Fractional Differential Inclusions with Impulses at Variable Times

Mouffak Benchohra and Farida Berhoun
Université de Sidi Bel Abbes
Laboratoire de Mathématiques
BP 89, 22000 Sidi Bel Abbes, Algeria
benchohra@univ-sba.dz, berhoun22@yahoo.fr

Juan J. Nieto
Universidad de Santiago de Compostela
Departamento de Análisis Matematico
Facultad de Matemáticas
Santiago de Compostela, Spain
juanjose.nieto.roig@usc.es

Abstract
In this paper, we establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential inclusions with variable times involving the Caputo fractional derivative.

AMS Subject Classifications: 26A33, 34A37, 34A60.
Keywords: Initial value problem, impulses, variable times, Caputo fractional derivative, fractional integral, differential inclusion.

This paper is dedicated to Johnny Henderson on the occasion of his 61st birthday.

1 Introduction

This paper deals with the existence of solutions for the initial value problems (IVP for short), for fractional order $1 < \alpha \leq 2$ differential inclusions with impulsive effects

\[ ^cD^\alpha y(t) \in F(t, y(t)), \text{ for a.e., } t \in J = [0, T], t \neq \tau_k(y(t)), k = 1, \ldots, m, \]  

Received July 13, 2011; Accepted August 19, 2011
Communicated by Martin Bohner
\( y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)), \quad k = 1, \ldots, m, \) 
\( y'(t^+) = \mathcal{T}_k(y(t)), \quad t = \tau_k(y(t)), \quad k = 1, \ldots, m, \) 
\( y(0) = y_0, \quad y'(0) = \overline{y_0}, \)

where \(^c D^\alpha\) is the Caputo fractional derivative, \( F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R}) \) is a compact convex valued multivalued map, \( \mathcal{P}(\mathbb{R}) \) is the family of all nonempty subsets of \( \mathbb{R} \), \( \tau_k : \mathbb{R} \to (0, T) \), \( I_k, \mathcal{T}_k : \mathbb{R} \to \mathbb{R}, \ k = 1, \ldots, m \) are given continuous functions and \( y_0, \overline{y_0} \in \mathbb{R} \).

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [21, 22, 26, 32, 38]). There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas et al [28], Lakshmikantham et al [30], Miller and Ross [33], Samko et al [42] and the papers of Agarwal et al [1], Ahmad and Nieto [3], Belmekki et al [8, 9], Benchohra et al [10, 12, 14], Chang and Nieto [16], Delbosco and Rodino [18], Diethelm et al [19], Podlubny et al [41], and the references therein.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain \( y(0), \ y'(0), \) etc. the same requirements of boundary conditions. Caputo’s fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [25, 40].

Integer order Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Benchohra et al [13], Lakshmikantham et al [29], and Samoilenko and Perestyuk [43] and the references therein. Particular attention has been given to differential equations at variable moments of impulse; see for instance the papers by Bajo and Liz [6], Belarbi and Benchohra [7], Benchohra et al [11], Frigon and O’Regan [20], Graef and Ouahab [23], Nieto et al [34–37], Stamova [44], Zhano and Feng [46] and the references therein. Very recently Agarwal et al [2], and Benchohra and Slimani [15] have initiated the study of impulsive fractional differential equations at fixed moments. These results were extended to the multivalued case by Ait Dads et al [4]. The aim of this paper is to extend these studies when the impulses are variable. This paper is organized as follows. In Section 2 we present some backgrounds on fractional derivatives and integrals and multivalued analysis. Section 3 is devoted to the existence of solutions for problem (1.1)–(1.4). In Section 4 we present an example showing the applicability of our assumptions.
2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm

$$||y||_{\infty} := \sup\{|y(t)| : t \in J\}.$$  

In order to define the solution of (1.1)-(1.4) we shall consider the space of functions

$$\Omega = \{y : J \to \mathbb{R} : \text{there exist } 0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T \text{ such that } t_k = \tau_k(y(t_k)), y(t_k^+) \text{ exists, } k = 0, \ldots, m \text{ and } y_k \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, \ldots, m\},$$

where $y_k$ is the restriction of $y$ over $(t_k, t_{k+1}], k = 0, \ldots, m$.

