

Perturbations of Mathieu Equations with Parametric Excitation of Large Period

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Abstract

We consider a linear differential system of Mathieu equations with periodic coefficients over periodic closed orbits and we prove that, arbitrarily close to this system, there is a linear differential system of Hamiltonian damped Mathieu equations with periodic coefficients over periodic closed orbits such that, all but a finite number of closed periodic coefficients, have unstable solutions. The perturbations will be performed in the periodic coefficients.

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1 Introduction

1.1 The Mathieu Equation

Let $\psi: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable τ -periodic function, i.e., $\psi(t + \tau) = \psi(t)$ for some $\tau > 0$ and all $t \in \mathbb{R}$. The *Hill's equation* is defined by

$$\ddot{y} + \psi(t)y = 0, \quad t \geq 0. \quad (1.1)$$

Fixing the parameters $\omega, \epsilon > 0$ and letting $\psi(t) = \cos(2\pi t)$ we obtain a very special case of a Hill equation - the *Mathieu equation*:

$$\ddot{y} + (\omega^2 + \epsilon \cos(2\pi t))y = 0, \quad t \geq 0. \quad (1.2)$$

These type of second-order differential equations, were introduced by Mathieu [11] in the study of oscillations. The parameters ω^2 and $\epsilon \cos(2\pi t)$ represent respectively, the frequency of the oscillation and the parametric excitation of strength ϵ .

In what follows we generalize and assume that the parametric excitation of strength ϵ can take the form $\epsilon\psi(t)$, where $\psi(t)$ is any function in the broader setting of smooth periodic functions. So, from now on we call the differential equation of second order

$$\ddot{y} + (\omega^2 + \epsilon\psi(t))y = 0, \quad t \geq 0, \quad (1.3)$$

a Mathieu equation.

More generally, by inputting a first order term, we get the *damped Mathieu equation* (see [13]) which is defined by

$$\ddot{y} + \gamma\dot{y} + (\omega^2 + \epsilon\psi(t))y = 0, \quad t \geq 0, \quad (1.4)$$

where $\gamma > 0$ is very small and, for this reason, (1.4) can be seen as a small perturbation of (1.3). In this paper we consider damped Mathieu equations with periodic coefficients. In fact, we say that

$$\ddot{y} + \Gamma(t)\dot{y} + (\omega^2 + \epsilon\Psi(t))y = 0, \quad t \geq 0, \quad (1.5)$$

is a *damped Mathieu equation with periodic coefficients* $\Gamma(t)$ and $\Psi(t)$ if $\Gamma(t)$ and $\Psi(t)$ are two τ -periodic smooth real functions with $\Gamma(t)$ very close to zero. In the sequel we will consider perturbations of the equation (1.3) and these perturbations are defined by damped Mathieu equations with periodic coefficients $\Gamma(t)$ and $\Psi(t)$, where $\Gamma(t)$, as we mentioned above, is close to zero and $\Psi(t)$ is close to $\psi(t)$. Actually, the central question of this paper is to fix an equation of type (1.3) and then, keeping the parameters ω and ϵ fixed and allowing some perturbations of the periodic coefficients $\Gamma(t)$ and $\Psi(t)$, understand the typical dynamical behavior of its solutions. To be more precise we will make use of an abstract and very general result about the C^0 perturbations of the dynamical cocycle along closed orbits (see [2, 5]) and connect it with the second-order differential equations of type (1.3) and (1.5) via its representation as linear differential systems.

This approach of studying the linear differential systems reveal to be the right framework in order to understand the dynamics of some type of differential equations, see e.g. [6, 7] and the references therein, where applications to the Schrödinger operator are presented.

We end the introduction by observing that we believe that the techniques developed in this paper can be upgraded in order to obtain similar results in the general context of differential equation of higher order. Actually, this is the subject of a work in progress.

1.2 Linear Differential Systems

In order to study families of linear nonautonomous differential equations like equations (1.3), (1.4) and (1.5) it is natural to write them in the terminology of linear differential

systems; let M be a compact, boundaryless Hausdorff manifold with dimension $n \geq 2$ and $\varphi^t: M \rightarrow M$ be a flow. A two-dimensional cocycle based on φ^t is defined by a flow $\Phi^t(p)$ differentiable on the time parameter $t \in \mathbb{R}$, acting on the general linear group $gl(2, \mathbb{R})$ (i.e., the set of 2×2 matrices with real entries and nonzero determinant). Together they form the linear skew-product flow

$$\Upsilon^t: \begin{array}{ccc} M \times \mathbb{R}^2 & \longrightarrow & M \times \mathbb{R}^2 \\ (p, v) & \longmapsto & (\varphi^t(p), \Phi^t(p) \cdot v). \end{array}$$

