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Existence and Uniqueness of Anti-Periodic Solutions for Nonlinear Higher-Order Differential Equations with Two Deviating Arguments

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Abstract

In this paper, we use the Leray–Schauder degree theory to establish new results on the existence and uniqueness of anti-periodic solutions for a class of nonlinear nth-order differential equations with two deviating arguments of the form

$$x^{(n)}(t) + f(t, x^{(n-1)}(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t).$$

As an application, we also give an example to demonstrate our results.

AMS Subject Classifications: 34C25, 34D40.
Keywords: Higher-order differential equations, deviating argument, anti-periodic solution, existence and uniqueness, Leray–Schauder degree.

1 Introduction

During the past twenty years, anti-periodic problems of nonlinear differential equations have been extensively studied by many authors, see [1–5] and references therein. For example, anti-periodic trigonometric polynomials are important in the study of interpolation problems [6, 7], and anti-periodic wavelets are discussed in [8]. Recently, anti-periodic boundary conditions have been considered for the Schrodinger and Hill differential operator [9, 10]. Also anti-periodic boundary conditions appear in the study of difference equations [11, 12]. Moreover, anti-periodic boundary conditions appear in physics in a variety of situations [13–15].

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In this paper, we discuss the existence and uniqueness of anti-periodic solutions for a class of nonlinear nth-order differential equations with two deviating arguments of the form

$$x^{(n)}(t) + f(t, x^{(n-1)}(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t). \tag{1.1}$$

where $\tau_i, e : \mathbb{R} \to \mathbb{R}$ and $f, g_i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, τ_i and e are T-periodic, f and g_i are T-periodic in their first arguments, $n \geq 2$ is an integer, $T > 0$ and $i = 1, 2$.

Clearly, when $n = 2$ and $f(t, x(t)) = f(x(t))$, (1.1) reduces to

$$x''(t) + f(x'(t)) + g_1(t, x(t - \tau_1(t))) + g_2(t, x(t - \tau_2(t))) = e(t), \tag{1.2}$$

which has been known as the delayed Rayleigh equation with two deviating arguments. Therefore, we can consider (1.1) as a high-order Rayleigh equation with two deviating arguments. The dynamic behaviors of Rayleigh equation and Rayleigh system have been widely investigated [16–19] due to their application in many fields such as physics, mechanics and the engineering technique fields. In such applications, it is important to know the existence of periodic solutions of Rayleigh equation, and some results on existence of periodic solutions were obtained in [20–23]. However, to the best of our knowledge, few authors have considered the existence and uniqueness of anti-periodic solutions for (1.1). This equation can stand for analog voltage transmission, and voltage transmission process is often an anti-periodic process. Thus, it is worth continuing the investigation of the existence and uniqueness of anti-periodic solutions of (1.1).

The main purpose of this paper is to establish sufficient conditions for the existence and uniqueness of anti-periodic solutions of (1.1). Our results are different from those of the references listed above. In particular, an example is also given to illustrate the effectiveness of our results.

It is convenient to introduce the following assumptions:

(A_0) Assume that there exist nonnegative constants C_1 such that

$$| f(t, x_1) - f(t, x_2) | \leq C_1 | x_1 - x_2 | \text{ for all } t, x_1, x_2 \in \mathbb{R}.$$

$(\widetilde{A}_0)$ Assume that there exist nonnegative constants C_2 such that

$$f(t, u) = f(u),$$

$$C_2 | x_1 - x_2 |^2 \leq (x_1 - x_2)(f(x_1) - f(x_2)) \text{ for all } x_1, x_2, u \in \mathbb{R}.$$

(A_1) for all $t, x \in \mathbb{R},\ i = 1, 2,$

$$f\left(t + \frac{T}{2}, -x\right) = -f(t, x), \quad g_i\left(t + \frac{T}{2}, -x\right) = -g_i(t, x),$$

$$e\left(t + \frac{T}{2}\right) = -e(t), \quad \tau_i\left(t + \frac{T}{2}\right) = \tau_i(t).$$

2 Preliminary Results

For convenience, we introduce a continuation theorem [24] as follows.

Lemma 2.1. *Let Ω be open bounded in a linear normal space X. Suppose that $\widetilde{f}$ is a complete continuous field on $\overline{\Omega}$. Moreover, assume that the Leray–Schauder degree*

$$\deg\{\widetilde{f}, \Omega, p\} \neq 0, \quad for\ p \in X \setminus \widetilde{f}(\partial\Omega).$$

Then equation $\widetilde{f}(x) = p$ has at least one solution in Ω.