$L^1(J, \mathbb{R})$ denotes the Banach space of measurable functions $x : J \to \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_0^T |x(t)| dt \quad \text{for all } x \in L^1(J, \mathbb{R}).$$

Let $P_{c}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ closed}\}$, $P_{b}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ bounded}\}$, $P_{cp}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ compact}\}$ and $P_{cp,c}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ compact and convex}\}$. A multivalued map $G : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in \mathbb{R}$, $G$ is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in $\mathbb{R}$ for all $B \in P_b(\mathbb{R})$ (i.e. $\sup \{\sup \{|y| : y \in G(x)\} : x \in B\} < \infty$). $G$ is called upper semicontinuous (u.s.c.) on $\mathbb{R}$ if for each $x_0 \in \mathbb{R}$, the set $G(x_0)$ is a nonempty closed subset of $\mathbb{R}$, and if for each open set $N$ of $\mathbb{R}$ containing $G(x_0)$, there exists an open neighborhood $N_0$ of $x_0$ such that $G(N_0) \subseteq N$. $G$ is said to be completely continuous if $G(B)$ is relatively compact for every $B \in P_b(\mathbb{R})$. If the multivalued map $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph (i.e. $x_n \to x_*$, $y_n \to y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). $G$ has a fixed point if there is $x \in \mathbb{R}$ such that $x \in G(x)$. The fixed point set of the multivalued operator $G$ will be denoted by $FixG$. A multivalued map $G : [0, 1] \to P_{c}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \mapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Frankowska [5], Deimling [17] and Hu and Papageorgiou [27].

Definition 2.1. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be $L^1$-Carathéodory if

(i) $t \mapsto F(t, u)$ is measurable for each $u \in \mathbb{R}$;

(ii) $u \mapsto F(t, u)$ is upper semicontinuous for each $t \in J$;

(iii) $F(t, \cdot)$ is compact valued and measurable.

(iv) $F(t, \cdot)$ is $L^1$-Carathéodory.
(ii) \( u \mapsto F(t, u) \) is upper semicontinuous for almost all \( t \in J \);

(iii) for each \( q > 0 \), there exists \( \varphi_q \in L^1(J, \mathbb{R}_+) \) such that

\[
\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq \varphi_q(t) \quad \text{for all } |u| \leq q \text{ and for a.e. } t \in J.
\]

For each \( x \in \Omega \), define the set of selections of \( F \) by

\[
S_{F,x} = \{v \in L^1(J, \mathbb{R}) : v(t) \in F(t, x(t)) \text{ a.e. } t \in J\}.
\]

**Lemma 2.2** (See [31]). Let \( F : J \times \mathbb{R} \rightarrow P_{cp,c}(\mathbb{R}) \) be an \( L^1 \)-Carathéodory multivalued map and let \( L \) be a linear continuous mapping from \( L^1(J, \mathbb{R}) \) to \( C(J, \mathbb{R}) \), then the operator

\[
L \circ S_F : C(J, \mathbb{R}) \rightarrow P_{cp,c}(C(J, \mathbb{R})),
\]

\[
x \mapsto (L \circ S_F)(x) := L(S_{F,x})
\]

is a closed graph operator in \( C(J, \mathbb{R}) \times C(J, \mathbb{R}) \).

**Definition 2.3** (See [28, 39]). The fractional (arbitrary) order integral of the function \( h \in L^1([a,b], \mathbb{R}_+) \) of order \( \alpha \in \mathbb{R}_+ \) is defined by

\[
I^\alpha_a h(t) = \int_a^t (t-s)^{\alpha-1} \Gamma(\alpha) h(s) ds,
\]

where \( \Gamma \) is the gamma function. When \( a = 0 \), we write \( I^\alpha h(t) = [h \ast \varphi_\alpha](t) \), where \( \varphi_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \) for \( t > 0 \), and \( \varphi_\alpha(t) = 0 \) for \( t \leq 0 \).

\( \varphi_\alpha(t) \rightarrow \delta(t) \) as \( \alpha \rightarrow 1 \), where \( \delta \) is the delta function.

**Definition 2.4** (See [28, 39]). For a function \( h \) given on the interval \([a,b] \), the \( \alpha \)th Riemann–Liouville fractional-order derivative of \( h \), is defined by

\[
(D^\alpha_{a+} h)(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right) \left( \int_a^t (t-s)^{n-\alpha-1} h(s) ds \right)
\]

Here \( n = [\alpha] + 1 \) and \( [\alpha] \) denotes the integer part of \( \alpha \).