The linear flow Φ^t satisfy the so-called *cocycle identity*, that is, for all $t, s \in \mathbb{R}$ and $p \in M$,

$$\Phi^{t+s}(p) = \Phi^s(\varphi^t(p)) \cdot \Phi^t(p).$$

Let $\mathfrak{gl}(2, \mathbb{R})$ denote the Lie algebra associated to the Lie group $gl(2, \mathbb{R})$. If we define a map $A: M \rightarrow \mathfrak{gl}(2, \mathbb{R})$ in a point $p \in M$ by:

$$A(p) = \dot{\Phi}^s(p)|_{s=0}$$

and along the orbit $\varphi^t(p)$ by:

$$A(\varphi^t(p)) = \dot{\Phi}^s(p)|_{s=t} \cdot [\Phi^t(p)]^{-1}, \tag{1.6}$$

then $\Phi^t(p)$ will be the solution of the *linear variational equation*:

$$\dot{u}(s)|_{s=t} = A(\varphi^t(p)) \cdot u(t), \tag{1.7}$$

and $\Phi^t(p)$ is called the *fundamental matrix* or *fundamental solution*. Given a cocycle Φ^t we can induce the associated infinitesimal generator A by using (1.6) and given A we can recover the cocycle by solving the linear variational equation (1.7), from which we get Φ_A^t . A two-dimensional *linear differential system* is a three-tuple (M, φ^t, A) ; φ^t defines the action in the *base* and A the action in the *fiber*.

In what follows we will be interested in cocycles evolving on the linear groups $sl(2, \mathbb{R})$ and $gl(2, \mathbb{R})$. Recall that $sl(2, \mathbb{R})$ denotes the *special linear group* of matrices with real entries and with determinant equal to one, i.e., they are *area-preserving* or *Hamiltonian*.

Using Liouville's formula we can relate the trace of $A \in \mathfrak{gl}(2, \mathbb{R})$ with the determinant of Φ_A^t by:

$$\exp \left(\int_0^t \text{Tr} A(\varphi^s(p)) ds \right) = \det \Phi^t(p), \tag{1.8}$$

where $\text{Tr}(A)$ denoted the trace of A .

If Φ_A^t evolves on $sl(2, \mathbb{R})$, then the associated infinitesimal generator A is *traceless* (cf. (1.8)) i.e., $\text{Tr}(A) = 0$. We topologize the space of linear differential systems in the following way; given two functions $A, B: M \rightarrow \mathfrak{gl}(2, \mathbb{R})$ over $\varphi^t: M \rightarrow M$ we can measure its distance by

$$\max_{p \in M} \|A(p) - B(p)\|,$$

where $\|\cdot\|$ denotes the standard uniform norm.

If the function $A: M \rightarrow \mathfrak{gl}(2, \mathbb{R})$ varies continuously with the space variable p (for all $p \in M$) we say that the linear differential system is *continuous*. If the function $A: M \rightarrow \mathfrak{gl}(2, \mathbb{R})$ is bounded with respect to the space variable p (for all $p \in M$) we say that the linear differential systems is *bounded*. In this paper we deal with linear differential systems with Mathieu equations in the fiber and also its perturbations (cf. §1.4), the main results in this paper are true for both continuous and bounded settings.

1.3 Writing Equations as Linear Differential Systems

Given a flow $\varphi^t: M \rightarrow M$ we will denote by $\text{Per}(\varphi^t)$ the set of its *closed orbits*. Let $\psi(t)$ be a τ -periodic differentiable coefficient and $p \in \text{Per}(\varphi^t)$ be a closed orbit of period $\pi(p) = \tau$. As a standard reduction on the order of the differential equation via the *momentum* $\dot{y} = z$ we write

$$\begin{cases} \dot{y} = z \\ \dot{z} = -(\omega^2 + \epsilon\psi(t))y \end{cases}$$

then we obtain

$$\dot{x} = A(t) \cdot x, \tag{1.9}$$

where $\dot{x} = (\dot{y}, \dot{z})$ and $A(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2 - \epsilon\psi(t) & 0 \end{pmatrix}$.

Once we have the linear variational equation (1.9) we are interested in the behavior of the linear differential system induced by the traceless system $A(t)$, i.e., the dynamics of the fundamental matrix solution $\Phi_A^t(p)$.

The eigenvalues of $\Phi_A^{\pi(p)}(p)$ are called the *characteristic multipliers* of the system A . It is trivial to see that the product of the characteristic multipliers (or the characteristic multiplier if there is only one) of the system (1.9) is equal to one. If they are nonreal conjugates the solution is called *stable* (and the point p *elliptic*), and if one of them is larger than one (in modulus) the solution is called *unstable* (and the point p *hyperbolic*).

The regions of stability *versus* instability depending on the two parameters of the equation (1.2) are well-known (see e.g., [8, Figure 8.13]). The tongues are formed by *parabolic* points for $\Phi_A^{\pi(p)}(p)$ (corresponding to characteristic multipliers equal to 1 or -1).