Let $u(t) : \mathbb{R} \to \mathbb{R}$ be continuous in t. $u(t)$ is said to be anti-periodic on $\mathbb{R}$ if,

$$u(t+T) = u(t), u(t+\frac{T}{2}) = -u(t) \text{ for all } t \in \mathbb{R}.$$

For ease of exposition, throughout this paper we will adopt the following notations:

$$C_T^k := \{x \in C^k(\mathbb{R}, \mathbb{R}), x \text{ is } T\text{-periodic}\}, \quad k \in \{0, 1, 2, \ldots\},$$

$$|x|_q = \left(\int_0^T |x(t)|^q dt\right)^{\frac{1}{q}}, \quad |x|_\infty = \max_{t\in[0,T]} |x(t)|, \quad |x^{(k)}|_\infty = \max_{t\in[0,T]} |x^{(k)}(t)|.$$

$$C_T^{k,\frac{1}{2}} = \left\{x \in C_T^k,\ x\left(t + \frac{T}{2}\right) = -x(t) \text{ for all } t \in \mathbb{R}\right\}$$

which is a linear normal space endowed with the norm $\|\cdot\|$ defined by

$$||x|| = \max_{t\in[0,T]} \{|x|_\infty, |x'|_\infty, \ldots, |x^{(k)}|_\infty\} \text{ for all } x \in C_T^{k,\frac{1}{2}}.$$

The following lemmas will be useful for proving our main results in Section 3.

Lemma 2.2. *If $x \in C^2(\mathbb{R}, \mathbb{R})$ with $x(t+T) = x(t)$, then*

$$|\ x'(t)\ |_2^2 \leq \left(\frac{T}{2\pi}\right)^2 |\ x''(t)\ |_2^2\ . \tag{2.1}$$

Proof. Lemma 2.2 is known as Wirtinger inequality, and see [25] for the proof of it. □

Lemma 2.3. *Suppose one of the following conditions is satisfied:*

(A_2) (A_0) *holds and there exist nonnegative constants b_1, b_2 such that*

$$C_1\frac{T}{2\pi} + (b_1 + b_2)\frac{T^n}{2(2\pi)^{n-1}} < 1 \textit{ and}$$

$$|g_i(t,x_1)-g_i(t,x_2)|\le b_i|x_1-x_2| \textit{ for all } t,x_i\in\mathbb{R}, i=1,2;$$

(A_3) $(\widetilde{A}_0)$ *holds and there exist nonnegative constants* b_1, b_2 *such that*

$$0\le (b_1+b_2)<\frac{2C_2(2\pi)^{n-2}}{T^{n-1}} \textit{ and}$$
$$|g_i(t,x_1)-g_i(t,x_2)|\le b_i|x_1-x_2| \textit{ for all } t,x_i\in\mathbb{R},\ i=1,\ 2.$$

Then (1.1) *has at most one anti-periodic solution.*

Proof. Suppose that x_1 and x_2 are two anti-periodic solutions of (1.1). Then, we have

$$\begin{aligned}&(x_1(t)-x_2(t))^{(n)}+(f(t,x_1^{(n-1)}(t))-f(t,x_2^{(n-1)}(t)))+(g_1(t,x_1(t-\tau_1(t)))\\&\quad-g_1(t,x_2(t-\tau_1(t))))+(g_2(t,x_1(t-\tau_2(t)))-g_2(t,x_2(t-\tau_2(t))))=0.\end{aligned}\tag{2.2}$$

Set $Z(t)=x_1(t)-x_2(t)$. Then, from (2.2), we obtain

$$\begin{aligned}&Z^{(n)}(t)+(f(t,x_1^{(n-1)}(t))-f(t,x_2^{(n-1)}(t)))+(g_1(t,x_1(t-\tau_1(t)))\\&\quad-g_1(t,x_2(t-\tau_1(t))))+(g_2(t,x_1(t-\tau_2(t)))-g_2(t,x_2(t-\tau_2(t))))=0.\end{aligned}\tag{2.3}$$

Since $Z(t)=x_1(t)-x_2(t)$ is an anti-periodic function on $\mathbb{R}$, we have

$$\int_0^T Z(t)dt=\int_0^{\frac{T}{2}}Z(t)dt+\int_{\frac{T}{2}}^T Z(t)dt=\int_0^{\frac{T}{2}}Z\left(t+\frac{T}{2}\right)dt+\int_0^{\frac{T}{2}}Z(t)dt=0.\tag{2.4}$$