**Definition 2.5** (See [28, 39]). For a function \( h \) given on the interval \([a,b] \), the \( \alpha \)th Caputo fractional-order derivative of \( h \), is defined by

\[
(\,^cD^\alpha_{a+} h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds
\]

Here \( n = [\alpha] + 1 \) and \( [\alpha] \) denotes the integer part of \( \alpha \).
3 Existence of Solutions

Let us define what we mean by a solution of problem (1.1)-(1.4).

**Definition 3.1.** A function \( y \in \Omega \) is said to be a solution of (1.1)-(1.4) if there exist a function \( v \in L^1(J, \mathbb{R}) \) such that \( v(t) \in F(t, y(t)) \) a.e. \( t \in J \) and satisfies the equation \( cD^\alpha_t y(t) = v(t) \), for a.e. \( t \in (t_k, t_{k+1}) \), \( t \neq \tau_k(y(t)) \), \( k = 1, \ldots, m \), and conditions \( y(t^+) = I_k(y(t)) \), \( y'(t^+) = \overline{I}_k(y(t)) \), \( t = \tau_k(y(t)) \), \( k = 1, \ldots, m \), and \( y(0) = y_0 \), \( y'(0) = \overline{y}_0 \) are satisfied.

For the existence of solutions for the problem (1.1)-(1.4), we need the following auxiliary lemmas.

**Lemma 3.2** (See [45]). Let \( \alpha > 0 \), then the linear differential equation

\[
cD^\alpha h(t) = 0
\]

has solutions \( h(t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1} \), \( c_i \in \mathbb{R} \), \( i = 0, 1, 2, \ldots, n-1 \), \( n = [\alpha] + 1 \).

**Lemma 3.3** (See [45]). Let \( \alpha > 0 \), then

\[
\Gamma^\alpha cD^\alpha h(t) = h(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}
\]

for some \( c_i \in \mathbb{R} \), \( i = 0, 1, 2, \ldots, n-1 \), \( n = [\alpha] + 1 \).

Let us introduce the following hypotheses which are assumed hereafter:

(H1) The function \( F : J \times \mathbb{R} \to P_{cp,c}(\mathbb{R}) \) is \( L^1 \)-Carathéodory;

(H2) There exist a continuous non-decreasing function \( \psi : [0, \infty) \to (0, \infty) \) and a function \( p \in C(J, \mathbb{R}^+ \) such that

\[
\|F(t, u)\| \leq p(t)\psi(|u|) \text{ for each } (t, u) \in J \times \mathbb{R};
\]

(H3) The functions \( I_k, \overline{I}_k : \mathbb{R} \to \mathbb{R}, k = 1, \ldots, m \) are continuous and nondecreasing;

(H4) There exists a number \( M > 0 \) such that

\[
\min \left\{ \frac{M}{|y_0| + T|\overline{y}_1| + \frac{p^0 T^\alpha}{\Gamma(\alpha+1)} \psi(M)}, \frac{M}{|I_k(M)| + T|\overline{I}_k(M)| + \frac{p^0 T^\alpha}{\Gamma(\alpha+1)} \psi(M)} \right\} > 1,
\]

where

\[
p^0 = \sup\{p(t) : t \in J\};
\]
(H₅) The functions $\tau_k \in C^1(\mathbb{R}, \mathbb{R})$ for $k = 1, \ldots, m$. Moreover

$$0 < \tau_1(y) < \ldots < \tau_m(y) < T \text{ for all } y \in \mathbb{R};$$

(H₆) For all $y \in \mathbb{R}$

$$\tau_k(I_k(y)) \leq \tau_k(y) < \tau_{k+1}(I_k(y)) \text{ for } k = 1, \ldots, m;$$

(H₇) For $a \in J$ fixed, $y \in \Omega, v \in S_{F,y}$ and for each $t \in J$ we have

$$\tau_k'(y(t)) \int_a^t (t-s)^{\alpha-2}v(s)ds \neq \Gamma(\alpha - 1) \text{ for } k = 1, \ldots, m.$$

**Theorem 3.4.** Assume that hypotheses (H₁)–(H₇) hold. Then the IVP (1.1)–(1.4) has at least one solution.

**Proof.** The proof will be given in several steps.

**Step 1:** Consider the problem

$$^cD^\alpha y(t) \in F(t, y(t)), \text{ for a.e., } t \in J \quad (3.1)$$

$$y(0) = y_0, \ y'(0) = \overline{y}_0. \quad (3.2)$$

Transform the problem (3.1)–(3.2) into a fixed point problem. Consider the operator $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ defined by

$$N(y) = \{ h \in C(J, \mathbb{R}) : h(t) = y_0 + \overline{y}_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}v(s)ds, \ v \in S_{F,y} \}.$$

We shall show that $N$ satisfies the assumptions of nonlinear alternative of Leray–Schauder type.