More generally, given any $A(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & \zeta(t) \end{pmatrix} \in \mathfrak{gl}(2, \mathbb{R})$, where $\alpha(t)$, $\beta(t)$, $\gamma(t)$ and $\zeta(t)$ are smooth τ -periodic coefficients, the second-order differential equation associated to it is equal to

$$\ddot{y} - \left(\text{Tr}(A) + \frac{\dot{\beta}}{\beta} \right) \dot{y} + \left(\det(A) - \dot{\alpha} + \frac{\dot{\beta}\alpha}{\beta} \right) y = 0. \tag{1.10}$$

Let us see how we derived (1.10); we have

$$\begin{cases} \dot{y} = \alpha(t)y + \beta(t)z \\ \dot{z} = \gamma(t)y + \zeta(t)z \end{cases},$$

now, taking derivatives with respect to (w.r.t.) time, we obtain

$$\begin{aligned} \ddot{y} &= \dot{\alpha}y + \alpha\dot{y} + \dot{\beta}z + \beta\dot{z} = \dot{\alpha}y + \alpha\dot{y} + \dot{\beta}\left(\frac{\dot{y} - \alpha y}{\beta}\right) + \beta(\gamma y + \zeta z) \\ &= \left(\alpha + \frac{\dot{\beta}}{\beta}\right)\dot{y} + \left(\dot{\alpha} - \frac{\dot{\beta}\alpha}{\beta} + \beta\gamma\right)y + \beta\zeta\left(\frac{\dot{y} - \alpha y}{\beta}\right) \\ &= \left(\alpha + \frac{\dot{\beta}}{\beta} + \zeta\right)\dot{y} + \left(\dot{\alpha} - \frac{\dot{\beta}\alpha}{\beta} + \beta\gamma - \zeta\alpha\right)y \\ &= \left(\text{Tr}(A) + \frac{\dot{\beta}}{\beta}\right)\dot{y} + \left(-\det(A) + \dot{\alpha} - \frac{\dot{\beta}\alpha}{\beta}\right)y. \end{aligned}$$

One trivial observation can be taken from equation (1.10); if $\alpha(t)$ and $\beta(t)$ are constant, thus $\dot{y} = ay + bz$, for $a, b \in \mathbb{R}$, then (1.10) is reduced to the simple form $\ddot{y} - \text{Tr}(A)\dot{y} + \det(A)y = 0$. For example, we can easily check that equation (1.3) has this form if we take $\zeta(t) = -a$ and $\gamma(t) = b^{-1}(-\omega^2 - \epsilon\psi(t) - a^2)$.

If we consider a traceless system $A(t)$, then $\alpha(t) + \zeta(t) = 0$ and so (1.10) become:

$$\ddot{y} - \frac{\dot{\beta}}{\beta}\dot{y} + \left(\det(A) - \dot{\alpha} + \frac{\dot{\beta}\alpha}{\beta}\right)y = 0. \quad (1.11)$$

If, in equation (1.5), we let $\dot{y} = z$ then,

$$\begin{cases} \dot{y} = z \\ \dot{z} = -(\omega^2 + \epsilon\Psi(t))y - \Gamma(t)z \end{cases}$$

and so $\dot{x} = C(t) \cdot x$, where $\dot{x} = (\dot{y}, \dot{z})$ and the nontraceless system

$$C(t) = \begin{pmatrix} 0 & 1 \\ -\omega^2 - \epsilon\Psi(t) & -\Gamma(t) \end{pmatrix}.$$

On the other hand if, in equation (1.3), we let $\dot{y} = \alpha(t)y + bz$ then, using (1.11) and forcing the traceless condition $\zeta(t) = -\alpha(t)$, we obtain the traceless linear differential system $\mathcal{A}(t) = \begin{pmatrix} \alpha(t) & b \\ \gamma(t) & -\alpha(t) \end{pmatrix}$, where $b \in \mathbb{R}$. Now, since we have two-degrees of freedom, given any smooth τ -periodic function $\alpha(t)$, we define the τ -periodic function:

$$\gamma(t) := b^{-1}(-\omega^2 - \epsilon\psi(t) - \dot{\alpha}(t) - \alpha^2(t)). \quad (1.12)$$

Observe that, if we consider $\dot{y} = \alpha(t)y + bz$ such that $\alpha(t)$ is C^1 -close¹ to the zero function and b is close to 1, then $\mathcal{A}(t)$ can be seen as a small perturbation of $A(t)$, defined in (1.9), and both are traceless. In fact, what we allow here is a small perturbation of the momentum $\dot{y} = z$.

Taking this last paragraph into account, we proceed in the same way in equation (1.5) and we let $\dot{y} = \alpha(t)y + \beta(t)z$ where $\alpha(t)$ is C^1 -close to the zero function and $\beta(t) = e^{-\int_0^t \Gamma(s)ds}$. Finally, we define the τ -periodic function $\gamma(t)$ by:

$$\gamma(t) := \beta^{-1}(t) \left(-\omega^2 - \epsilon\psi(t) - \dot{\alpha}(t) - \alpha^2(t) + \frac{\dot{\beta}(t)\alpha(t)}{\beta(t)} \right).$$

With this choice for \dot{y} we obtain a **traceless** linear differential system induced by equation (1.5) very close to $C(t)$.