It follows that there exists a constant $\widetilde{\gamma}\in(0,T)$ such that

$$Z(\widetilde{\gamma})=0.\tag{2.5}$$

Then, we have

$$|Z(t)|=\left|Z(\widetilde{\gamma})+\int_{\widetilde{\gamma}}^t Z'(s)ds\right|\le\int_{\widetilde{\gamma}}^t|Z'(s)|ds,\ \ t\in[\widetilde{\gamma},\widetilde{\gamma}+T],\tag{2.6}$$

and

$$\begin{aligned}|Z(t)|=|Z(t-T)|&=\left|Z(\widetilde{\gamma})-\int_{t-T}^{\widetilde{\gamma}}Z'(s)ds\right|\\&\le\int_{t-T}^{\widetilde{\gamma}}|Z'(s)|ds,\ \ t\in[\widetilde{\gamma},\widetilde{\gamma}+T].\end{aligned}\tag{2.7}$$

Combining the above two inequalities, we obtain

$$\begin{aligned}|Z|_\infty&=\max_{t\in[0,T]}|Z(t)|=\max_{t\in[\widetilde{\gamma},\widetilde{\gamma}+T]}|Z(t)|\\&\le\max_{t\in[\widetilde{\gamma},\widetilde{\gamma}+T]}\left\{\frac{1}{2}\left(\int_{\widetilde{\gamma}}^t|Z'(s)|ds+\int_{t-T}^{\widetilde{\gamma}}|Z'(s)|ds\right)\right\}\\&\le\frac{1}{2}\int_0^T|Z'(s)|ds\le\frac{1}{2}\sqrt{T}|Z'|_2.\end{aligned}\tag{2.8}$$

Now suppose that (A_2) (or(A_3)) holds. We shall consider two cases as follows.

Case (i) If (A_2) holds, multiplying both sides of (2.3) by $Z^{(n)}(t)$ and then integrating them from 0 to T, we have

$$\begin{aligned}
|Z^{(n)}|_2^2 &= \int_0^T |Z^{(n)}(t)|^2 dt \\
&= -\int_0^T (f(t, x_1^{(n-1)}(t)) - f(t, x_2^{(n-1)}(t)))Z^{(n)}(t)dt \\
&\quad -\int_0^T (g_1(t, x_1(t-\tau_1(t))) - g_1(t, x_2(t-\tau_1(t))))Z^{(n)}(t)dt \\
&\quad -\int_0^T (g_2(t, x_1(t-\tau_2(t))) - g_2(t, x_2(t-\tau_2(t))))Z^{(n)}(t)dt \\
&\le C_1 \int_0^T |x_1^{(n-1)}(t) - x_2^{(n-1)}(t)||Z^{(n)}(t)|dt \\
&\quad +b_1 \int_0^T |x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))||Z^{(n)}(t)|dt \\
&\quad +b_2 \int_0^T |x_1(t-\tau_2(t)) - x_2(t-\tau_2(t))||Z^{(n)}(t)|dt.
\end{aligned}$$

From this, (2.1), (2.8) and the Schwarz inequality, we have

$$\begin{aligned}
|Z^{(n)}|_2^2 &\le C_1 \left(\int_0^T |x_1^{(n-1)}(t) - x_2^{(n-1)}(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_0^T |Z^{(n)}(t)|^2 dt\right)^{\frac{1}{2}} \\
&\quad +(b_1+b_2)|Z|_\infty \int_0^T 1 \times |Z^{(n)}(t)|dt \\
&\le C_1|Z^{(n-1)}|_2|Z^{(n)}|_2 + (b_1+b_2)|Z|_\infty \left(\int_0^T 1^2 dt\right)^{\frac{1}{2}} \left(\int_0^T |Z^{(n)}(t)|^2 dt\right)^{\frac{1}{2}} \\
&\le C_1|Z^{(n-1)}|_2|Z^{(n)}|_2 + (b_1+b_2)|Z|_\infty \sqrt{T}|Z^{(n)}|_2 \\
&\le C_1\frac{T}{2\pi}|Z^{(n)}|_2^2 + \frac{(b_1+b_2)}{2}\sqrt{T}|Z'|_2\sqrt{T}|Z^{(n)}|_2 \\
&\le \left(C_1\frac{T}{2\pi} + \frac{(b_1+b_2)}{2}\frac{T^n}{(2\pi)^{n-1}}\right)|Z^{(n)}|_2^2.
\end{aligned}$$

