\Gamma(\lambda)}{\Gamma(\lambda-\rho)} t^{\lambda-\rho-1}$.

The following lemma is crucial in finding an integral representation of the boundary value problem (1.5).

Lemma 2.4 (See [21]). *For $\alpha > 0$, the general solution of the fractional differential equation ${}^cD^\alpha u(t) = 0$ is given by*

$$u(t) = c_0 + c_1t + c_2t^2 + \ldots + c_{n-1}t^{n-1}, \tag{2.3}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1 (n = [\alpha] + 1)$.

Lemma 2.5 (See [21]). *Assume that $u \in C(0, 1) \cap L(0, 1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $C(0, 1) \cap L(0, 1)$. Then*

$$I^\alpha\, {}^cD^\alpha u(t) = u(t) + c_0 + c_1t + c_2t^2 + \ldots + c_{n-1}t^{n-1}, \tag{2.4}$$

where $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, n-1 (n = [\alpha] + 1)$.

To study the nonlinear problem (1.5), we first consider the associated linear problem and obtain its solution.

Lemma 2.6. *Let $y(t) \in C[0, 1]$ and $3 < \alpha \le 4$. The fractional boundary value problem*

$$\begin{cases} {}^cD^\alpha u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u'(0) = u''(0) = 0, u(1) = u(\xi), \end{cases} \tag{2.5}$$

has a unique solution

$$u(t) = \int_0^1 G(t, s)y(s)ds, \tag{2.6}$$

where

$$G(t, s) = \frac{1}{(1-\xi^3)\Gamma(\alpha)} \begin{cases} t^3[(1-s)^{\alpha-1} - (\xi-s)^{\alpha-1}] - (1-\xi^3) \\ (t-s)^{\alpha-1}, (0 \le s \le t \le 1, s \le \xi), \\ t^3(1-s)^{\alpha-1} - (1-\xi^3)(t-s)^{\alpha-1}, \\ (0 < \xi \le s \le t \le 1), \\ t^3(1-s)^{\alpha-1} - t^3(\xi-s)^{\alpha-1}, \\ (0 \le t \le s \le \xi < 1), \\ t^3(1-s)^{\alpha-1}, \\ (0 \le t \le s \le 1, \xi \le s). \end{cases}$$

Proof. At first, Lemma 2.4 implies that

$$u(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 - \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds, \tag{2.7}$$

where $c_0, c_1, c_2, c_3 \in R$ are arbitrary constants. By Lemma 2.3, we obtain

$$u'(t) = c_1 + 2c_2 t + 3c_3 t^2 - \int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)} y(s)ds, \tag{2.8}$$

$$u''(t) = 2c_2 + 6c_3 t - \int_0^t \frac{(t-s)^{\alpha-3}}{\Gamma(\alpha-2)} y(s)ds. \tag{2.9}$$

Applying the boundary conditions of (2.5), we know that $c_0 = c_1 = c_2 = 0$, and

$$c_3 = \frac{1}{1-\xi^3} \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \right].$$

Then, the unique solution of (2.5) is given by

$$u(t) = \frac{t^3}{1-\xi^3} \left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds - \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds \right]$$
$$- \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds = \int_0^1 G(t,s)y(s)ds.$$

This completes the proof. □

3 Main Results

In this section, we impose growth conditions on f which allow us to apply the Schauder fixed-point theorem to establish an existence result for solutions for problem (1.5). We define the space

$$X = \{u(t) | u(t) \in C[0,1] \text{ and } {}^cD^\beta u(t) \in C[0,1]\}$$

endowed with the norm $||u||_X = \max_{t\in[0,1]} |u(t)| + \max_{t\in[0,1]} |{}^cD^\beta u(t)|$. Then clearly $(X, ||\cdot||)$ is a Banach space.

As proved in reference [27], the solution of the boundary value problem (1.5) is equivalent to that of the integral equation

$$u(t) = \int_0^1 G(t,s) f(s, u(s), {}^c D^\beta u(s))ds. \tag{3.1}$$

Theorem 3.1. *Let f be a continuous function. Suppose that one of the following conditions is satisfied.*

(A_1) *There exist a nonnegative function* $m(t) \in L^1([0,1])$ *such that* $|f(t,x,y)| \leq m(t) + c_1|x|^p + c_2|y|^q$ *for all* $t \in [0,1]$, $c_1, c_2 \geq 0, 0 < p, q < 1$.