1.4 On the Neighborhoods of the Equation (1.3)

In this work we are going to perturb the equation (1.3) allowing that these perturbations being of the form (1.5). Moreover, we assure that, for a certain perturbation of the momentum, $\dot{y} = \alpha(t)y + \beta(t)z$, the equation (1.5) has a traceless linear differential system associated.

Fix positive real numbers ω, ϵ and δ and a τ -periodic smooth function $\psi(t)$. This elements give rise to an equation $\mathcal{M} = \mathcal{M}(\omega, \epsilon, \psi(t))$ of the form (1.3). Let A be the linear differential system induced by equation (1.3) with $\dot{y} = z$. We say that $\mathcal{U}(\mathcal{M}, \delta)$ is a δ -neighborhood of \mathcal{M} if:

- (i) there exists a choice for \dot{y} ; $\dot{y} = \alpha(t)y + \beta(t)z$, where $\alpha(t)$ is C^1 -close to the zero function and $\beta(t)$ is C^1 -close to the function $f(t) = 1$;
- (ii) there exists a τ -periodic coefficient

$$\gamma(t) = \beta^{-1}(t) \left(-\omega^2 - \epsilon\psi(t) - \dot{\alpha}(t) - \alpha^2(t) + \frac{\dot{\beta}(t)\alpha(t)}{\beta(t)} \right),$$

C^0 -close to $\omega^2 - \epsilon\psi(t)$ such that the linear differential system

$$\mathcal{C}(t) = \begin{pmatrix} \alpha(t) & \beta(t) \\ \gamma(t) & -\alpha(t) \end{pmatrix}$$

satisfy

$$\|A(t) - \mathcal{C}(t)\| < \delta.$$

¹The δ - C^1 -closeness of two functions $f(t)$ and $g(t)$ means that $|f(t) - g(t)| < \delta$ and $|\dot{f}(t) - \dot{g}(t)| < \delta$ for all t .

More generally, we can define a δ -neighborhood of a linear differential system with equations (1.3) on the fiber in the following way.

Definition 1.1. Fix a base flow $\varphi^t: M \rightarrow M$ and, for each $p \in \text{Per}(\varphi^t)$ of period $\pi(p)$, take $\pi(p)$ -periodic smooth coefficients $\psi_p(t)$. Fix also $\omega, \epsilon > 0$. Let $A: \text{Per}(\varphi^t) \rightarrow \mathfrak{sl}(2, \mathbb{R})$ be the linear differential system defined by:

$$A(p, t) := \begin{pmatrix} 0 & 1 \\ -\omega^2 - \epsilon\psi_p(t) & 0 \end{pmatrix}$$

We say that $\mathcal{U}(A, \delta)$ is a δ -neighborhood of A if, for each $p \in \text{Per}(\varphi^t)$,

- (i) there exists a choice for \dot{y} ; $\dot{y} = \alpha_p(t)y + \beta_p(t)z$ where $\alpha_p(t)$ is C^1 -close to the zero function (w.r.t. t) and $\beta_p(t)$ is C^1 -close (w.r.t. t) to the function $f(t) = 1$;
- (ii) there exists a $\pi(p)$ -periodic coefficient $\gamma: \text{Per}(\varphi^t) \times \mathbb{R}$ defined by

$$\gamma_p(t) = \beta_p^{-1}(t) \left(-\omega^2 - \epsilon\psi_p(t) - \dot{\alpha}_p(t) - \alpha_p^2(t) + \frac{\dot{\beta}_p(t)\alpha_p(t)}{\beta_p(t)} \right)$$

and C^0 -close to $\omega^2 - \epsilon\psi_p(t)$ such that the linear differential system

$$\mathcal{C}(p, t) = \begin{pmatrix} \alpha_p(t) & \beta_p(t) \\ \gamma_p(t) & -\alpha_p(t) \end{pmatrix}$$

satisfy $\|A(p, t) - \mathcal{C}(p, t)\| < \delta$, where $A(p, t)$ is the induced linear differential system by equation (1.3) with $\dot{y} = z$ and a $\pi(t)$ -periodic coefficient $\psi_p(t)$.

1.5 Hyperbolicity and Dominated Splitting

Let (M, φ^t, A) be a linear differential system. The set $\Lambda \subseteq M$ is said to be a *uniformly hyperbolic set* if there exists a uniform $\ell \in \mathbb{N}$ such that, for any $p \in \Lambda$, there is a $\Phi_A^t(p)$ -invariant decomposition $\mathbb{R}_p^2 = N_p^u \oplus N_p^s$ satisfying the following inequalities:

$$\|\Phi_A^{-\ell}(p)|_{N_p^u}\| \leq \frac{1}{2} \text{ and } \|\Phi_A^\ell(p)|_{N_p^s}\| \leq \frac{1}{2}.$$

If $\Lambda = M$, then we say that (M, φ^t, A) is *uniformly hyperbolic*. The concept of uniform hyperbolicity is equivalent to the *exponential dichotomy* concept, see [9] for details.