It follows from $C_1\frac{T}{2\pi} + \frac{(b_1+b_2)}{2}\frac{T^n}{(2\pi)^{n-1}} < 1$ that

$$Z^{(n)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}. \tag{2.9}$$

Since $Z^{(n-2)}(0) = Z^{(n-2)}(T)$, there exists a constant $\xi_{n-1} \in [0, T]$ with $Z^{(n-1)}(\xi_{n-1})$ $= 0$, then, in view of (2.9), we get

$$Z^{(n-1)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}. \tag{2.10}$$

By using a similar argument as in the proof of (2.10), in view of (2.5), we can show

$$Z(t) \equiv Z'(t) \equiv \ldots \equiv Z^{(n-2)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}.$$

Thus, $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$. Therefore, (1.1) has at most one anti-periodic solution.

Case (ii) If (A_3) holds, multiplying both sides of (2.3) by $Z^{(n-1)}(t)$ and then integrating them from 0 to T, together with (2.8), we have

$$\begin{aligned}
C_2|Z^{(n-1)}|_2^2 &= \int_0^T C_2|x_1^{n-1}(t) - x_2^{n-1}(t)|^2 dt \\
&\le \int_0^T (f(x_1^{n-1}(t)) - f(x_2^{n-1}(t)))(x_1^{n-1}(t) - x_2^{n-1}(t))dt \\
&= -\int_0^T Z^{(n)}(t)Z^{(n-1)}(t)dt - \int_0^T (g_1(t, x_1(t-\tau_1(t))) \\
&\quad - g_1(t, x_2(t-\tau_1(t))))Z^{(n-1)}(t)dt - \int_0^T (g_2(t, x_1(t-\tau_2(t))) \\
&\quad - g_2(t, x_2(t-\tau_2(t))))Z^{(n-1)}(t)dt \\
&= -\int_0^T (g_1(t, x_1(t-\tau_1(t))) - g_1(t, x_2(t-\tau_1(t))))Z^{(n-1)}(t)dt \qquad (2.11) \\
&\quad - \int_0^T (g_2(t, x_1(t-\tau_2(t))) - g_2(t, x_2(t-\tau_2(t))))Z^{(n-1)}(t)dt \\
&\le b_1 \int_0^T |x_1(t-\tau_1(t)) - x_2(t-\tau_1(t))||Z^{(n-1)}(t)|dt \\
&\quad + b_2 \int_0^T |x_1(t-\tau_2(t)) - x_2(t-\tau_2(t))||Z^{(n-1)}(t)|dt \\
&\le (b_1 + b_2)|Z|_\infty \sqrt{T}|Z^{(n-1)}|_2 \\
&\le \frac{b_1 + b_2}{2} \frac{T^{n-1}}{(2\pi)^{n-2}} |Z^{(n-1)}|_2^2.
\end{aligned}$$

By using a similar argument as in the proof of Case (i), in view of (2.5), (A_3) and (2.11), we obtain

$$Z(t) \equiv Z'(t) \equiv \ldots \equiv Z^{(n-2)}(t) \equiv 0 \text{ for all } t \in \mathbb{R}.$$

Thus, $x_1(t) \equiv x_2(t)$ for all $t \in \mathbb{R}$. Therefore, (1.1) has at most one anti-periodic solution. The proof of Lemma 2.3, is now complete. □

Remark 2.4. If $f'(x) > C_2$ for all $t \in \mathbb{R}$, one can see that $f(x)$ satisfies the assumption $(\widetilde{A_0})$.

3 Main Results

In this section, we establish some sufficient conditions for the existence and uniqueness of anti-periodic solutions for (1.1).

Theorem 3.1. *Let condition* (A_1) *hold. Assume that condition* (A_2) *or condition* (A_3) *is satisfied. Then* (1.1) *has a unique anti-periodic solution.*

Proof. Consider the auxiliary equation of (1.1) as the following:

$$\begin{aligned} x^{(n)}(t) &= -\lambda f(t, x^{(n-1)}(t)) - \lambda g_1(t, x(t-\tau_1(t))) - \lambda g_2(t, x(t-\tau_2(t))) + \lambda e(t) \\ &= \lambda Q(t, x(t), x^{(n-1)}(t)), \quad \lambda \in (0, 1]. \end{aligned} \tag{3.1}$$

By Lemma 2.3, together with (A_2) and (A_3), it is easy to see that (1.1) has at most one anti-periodic solution. Thus, to prove Theorem 3.1, it suffices to show that (1.1) has at least one anti-periodic solution. To do this, we shall apply Lemma 2.1. Firstly, we will claim that the set of all possible anti-periodic solutions of (3.1) is bounded.