(A_2) $|f(t,x,y)| \leq c_1|x|^p + c_2|y|^q$ *for all* $t \in [0,1]$, $c_1, c_2 > 0, p, q > 1$.

Then problem (1.5) *has a solution.*

Proof. Let $T : X \to X$ be the operator defined as

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s),{}^c D^\beta u(s))ds, \quad t \in [0,1]. \tag{3.2}$$

We shall prove this result by using the Schauder fixed-point theorem. First, let the condition (A_1) be satisfied. Define

$$U = \{u(t)|u(t) \in X, ||u(t)||_X \leq R, \quad t \in [0,1]\},$$

where

$$R \geq \max\{(9c_1A)^{\frac{1}{1-p}}, (9c_2A)^{\frac{1}{1-q}}, 3l\},$$

and

$$A = \frac{1}{|1-\xi^3|\Gamma(\alpha+1)}, \quad l = \max_{t\in[0,1]} \int_0^1 |G(t,s)m(s)|ds.$$

Observe that U is the ball in the Banach space X.

Now we prove that $T : U \to U$. For any $u \in U$, applying Lemma 2.3, we have

$$\begin{aligned}
|Tu(t)| &= \left|\int_0^1 G(t,s)f(s,u(s),{}^c D^\beta u(s))ds\right| \\
&\leq \int_0^1 |G(t,s)m(s)|ds + (c_1R^p + c_2R^q)\int_0^1 |G(t,s)|ds \\
&= \int_0^1 |G(t,s)m(s)|ds + (c_1R^p + c_2R^q)\left(\frac{t^3}{|1-\xi^3|}\left[\int_0^1 \frac{(1-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right.\right. \\
&\quad \left.\left. + \int_0^\xi \frac{(\xi-s)^{\alpha-1}}{\Gamma(\alpha)}ds\right] + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}y(s)ds\right) \\
&\leq \int_0^1 |G(t,s)m(s)|ds + (c_1R^p + c_2R^q)\left[\frac{1+\xi^\alpha}{|1-\xi^3|\Gamma(\alpha+1)} + \frac{1}{\Gamma(\alpha+1)}\right] \\
&\leq \int_0^1 |G(t,s)m(s)|ds + 3(c_1R^p + c_2R^q)A,
\end{aligned}$$

$$\begin{aligned}
|{}^cD^\beta Tu(t)| &= |{}^cD^\beta I^\alpha f(t,u(t),{}^c D^\beta u(t)) - I^\alpha f(1,u(1),{}^c D^\beta u(1)){}^cD^\beta t^{\alpha-1}| \\
&= \left|I^{\alpha-\beta} f(t,u(t),{}^c D^\beta u(t)) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)}t^{\alpha-\beta-1} I^\alpha f(1,u(1),{}^c D^\beta u(1))\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\alpha-\beta)}\left[\int_0^t (t-s)^{\alpha-\beta-1} m(s)ds + (c_1R^p + c_2R^q)\int_0^t (t-s)^{\alpha-\beta-1}ds\right.\\
&\quad \left. + \int_0^1 (1-s)^{\alpha-\beta-1} m(s)ds + (c_1R^p + c_2R^q)\int_0^1 (1-s)^{\alpha-\beta-1}ds\right]\\
&\leq \frac{2}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1} m(s)ds + (c_1R^p + c_2R^q)\frac{2}{\Gamma(\alpha-\beta+1)}.
\end{aligned}
$$

Hence, $||Tu(t)||_X \leq l + 3(c_1R^p + c_2R^q)A \leq R/3 + R/3 + R/3 = R$. Notice that $Tu(t), ^c D^\beta Tu(t)$ are continuous on $[0,1]$. Thus, we have $T : U \to U$.

Under assumption (A_2), choosing

$$0 \leq R \leq \min\{(6c_1A)^{\frac{1}{1-p}}, (6c_2A)^{\frac{1}{1-q}}\}.$$

Repeating arguments similar to that above we can arrive at

$$||Tu(t)||_X = 3(c_1R^p + c_2R^q)A \leq R/2 + R/2 = R. \tag{3.3}$$

Consequently we have $T : U \to U$.