More generally, a φ^t -invariant set $\Lambda_\ell \subseteq M$ is said to have an ℓ -dominated splitting for (M, φ^t, A) , where $p \in \Lambda_\ell$, if for any $p \in \Lambda_\ell$, there is a $\Phi_A^t(p)$ -invariant decomposition $\mathbb{R}_p^2 = N_p^u \oplus N_p^s$ satisfying,

$$\frac{\|\Phi_A^\ell(q)|_{N_q^s}\|}{\|\Phi_A^\ell(q)|_{N_q^u}\|} \leq \frac{1}{2}.$$

When the fiber N_p^u dominates N_p^s both fibers may contract, however N_p^s is most contracting than N_p^u . On the other hand if both fibers expand, N_p^s is less expanding than N_p^u (see [4] for more details on dominated splitting). In rough terms, in the two-dimensional Hamiltonian context, hyperbolicity is tantamount to dominated splitting. This is the content of the following lemma whose proof is elementary (see [1, §4.1]).

Lemma 1.2. *If (M, φ^t, A) is a linear differential system, $A \in \mathfrak{sl}(2, \mathbb{R})$ and Λ is a set with dominated splitting, then Λ is hyperbolic.*

1.6 An Abstract General Setting and Statement of the Result

In order to state our results with great generality we will use the nomenclature developed in [5] (for the discrete case) and then generalized to the continuous-time volume-preserving case in ([2]); Let $\Sigma \subset \text{Per}(\varphi^t)$ be a set formed by a countable union of closed orbits of $\varphi^t: M \rightarrow M$. The linear differential system $\mathcal{A} = (\Sigma, \varphi^t, A)$ is *bounded* if there exists $K > 0$ such that $\|A(x)\| \leq K$, for all $x \in \Sigma$. The linear differential system \mathcal{A} is said to be a *large period system* if the number of closed orbits of Σ with period less or equal to τ is finite, for any $\tau > 0$.

Example 1.3. An Anosov flow (see e.g., [4]) is an example of a flow with the **large period property** described above. Fix $\omega, \epsilon > 0$ and take a Mathieu equation

$$\ddot{y} + \left(\omega^2 + \epsilon \cos \left(\frac{2\pi}{\pi(p)} t \right) \right) y = 0,$$

over each closed orbit p of period $\pi(p)$ of an Anosov flow to obtain $A: \text{Per}(\varphi^t) \rightarrow \mathfrak{sl}(2, \mathbb{R})$ which is an example of a bounded and large period system.

A linear differential system $\mathcal{B} = (\Sigma, \varphi^t, B)$ is a *conservative perturbation* of a bounded system \mathcal{A} if, for every $\epsilon > 0$, $\|A(x) - B(x)\| < \epsilon$, up to points x belonging to a finite number of orbits, and \mathcal{B} is conservative i.e., if $\text{Tr}(A) = \text{Tr}(B)$.

A direct application of the Gronwall inequality gives that

$$\|\Phi_A^t(x) - \Phi_B^t(x)\| \leq e^{K|t|} \|A(x) - B(x)\|.$$

In particular Φ_B^1 is a perturbation of Φ_A^1 in the sense introduced in [5] for the discrete case.

A bounded linear differential system \mathcal{A} is said to be *strictly without dominated decomposition* if the only invariant subsets of Σ that admit a dominated splitting for Φ_A^t are finite sets.

Let us now present a key result about linear differential systems which is the conservative flow version of [5, Theorem 2.2].

Theorem 1.4 (See [2, Theorem 4.1]). *Suppose \mathcal{A} is a conservative, large period and bounded linear differential system. If \mathcal{A} is strictly without dominated decomposition then there exist a conservative perturbation \mathcal{B} of \mathcal{A} and an infinite set $\Sigma' \subset \Sigma$ which is φ^t -invariant such that for every $x \in \Sigma'$ the linear map $\Phi_B^{\pi(x)}(x)$ has all eigenvalues real and with the same modulus (thus equal to 1 or to -1).*

Once we develop the perturbation framework, specific of the conservative-flow context, previous theorem follows directly from [5, Theorem 2.2]. In the present paper we can obtain a similar result, restricted to our setting, as long as we show how to perform all the perturbations in our setting. These perturbations will be the content of §2.

Fix $\omega, \epsilon > 0$, a base flow $\varphi^t: M \rightarrow M$ and, for each $p \in \text{Per}(\varphi^t)$, a family of $\pi(p)$ -periodic smooth coefficients $\{\psi_p(t)\}$. Let

$$\mathcal{A} = \mathcal{A}(p, t, \omega, \epsilon, M, \varphi^t, A, \psi_p(t)) \tag{1.13}$$

denote the *Mathieu linear differential system* induced by the Mathieu equation $\ddot{y} + (\omega^2 + \epsilon\psi_p(t))y = 0$. From now on we assume that our Mathieu linear differential systems are bounded, with large period and strictly without dominated decomposition.