Let $x \in C_T^{n-1,\frac{1}{2}}$ be an arbitrary anti-periodic solution of (3.1). Then, by using a similar argument as that in the proof of (2.8), we have

$$|x|_\infty \leq \frac{1}{2}\sqrt{T}|x'|_2. \tag{3.2}$$

In view of (A_2) and (A_3), we consider two cases as follows.

Case (i) If (A_2) holds, multiplying both sides of (3.1) by $x^{(n)}(t)$ and then integrating them from 0 to T, in view of (2.1), (3.2), (A_2) and the inequality of Schwarz, we obtain

$$\begin{aligned} |x^{(n)}|_2^2 &= \int_0^T |x^{(n)}|^2 dt \\ &= -\lambda \int_0^T f(t, x^{(n-1)}(t)) x^{(n)}(t) dt - \lambda \int_0^T g_1(t, x(t-\tau_1(t))) x^{(n)}(t) dt \\ &\quad - \lambda \int_0^T g_2(t, x(t-\tau_2(t))) x^{(n)}(t) dt + \lambda \int_0^T e(t) x^{(n)}(t) dt \\ &\leq \int_0^T |f(t, x^{(n-1)}(t)) - f(t, 0) + f(t, 0)||x^{(n)}(t)| dt \\ &\quad + \int_0^T [|g_1(t, x(t-\tau_1(t))) - g_1(t, 0)| + |g_1(t, 0)|]|x^{(n)}(t)| dt \\ &\quad + \int_0^T [|g_2(t, x(t-\tau_2(t))) - g_2(t, 0)| + |g_2(t, 0)|]|x^{(n)}(t)| dt \\ &\quad + \int_0^T e(t) x^{(n)}(t) dt \\ &\leq C_1 |x^{(n-1)}|_2 |x^{(n)}|_2 + b_1 \int_0^T |x(t-\tau_1(t))||x^{(n)}(t)| dt \end{aligned}$$

$$
\begin{aligned}
&+ b_2 \int_0^T |x(t-\tau_2(t))||x^{(n)}(t)|dt + \int_0^T [|f(t,0)| + |g_1(t,0)| \\
&+ |g_2(t,0)|]|x^{(n)}(t)|dt + \int_0^T |e(t)||x^{(n)}(t)|dt \\
\le\ & C_1 \frac{T}{2\pi}|x^{(n)}|_2^2 + (b_1+b_2)|x|_\infty \sqrt{T}|x^{(n)}|_2 + [\max\{|f(t,0)| \\
&+ |g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]\sqrt{T}|x^{(n)}|_2 \\
\le\ & C_1 \frac{T}{2\pi}|x^{(n)}|_2^2 + \frac{b_1+b_2}{2}\sqrt{T}|x'|_2\sqrt{T}|x^{(n)}|_2 + [\max\{|f(t,0)| \\
&+ |g_1(t,0)| + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]\sqrt{T}|x^{(n)}|_2 \\
\le\ & C_1 \frac{T}{2\pi}|x^{(n)}|_2^2 + \frac{b_1+b_2}{2}\frac{T^n}{(2\pi)^{n-1}}|x^{(n)}|_2^2 + [\max\{|f(t,0)| + |g_1(t,0)| \\
&+ |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]\sqrt{T}|x^{(n)}|_2,
\end{aligned} \tag{3.3}
$$

which, together with (A_2), implies that there exists a positive constant D_1

$$|x^{(j)}|_2 \le \left(\frac{T}{2\pi}\right)^{n-j}|x^n|_2 < D_1, \ \ j = 1, 2, \ldots, n. \tag{3.4}$$

Since $x^{(j)}(0) = x^{(j)}(T)(j = 1, 2, \ldots, n-1)$, it follows that there exists a constant $\eta_j \in [0, T]$ such that

$$x^{(j+1)}(\eta_j) = 0,$$

and

$$
\begin{aligned}
|x^{(j+1)}(t)| &= \left| x^{(j+1)}(\eta_j) + \int_{\eta_j}^t x^{(j+2)}(s)ds \right| \\
&\le \int_0^T |x^{(j+2)}(t)|dt \le \sqrt{T}|x^{(j+2)}|_2,
\end{aligned} \tag{3.5}
$$

where $j = 1, 2, \ldots, n-2$, $t \in [0, T]$.