**Claim 1:** $N(y)$ is convex for each $y \in C(J, \mathbb{R})$.

Indeed, if $h_1, h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = y_0 + \overline{y}_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}v_i(s)ds \ (i = 1, 2).$$

Let $0 \leq d \leq 1$. Then for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = y_0 + \overline{y}_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}[dv_1(s) + (1-d)v_2(s)]ds.$$
Since $S_{F,y}$ is convex (because $F$ has convex values), then
\[ dh_1 + (1 - d)h_2 \in N(y). \]

**Claim 2:** $N$ maps bounded sets into bounded sets in $C(J, \mathbb{R})$.

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant $\ell$ such that for each $y \in B_q = \{ y \in C(J, \mathbb{R}) : \| y \|_\infty \leq q \}$, we have $\| N(y) \|_\infty \leq \ell$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that
\[ h(t) = y_0 + y_0 t + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} v(s) ds \]

By (H2) we have for each $t \in J$,
\[
|h(t)| \leq |y_0| + T|y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} |v(s)| ds
\leq |y_0| + T|y_0| + \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} p(s) \psi(|y(s)|) ds
\leq |y_0| + T|y_0| + \frac{p^\beta \psi(q)}{T^\alpha p^\beta \psi(q)} \int_0^t (t - s)^{\alpha - 1} ds
\leq |y_0| + T|y_0| + \frac{\psi(q)}{T^\alpha (\alpha + 1)} := \ell.
\]

**Claim 3:** $N$ maps bounded sets into equicontinuous sets of $C(J, \mathbb{R})$.

Let $\tilde{t}_1, \tilde{t}_2 \in J$, $\tilde{t}_1 < \tilde{t}_2$, $B_q$ a bounded set of $C(J, \mathbb{R})$ as in Claim 2 and let $y \in B_q$ and $h \in N(y)$. Then
\[
|h(\tilde{t}_2) - h(\tilde{t}_1)| \leq \frac{1}{\psi(q)} (\tilde{t}_2 - \tilde{t}_1) + \frac{1}{\Gamma(\alpha)} \int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha - 1} |v(s)| ds
+ \frac{1}{\Gamma(\alpha)} \int_0^{\tilde{t}_1} [((\tilde{t}_2 - s)^{\alpha - 1} - (\tilde{t}_1 - s)^{\alpha - 1}) |v(s)| ds
\leq \frac{\psi(q)}{\Gamma(\alpha)} \int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha - 1} p(s) ds
+ \frac{\psi(q)}{T^\alpha p^\beta \psi(q)} \int_0^{\tilde{t}_1} [((\tilde{t}_2 - s)^{\alpha - 1} - (\tilde{t}_1 - s)^{\alpha - 1}) p(s) ds.
\]

As $\tilde{t}_1 \to \tilde{t}_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that $N : C(J, \mathbb{R}) \to \mathcal{P}(C(J, \mathbb{R}))$ is completely continuous.

**Claim 4:** $N$ has a closed graph.
Let \( y_n \to y_\ast \), \( h_n \in N(y_n) \) and \( h_n \to h_\ast \). We need to show that \( h_\ast \in N(y_\ast) \).

\( h_n \in N(y_n) \) means that there exists \( v_n \in S_{F,y_n} \) such that, for each \( t \in J \)

\[
h_n(t) = y_0 + \overline{y_0} t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_n(s) ds.
\]

We must show that there exist \( v_\ast \in S_{F,y_\ast} \) such that for each \( t \in J \),

\[
h_\ast(t) = y_0 + \overline{y_0} t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_\ast(s) ds.
\]

Clearly \( \|(h_n - \phi) - (h_\ast - \phi)\|_\infty \to 0 \) as \( n \to \infty \), where \( \phi(t) = y_0 + t \overline{y_0} \).