We now state our main result.

Theorem 1.5. *Let \mathcal{A} denote a Mathieu linear differential system as in (1.13). Fix $\delta > 0$. Then there exist $\ell \in \mathbb{N}$ and $T > 0$ such that any closed orbit p of period $\pi(p) \geq T$ satisfy one of the two properties:*

1. *the φ^t -orbit of p is **hyperbolic**² with strength ℓ or else*
2. *there is a system $B \in \mathcal{U}(A, \delta)$ such that the φ^t -orbit of p is **parabolic** for B .*

A few words about previous theorem. Condition (1) says that the Mathieu system with $\pi(p)$ -periodic coefficient over p is *unstable*. Even more, the degree of instability shared by all the Mathieu equations (over points p satisfying condition (1)) is uniform. In this case the characteristic multipliers are uniformly “far” from the tongues of instability, that is, the characteristic multipliers are uniformly away from 1 and -1 . Condition (2) says that, a perturbation can be made in order to the characteristic multipliers fall into the boundary of instability, this perturbation is a Hamiltonian damped Mathieu system with two large periodic coefficients. In conclusion, near any Mathieu system with parametric excitations of large period, there exists a perturbation of the two coefficients such that, if we neglect a finite number of closed orbits, any system over any closed orbit p is *unstable*. In other words stability tends to disappear when the period of the coefficient increases and some first order term with a small periodic coefficient is introduced.

²In fact we obtain that the φ^t -orbit has an ℓ -dominated splitting. Then, by Lemma 1.2, we obtain hyperbolicity.

We end this section by noting that, Theorem 1.4 and the perturbations to be developed in §2, allows us to obtain item (2) of Theorem 1.5 if we have the strictly without domination property. In brief terms weak hyperbolicity can be made parabolic (see paragraph after the proof of Lemma 2.1) and ellipticity can also be made parabolic, under certain hypothesis, by small perturbations (cf. Lemma 2.2).

On the other hand, if we do not have this property, we must have a uniform dominated slitting, obtaining (1).

2 Construction of the Perturbations

2.1 Inducing Rotations

We are going to perform some perturbations defined by small rotations of solutions on the invariant eigendirections. Results of this type go back to Novikov [12] and Mañé [10]. See also [1] where perturbations of these type were made in the two-dimensional Hamiltonian setting of linear differential systems.

Lemma 2.1. *Let A denote the linear differential system as in (1.13). Fix $\delta > 0$. There exists $\eta > 0$, such that given any $p \in \text{Per}(\varphi^t)$ (with $\pi(p) > 1$), there exists $B \in \mathcal{U}(A, \delta)$ such that*

(A) B is supported in $\varphi^t(p)$ for $t \in [0, 1]$ and

(B) $\Phi_B^1(p) = \Phi_A^1(p) \cdot R_\eta$, where R_η is a rotation of angle η .

Proof. Let $\eta \in (0, 1)$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ be a bump-function defined by $g(t) = 0$ for $t < 0$, $g(t) = t$ for $t \in [\eta, 1 - \eta]$ and $g(t) = 1$ for $t \geq 1$. Fix a periodic point p with $\pi(p) > 1$ and define for all $t \geq 0$:

$$\Phi_A^t(p) = \Phi^t(p) = \begin{pmatrix} a(t) & b(t) \\ c(t) & d(t) \end{pmatrix} \text{ and } R_{\eta g(t)} = \begin{pmatrix} \cos(\eta g(t)) & -\sin(\eta g(t)) \\ \sin(\eta g(t)) & \cos(\eta g(t)) \end{pmatrix}.$$

We know that $u(t) = \Phi^t(p)$ is a solution of the linear variational equation $\dot{u}(t) = A(t) \cdot u(t)$. Take $\Phi^t(p) \cdot R_{\eta g(t)}$ and then compute the derivative w.r.t. time:

$$\begin{aligned} \frac{d}{dt}(\Phi^t(p) \cdot R_{\eta g(t)}) &= \dot{\Phi}^t(p) R_{\eta g(t)} + \Phi^t(p) \dot{R}_{\eta g(t)} \\ &= A(\varphi^t(p)) \Phi^t(p) R_{\eta g(t)} + \Phi^t(p) \dot{R}_{\eta g(t)} \\ &= A(\varphi^t(p)) \Phi^t(p) R_{\eta g(t)} + \Phi^t(p) \dot{R}_{\eta g(t)} R_{-\eta g(t)} [\Phi^t(p)]^{-1} \Phi^t(p) R_{\eta g(t)} \\ &= [A(\varphi^t(p)) + \Phi^t(p) \dot{R}_{\eta g(t)} R_{-\eta g(t)} [\Phi^t(p)]^{-1}] \cdot (\Phi^t(p) R_{\eta g(t)}). \end{aligned}$$