Together with (3.2) and (3.4), (3.5) implies that there exists a positive constant D_2

$$|x^{(j)}|_\infty \le \sqrt{T}|x^{(j+1)}|_2 \le D_2, \ \ j = 1, 2, \ldots, n-1, \tag{3.6}$$

which implies that, for all possible anti-periodic solutions $x(t)$ of (3.1), there exists a constant M_1 such that

$$\max_{1 \le j \le n-1} |x^{(j)}|_\infty < M_1. \tag{3.7}$$

Case (ii) If (A_3) holds, multiplying both sides of (3.1) by $x^{(n-1)}(t)$ and then integrating them from 0 to T, in view of (2.1), (3.2), (A_3) and the inequality of Schwarz, we obtain

$$\begin{aligned}
C_2|x^{(n-1)}|_2^2 =& \int_0^T C_2 x^{(n-1)}(t)x^{(n-1)}(t)dt \\
\leq& \int_0^T (f(x^{(n-1)}(t)) - f(0))x^{(n-1)}(t)dt \\
&- \int_0^T g_1(t, x(t-\tau_1(t)))x^{(n-1)}(t)dt \\
&- \int_0^T g_2(t, x(t-\tau_2(t)))x^{(n-1)}(t)dt \\
&+ \int_0^T e(t)x^{(n-1)}(t)dt - \int_0^T f(0)x^{(n-1)}(t)dt \\
\leq& \int_0^T |g_1(t, x(t-\tau_1(t))) - g_1(t,0)||x^{(n-1)}(t)|dt \\
&+ \int_0^T |g_2(t, x(t-\tau_2(t))) - g_2(t,0)||x^{(n-1)}(t)|dt \\
&+ \int_0^T |e(t)||x^{(n-1)}(t)|dt \\
&+ \int_0^T [|f(0)| + |g_1(t,0)| + |g_2(t,0)|]|x^{(n-1)}(t)|dt \\
\leq& b_1 \int_0^T |x(t-\tau_1(t))||x^{(n-1)}(t)|dt + b_2 \int_0^T |x(t-\tau_2(t))||x^{(n-1)}(t)|dt \\
&+ \int_0^T |e(t)||x^{(n-1)}(t)|dt \\
&+ \int_0^T [|f(0)| + |g_1(t,0)| + |g_2(t,0)|]|x^{(n-1)}(t)|dt \\
\leq& (b_1+b_2)|x|_\infty \sqrt{T}|x^{(n-1)}(t)|_2 + [\max\{|f(0)| + |g_1(t,0)| \\
&+ |g_2(t,0)| : 0 \leq t \leq T\} + |e|_\infty]\sqrt{T}|x^{(n-1)}(t)|_2 \\
\leq& \frac{b_1+b_2}{2} T|x'|_2|x^{(n-1)}(t)|_2 + [\max\{|f(0)| + |g_1(t,0)| \\
&+ |g_2(t,0)| : 0 \leq t \leq T\} + |e|_\infty]\sqrt{T}|x^{(n-1)}(t)|_2 \\
\leq& \frac{b_1+b_2}{2} \frac{T^n}{(2\pi)^{n-2}}|x^{(n-1)}(t)|_2^2 + [\max\{|f(0)| + |g_1(t,0)| \\
&+ |g_2(t,0)| : 0 \leq t \leq T\} + |e|_\infty]\sqrt{T}|x^{(n-1)}(t)|_2.
\end{aligned} \tag{3.8}$$

This implies that there exists a constant $\overline{D}_1 > 0$ such that

$$|x^{(j)}|_\infty \le \sqrt{T}|x^{(j+1)}|_2 \le \overline{D}_1, \ \ j = 1, 2, \dots, n-2. \tag{3.9}$$