Due to the continuity of G, f, it is easy to see that the operator T is continuous. In what follows we show that T is a completely continuous operator. For this we take $L = \max\limits_{t\in[0,1]} |f(t, u(t), ^c D^\beta u(t))|$ for any $u \in U$. Let $t, \varsigma \in [0,1]$ such that $t < \varsigma$, then we obtain

$$
\begin{aligned}
|Tu(t) - Tu(\varsigma)| &= \left|\int_0^1 (G(t,s) - G(\varsigma,s)) f(s, u(s), ^c D^\beta u(s))ds\right|\\
&\leq L\left[\int_0^t |G(t,s) - G(\varsigma,s)|ds + \int_t^\varsigma |G(t,s) - G(\varsigma,s)|ds\right.\\
&\quad \left. + \int_\varsigma^1 |G(t,s) - G(\varsigma,s)|ds\right]\\
&\leq \frac{L}{(1-\xi^3)\Gamma(\alpha)}\int_0^1 (\tau^3 - t^3)(1-s)^{\alpha-1}ds\\
&= \frac{L}{(1-\xi^3)\Gamma(\alpha+1)}(\tau^3 - t^3),
\end{aligned}
$$

$$
\begin{aligned}
&|^c D^\beta Tu(t) - ^c D^\beta Tu(\varsigma)|\\
&= \left| I^{\alpha-\beta} f(t, u(t), ^c D^\beta u(t)) - \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1} I^\alpha f(1, u(1), ^c D^\beta u(1))\right.\\
&\quad \left. - I^{\alpha-\beta} f(\varsigma, u(\varsigma), ^c D^\beta u(\varsigma)) + \frac{\Gamma(\alpha)}{\Gamma(\alpha-\beta)} \varsigma^{\alpha-\beta-1} I^\alpha f(1, u(1), ^c D^\beta u(1))\right|\\
&\leq \frac{1}{\Gamma(\alpha-\beta)}\left|\int_0^t (t-s)^{\alpha-\beta-1} f ds - \int_0^\varsigma (\varsigma-s)^{\alpha-\beta-1} f ds\right|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{L}{\Gamma(\alpha-\beta)}(\varsigma^{\alpha-\beta-1}-t^{\alpha-\beta-1})\int_0^1(1-s)^{\alpha-1}ds \\
\leq\ & \frac{L}{\Gamma(\alpha-\beta+1)}(\varsigma^{\alpha-\beta}-t^{\alpha-\beta})+\frac{L}{\alpha\Gamma(\alpha-\beta)}(\varsigma^{\alpha-\beta-1}-t^{\alpha-\beta-1}).
\end{aligned}
$$

Now, using the fact that the functions $\tau^3, t^3, \varsigma^{\alpha-\beta}, t^{\alpha-\beta}, \varsigma^{\alpha-\beta-1}, t^{\alpha-\beta-1}$ are uniformly continuous on the interval $[0,1]$, we conclude that TU is an equicontinuous set. Obviously it is uniformly bounded since $TU \subseteq U$. Thus, T is completely continuous. The Schauder fixed-point theorem implies the existence of solutions in U for the problem (1.5) and the theorem is proved. □

Remark 3.2. If we impose additionally some restriction on c_i in (A_1) and (A_2), the conclusion of Theorem 3.1 remains true for the nonstrict inequalities $p, q \leq 1$ and $p, q \geq 1$. For example, we suppose that $p, q \geq 1$ in (A_2), in addition, if $p = q = 1$, then $c_i \leq \dfrac{1}{6A}$. Without loss of generality, let $p = 1$ and $q > 1$, then we may choose

$$0 \leq R \leq (6c_2A)^{\frac{1}{1-q}}.$$

One can easily obtain the estimate (3.3). Further arguments such as that in Theorem 3.1 yield our desired result.

4 Example

Finally, we give an example to illustrate the result obtained in this paper.

Example 4.1. Consider the nonlinear fractional differential system

$$
\begin{cases}
{}^cD^{\frac{7}{2}}u(t)+\dfrac{\cos t}{(t+5)^2}(u^p+({}^cD^{\frac{4}{3}}u)^q)=0, \quad 0<t<1, \\
u(0)=u'(0)=u''(0)=0, u(1)=u(1/2),
\end{cases}
\tag{4.1}
$$

where $0 < p, q < 1$ or $p, q \geq 1$ are parameters.

Note that $m(t) = 0$, $c_i = 1/25$. By Theorem 3.1 the existence of solutions is obvious for $0 < p, q < 1$ or $p, q > 1$. Furthermore when $p, q \geq 1$, with the use of $\Gamma(9/2) = \dfrac{105}{16}\sqrt{\pi}$, a simple computation shows $\dfrac{1}{6A} \approx 1.696$, since $c_i \leq \dfrac{1}{6A}$, Remark 3.2 implies that problem (4.1) has a solution.

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