Define $B(\varphi^t(p)) = A(\varphi^t(p)) + H(\varphi^t(p))$ where:

$$H(\varphi^t(p)) = H(\eta, t) = \Phi^t(p) \dot{R}_{\eta g(t)} R_{-\eta g(t)} [\Phi^t(p)]^{-1}.$$

We conclude that $v(t) = \Phi^t(p) \cdot R_{\eta g(t)}$ is a solution of the linear variational equation

$$\dot{v}(s)|_{s=t} = [A(\varphi^t(p)) + H(\eta, t)]v(t). \quad (2.1)$$

Since,

$$\dot{R}_{\eta g(t)} \cdot R_{-\eta g(t)} = \eta \dot{g}(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

and since $\text{Tr}(A) = 0$ we use (1.8) and we obtain that

$$\begin{aligned} H(\eta, t) &= \frac{\eta \dot{g}(t)}{\det \Phi^t(p)} \begin{pmatrix} b(t)d(t) + a(t)c(t) & -b(t)^2 - a(t)^2 \\ d(t)^2 + c(t)^2 & -b(t)d(t) - a(t)c(t) \end{pmatrix} \\ &= \eta \dot{g}(t) \begin{pmatrix} b(t)d(t) + a(t)c(t) & -b(t)^2 - a(t)^2 \\ d(t)^2 + c(t)^2 & -b(t)d(t) - a(t)c(t) \end{pmatrix}. \end{aligned}$$

Hence $\text{Tr}(H(\eta, t)) = 0$, moreover since $\dot{g}(t) = 0$ for $t \notin]0, 1[$, its support is $\varphi^t(p)$ for $t \in [0, 1]$ and (A) is proved. Since $t \in [0, 1]$ and all the terms in the definition of $H(\eta, t)$ are uniformly bounded for all $p \in M$ (by definition of these type of systems), given any size of perturbation allowed by $\delta > 0$ we take $\eta(\delta)$ sufficiently small to guarantee that we get $\|A - B\| = \|H\| < \delta$.

Finally, for (B), we observe that $v(t) = \Phi_A^t(p) \cdot R_{\eta g(t)}$ is solution of (2.1). So for $t = 1$ we obtain $\Phi_B^1(p) = \Phi_A^1(p) \cdot R_\eta$.

To conclude the proof of the lemma we only need to see that B generates a traceless damped Mathieu system with two perturbed coefficients $\Gamma(t)$ and $\Psi(t)$ close to zero and $\psi(t)$ respectively.

$$\begin{aligned} B &= A(t) + H(\eta, t) \\ &= \begin{pmatrix} 0 & 1 \\ -\omega^2 - \epsilon\psi(t) & 0 \end{pmatrix} + \eta \dot{g}(t) \begin{pmatrix} b(t)d(t) + a(t)c(t) & -b(t)^2 - a(t)^2 \\ d(t)^2 + c(t)^2 & -b(t)d(t) - a(t)c(t) \end{pmatrix} \\ &= \begin{pmatrix} \eta \dot{g}(t)(b(t)d(t) + a(t)c(t)) & 1 - \eta \dot{g}(t)(b(t)^2 + a(t)^2) \\ -\omega^2 - \epsilon\psi(t) + \eta \dot{g}(t)(d(t)^2 + c(t)^2) & -\eta \dot{g}(t)(b(t)d(t) + a(t)c(t)) \end{pmatrix}. \end{aligned}$$

We abbreviate $F = b(t)d(t) + a(t)c(t)$, $G = b(t)^2 + a(t)^2$ and $H = d(t)^2 + c(t)^2$ and thus,

$$B = \begin{pmatrix} \eta \dot{g}(t)F & 1 - \eta \dot{g}(t)G \\ -\omega^2 - \epsilon\psi(t) + \eta \dot{g}(t)H & -\eta \dot{g}(t)F \end{pmatrix}.$$

Using equation (1.10) we obtain (denoting also by ' the derivative w.r.t. t):

$$\ddot{y} - \left(\frac{(1 - \eta \dot{g}G)'}{1 - \eta \dot{g}G} \right) \dot{y} + \left(\det(B) - (\eta \dot{g}F)' + \frac{(1 - \eta \dot{g}G)' \eta \dot{g}F}{1 - \eta \dot{g}G} \right) y = 0,$$

which is equivalent to

$$\ddot{y} + \left(\frac{\eta \ddot{g}G + \eta \dot{g}\dot{G}}{1 - \eta \dot{g}G} \right) \dot{y} + \left(\det(B) - \eta \ddot{g}F - \eta \dot{g}\dot{F} + \frac{(-\eta \ddot{g}G - \eta \dot{g}\dot{G})\eta \dot{g}F}{1 - \eta \dot{g}G} \right) y = 0.$$