Multiplying $x^{(n)}(t)$ and (3.1) and then integrating it from 0 to T, by (A_3), (3.2), (3.3), (3.9) and the inequality of Schwarz, we obtain

$$\begin{aligned}
|x^{(n)}|_2^2 &= \int_0^T |x^{(n)}(t)|^2 dt \\
&\le \int_0^T |g_1(t, x(t-\tau_1(t))) - g_1(t,0)||x^{(n)}(t)|dt + \int_0^T |g_2(t, x(t-\tau_2(t))) \\
&\quad - g_2(t,0)||x^{(n)}(t)|dt + \int_0^T |e(t)||x^{(n)}(t)|dt \\
&\quad + \int_0^T [|g_1(t,0)| + |g_2(t,0)|]|x^{(n)}(t)|dt \\
&\le b_1 \int_0^T |x(t-\tau_1(t))||x^{(n)}(t)|dt + b_2 \int_0^T |x(t-\tau_2(t))||x^{(n)}(t)|dt \\
&\quad + \int_0^T |e(t)||x^{(n)}(t)|dt + \int_0^T [|g_1(t,0)| + |g_2(t,0)|]|x^{(n)}(t)|dt \\
&\le (b_1+b_2)|x|_\infty \sqrt{T}|x^{(n)}(t)|_2 + [\max\{|g_1(t,0)| \\
&\quad + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]\sqrt{T}|x^{(n)}(t)|_2 \\
&\le \frac{b_1+b_2}{2} T|x'|_2|x^{(n)}(t)|_2 + [\max\{|g_1(t,0)| \\
&\quad + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]\sqrt{T}|x^{(n)}(t)|_2 \\
&\le \frac{b_1+b_2}{2} T\overline{D}_1|x^{(n)}(t)|_2 + [\max\{|g_1(t,0)| \\
&\quad + |g_2(t,0)| : 0 \le t \le T\} + |e|_\infty]\sqrt{T}|x^{(n)}(t)|_2,
\end{aligned}$$

it follows from (3.5) that there exists a positive constant $\overline{D}_2$ such that

$$|x^{(n-1)}(t)| \le \sqrt{T}|x^{(n)}|_2 \le \overline{D}_2. \tag{3.10}$$

Therefore, in view of (3.9) and (3.10), for all possible anti-periodic solutions x of (3.1), there exists a constant $\widetilde{M_1}$ such that

$$\max_{1 \le j \le n-1} |x^{(j)}|_\infty \le \widetilde{M_1}, \tag{3.11}$$

which, together with (3.7), implies that

$$\max_{1 \le j \le n-1} |x^{(j)}|_\infty \le M_1 + \widetilde{M_1} + 1 := M. \tag{3.12}$$

Set

$$\Omega = \left\{ x \in C_T^{n-1,\frac{1}{2}} = X : \max_{1\leq j\leq n-1} |x^{(j)}|_\infty < M \right\},$$

then we know that (3.1) has no anti-periodic solution on $\partial\Omega$ as $\lambda \in (0, 1]$.

Now, we consider the Fourier series expansion of a function $x \in C_T^{n-1,\frac{1}{2}}$, we have

$$x(t) = \sum_{i=0}^{\infty} \left[a_{2i+1} \cos \frac{2\pi(2i+1)t}{T} + b_{2i+1} \sin \frac{2\pi(2i+1)t}{T} \right].$$

Define an operator $L : C_T^{k,\frac{1}{2}} \to C_T^{k+1,\frac{1}{2}}$ by setting

$$\begin{aligned}(Lx)(t) &= \int_0^t x(s)ds - \frac{T}{2\pi} \sum_{i=0}^{\infty} \frac{b_{2i+1}}{2i+1} \\ &= \frac{T}{2\pi} \sum_{i=0}^{\infty} \left[\frac{a_{2i+1}}{(2i+1)} \sin \frac{2\pi(2i+1)t}{T} \right. \\ &\qquad \left. - \frac{b_{2i+1}}{(2i+1)} \cos \frac{2\pi(2i+1)t}{T} \right].\end{aligned} \tag{3.13}$$

Then

$$\frac{d(Lx)(t)}{dt} = x(t),$$

and

$$\begin{aligned}|(Lx)(t)| &\leq \int_0^T |x(s)|ds + \frac{T}{2\pi} \sum_{i=0}^{\infty} \frac{|b_{2i+1}|}{(2i+1)} \\ &\leq T||x|| + \frac{T}{2\pi} \left(\sum_{i=0}^{\infty} b_{2i+1}^2 \right)^{\frac{1}{2}} \left(\sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} \right)^{\frac{1}{2}}.\end{aligned} \tag{3.14}$$