Now, since

$$\det(B) = \omega^2 + \epsilon\psi(t) + \eta^2\dot{g}^2(GH - F^2) - \eta\dot{g}(H + \omega^2G + G\epsilon\psi(t)),$$

we define the τ -periodic coefficients by:

$$\Gamma_\eta(t) := \frac{\eta\ddot{g}G + \eta\dot{g}\dot{G}}{1 - \eta\dot{g}G}$$

and

$$\Theta_\eta(t) := -\eta\ddot{g}F - \frac{(\ddot{g}G + \dot{g}\dot{G})\eta^2\dot{g}F}{1 - \eta\dot{g}G} + \eta^2\dot{g}^2(GH - F^2) - \eta\dot{g}(\dot{F} + H + \omega^2G + G\epsilon\psi(t)).$$

Finally, we obtain,

$$\ddot{y} + \Gamma_\eta(t)\dot{y} + (\omega^2 + \epsilon\Psi_\eta(t))y = 0, \quad (2.2)$$

where $\Psi_\eta(t) := \psi(t) + \epsilon^{-1}\Theta_\eta(t)$. Observe that, if η is taken very small, then $\Gamma_\eta(t)$ is close to zero, moreover, since $\Theta_\eta(t) \xrightarrow{\eta \rightarrow 0} 0$, $\Psi_\eta(t)$ is close to $\psi(t)$. Then $B \in \mathcal{U}(A, \delta)$ according to Definition 1.1 and the lemma is proved. \square

We observe that condition (2) of Theorem 1.5 assures that there is a Hamiltonian damped Mathieu system B with perturbed coefficients, arbitrarily close to the original one, such that $\Phi_B^{\pi(p)}(p)$ has only real eigenvalues of the same modulus (thus equal to one). A perturbation similar to the one constructed in Lemma 2.1 can be done, by adding small directional homotheties, in order to obtain two real characteristic multipliers (hyperbolic case).

The next crucial lemma show how to create instability (parabolic or hyperbolic points), by small perturbations, once we are in the presence of an elliptic point and we have enough “time” to perturb.

Lemma 2.2. *Let \mathcal{A} denote the Mathieu linear differential system as in (1.13) and $\delta > 0$. Then there exists $T > 0$ such that for any closed orbit p of period $\pi(p) \geq T$ there is $B \in \mathcal{U}(A, \delta)$ satisfying*

- all the eigenvalues of $\Phi_B^{\pi(p)}$ are real, and
- $B = A$ outside a small neighbourhood of the orbit of p .

Proof. Fix a small $\delta > 0$. Let R_θ denote the plane rotation of angle θ . By [3, Lemma 6.6] there exists $N = N(\epsilon) \in \mathbb{N}$ satisfying the following: for any $k > N$ and for any $C_1, C_2, \dots, C_k \in sl(2, \mathbb{R})$ there are rotations $R_{\theta_1}, R_{\theta_2}, \dots, R_{\theta_k}$, with $|\theta_j| < \delta$ for all $j \in \{1, 2, \dots, k\}$, such that the linear map

$$C_k \cdot R_{\theta_k} \cdot C_{k-1} \cdot R_{\theta_{k-1}} \cdot \dots \cdot C_1 \cdot R_{\theta_1}$$

has real eigenvalues.

Let us fix a periodic orbit Ω and $p \in \Omega$ with $\pi(p) \geq N$. We assume that $\Phi_A^{\pi(p)}(p)$ has a complex eigenvalue. Assuming that $\pi(p) = k \in \mathbb{N}$, we consider, for each $j \in \{1, \dots, k\}$ the linear maps $C_j: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by

$$C_j = \Phi_A^1(\varphi^{j-1}(p)).$$

If $\pi(p) \notin \mathbb{N}$ we take $k = [\pi(p)]$, consider C_1, \dots, C_{k-1} as before and define

$$C_k: \begin{array}{ccc} \mathbb{R}_{\varphi^{k-1}(p)}^2 & \longrightarrow & \mathbb{R}_{\varphi^{\pi(p)}(p)}^2 \\ v & \longmapsto & \Phi_A^{1+\pi(p)-k}(\varphi^{k-1}(p)) \cdot v. \end{array}$$

In what follows, without loss of generality, we assume that $\pi(p) = k \in \mathbb{N}$.

We observe that each C_j can be identified with a linear map of $sl(2, \mathbb{R})$ and that $\Phi_A^{\pi(p)}(p) = C_k \cdot C_{k-1} \cdot (\dots) \cdot C_1$. Therefore, [3, Lemma 6.6] gives a family of rotations $R_{\theta_1}, R_{\theta_2}, \dots, R_{\theta_k}$ with the properties described above.

Now we apply Lemma 2.1 to each arc $\{\varphi^{j-1+t}(p): t \in [0, 1]\}$ and to the maps C_j and R_{θ_j} and realize the perturbations, i.e., find adequate coefficients $\Gamma_p^j(t)$ and $\Psi_p^j(t)$. \square

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