In view of

$$\left(\sum_{i=0}^{\infty} \frac{1}{(2i+1)^2} \right)^{\frac{1}{2}} = \frac{\pi}{2\sqrt{2}},$$

and the Parseval equality

$$\int_0^T |x(s)|^2 ds = \frac{T}{2} \sum_{i=0}^{\infty} (a_{2i+1}^2 + b_{2i+1}^2),$$

we obtain

$$
\begin{aligned}
|(Lx)(t)| &\leq T||x|| + \frac{T}{4\sqrt{2}} \left(\sum_{i=0}^{\infty} (a_{2i+1}^2 + b_{2i+1}^2) \right)^{\frac{1}{2}} \\
&\leq T||x|| + \frac{T}{4\sqrt{2}} \left(\frac{2}{T} \int_0^T |x(s)|^2 ds \right)^{\frac{1}{2}} \qquad (3.15) \\
&\leq \left(T + \frac{T}{4} \right) ||x||, \;\; l\, t \in [0, T].
\end{aligned}
$$

Thus, $|(Lx)(t)| \leq (T + \dfrac{T}{4})||x||$, and the operator L is continuous.

For all $x \in C_T^{n-1,\frac{1}{2}}$, from (A_1), we get

$$
Q_1 \left(t + \frac{T}{2}, x \left(t + \frac{T}{2} \right), x^{(n-1)} \left(t + \frac{T}{2} \right) \right) = -Q_1(t, x(t), x^{(n-1)}(t)).
$$

Therefore, $Q_1(t, x(t), x^{(n-1)}(t)) \in C_T^{0,\frac{1}{2}}$. Define a operator $F_\mu : \overline{\Omega} \rightarrow C_T^{n,\frac{1}{2}} \subset X$ by setting

$$
F_\mu(x) = L(\ldots L(L(Q_1(x)))) = \mu L^n(Q_1(x)), \; \mu \in [0, 1].
$$

It is easy to see from the Arzela–Ascoli lemma that F_μ is a compact homotopy, and the fixed point of F_1 on $\overline{\Omega}$ is the antiperiodic solution of (1.1).

Define the homotopic continuous field as follows

$$
H_\mu(x) : \overline{\Omega} \times [0, 1] \rightarrow C_T^{n-1,\frac{1}{2}}, \;\; H_\mu(x) = x - F_\mu(x).
$$

Together with (3.12), we have

$$
H_\mu(\partial\Omega) \neq 0, \;\; \mu \in [0, 1].
$$

Hence, using the homotopy invariance theorem, we obtain

$$
\deg\{x - F_1 x, \Omega, 0\} = \deg\{x, \Omega, 0\} \neq 0.
$$

By now we know that Ω satisfies all the requirement in Lemma 2.1, and then $x - F_1 x = 0$ has at least one solution in the Ω, i.e., F_1 has a fixed point on $\overline{\Omega}$. So, we have proved that (1.1) has a unique anti-periodic solution. This completes the proof. □

4 Example and Remark

In this section, we give an example to demonstrate the results obtained in previous sections.

Example 4.1. Let $g_1(t,x) = g_2(t,x) = (1+\sin^4(t))\dfrac{1}{12\pi}\sin x$. Then the Rayleigh equation

$$\begin{aligned} x'''(t) + \frac{1}{8}x''(t) + \frac{1}{8}e^{-|\sin t|}\sin x''(t) + g_1(t, x(t-\sin^2 t)) \\ + g_2(t, x(t-\cos^2 t)) = \frac{1}{6\pi}\cos t, \end{aligned} \tag{4.1}$$

has a unique anti-periodic solution with period 2π.

Proof. From (4.1), we have $f(t,x) = \dfrac{1}{8}x + \dfrac{1}{8}e^{-|\sin t|}\sin x$, then

$$|f(t,x_1) - f(t,x_2)| \le \frac{1}{4}|x_1 - x_2| \text{ for all } t,\ x_1,\ x_2 \in \mathbb{R}.$$

Thus, $b_1 = b_2 = \dfrac{1}{6\pi},\ C_1 = \dfrac{1}{4},\ \tau_1(t) = \sin^2 t,\ \tau_2(t) = \cos^2 t$ and $e(t) = \dfrac{1}{6\pi}\cos t$. It is obvious that the assumptions (A_1) and (A_2) hold. Therefore, in view of Theorem 3.1, (4.1) has a unique anti-periodic solution with period 2π. □

Remark 4.2. Since there exist no results for the uniqueness of anti-periodic solutions of the nth-order differential equations with two deviating arguments. One can easily see that all the results in [1–23, 26] and the references therein can not be applicable to (4.1) to obtain the existence and uniqueness of anti-periodic solutions with periodic 2π. This implies that the results of this paper are essentially new.

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