Consider the continuous linear operator \( \Gamma : L^1(J, \mathbb{R}) \to C(J, \mathbb{R}) \) defined by

\[
v \mapsto (\Gamma v)(t) = \int_0^t (t-s)^{\alpha-1} v(s) ds.
\]

From Lemma (2.2), it follows that \( \Gamma \circ S_F \) is a closed graph operator. Moreover, we have

\[
(h_n(t) - \phi(t)) \in \Gamma(S_{F,y_n}).
\]

Since \( y_n \to y_\ast \), it follows from Lemma 2.2 that

\[
h_\ast(t) = y_0 + \overline{y_0} t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v_\ast(s) ds.
\]

**Claim 5: A priori bounds on solutions.**

Let \( y \in C(J, \mathbb{R}) \) be such that \( y \in \lambda N(y) \) for some \( \lambda \in (0, 1) \). Then, there exists \( v \in L^1(J, \mathbb{R}) \) with \( v \in S_{F,y} \) such that, for each \( t \in J \),

\[
y(t) = \lambda y_0 + \lambda \overline{y_0} t + \frac{\lambda}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s) ds.
\]

From (H2)

\[
|y(t)| \leq |y_0| + T|\overline{y_0}| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |v(s)| ds
\]

\[
\leq |y_0| + T|\overline{y_0}| + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p(s) \psi(|y(s)|) ds
\]

\[
\leq |y_0| + T|\overline{y_0}| + \frac{\mu \psi(\|y\|_\infty)}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} ds
\]

\[
\leq |y_0| + T|\overline{y_0}| + \frac{T^\alpha \mu \psi(\|y\|_\infty)}{\Gamma(\alpha + 1)}.\]
Thus
\[
\|y\|_\infty \leq 1
\]
Then by (H_4), \(\|y\|_\infty \neq M\).
Let
\[
U = \{y \in C(J, \mathbb{R}) : \|y\|_\infty < M\}.
\]
The operator \(N : \overline{U} \to \mathcal{P}(C(J, \mathbb{R}))\) is upper semicontinuous and completely continuous. From the choice of \(U\), there is no \(y \in \partial U\) such that \(y \in \lambda N(y)\) for some \(\lambda \in (0, 1)\).
As a consequence of the nonlinear alternative of Leray–Schauder type [24], we deduce that \(N\) has a fixed point in \(\overline{U}\) which is a solution of the problem (3.1)–(3.2). Denote this solution \(y_1\). Define the function
\[
r_{k,1}(t) = \tau_k(y_1(t)) - t \text{ for } t \geq 0.
\]
(H_5) implies that
\[
r_{k,1}(0) \neq 0 \text{ for } k = 1, \ldots, m.
\]
If \(r_{k,1}(t) \neq 0\) on \(J\) for \(k = 1, \ldots, m\), then \(y_1\) is solution of the problem (1.1)–(1.4).
Now we consider the case when \(r_{1,1}(t) = 0\) for some \(t \in J\). Since \(r_{1,1}(0) \neq 0\) and \(r_{1,1}\) is continuous, there exists \(t_1 > 0\) such that
\[
r_{1,1}(t_1) = 0 \text{ and } r_{1,1}(t) \neq 0 \text{ for all } t \in [0, t_1).
\]
Thus by (H_5) we have
\[
r_{k,1}(t) \neq 0 \text{ for all } t \in [0, t_1) \text{ and } k = 1, \ldots, m.
\]

**Step 2:** Consider the problem
\[
cD_t^\alpha y(t) \in F(t, y(t)), \text{ for a.e., } t \in [t_1, T]
\]
\[
y(t_1^+) = I_1(y_1(t_1))
\]
\[
y'(t_1^+) = \overline{I}_1(y_1(t_1)).
\]
Consider the operator \(N_1 : C([t_1, T], \mathbb{R}) \to \mathcal{P}(C([t_1, T], \mathbb{R}))\) defined by
\[
N_1(y) = \{h \in C([t_1, T], \mathbb{R})\}
\]
with
\[
h(t) = I_1(y_1(t_1)) + (t - t_1)\overline{I}_1(y_1(t_1)) + \frac{1}{\Gamma(\alpha)} \int_{t_1}^t (t - s)^{\alpha - 1}v(s),
\]
and \( v \in S_{F,y} \). As in Step 1 we can show that \( N_1 \) satisfies the assumptions of the nonlinear alternative of Leray–Schauder type and deduce that the problem (3.3)–(3.5) has a solution denoted by \( y_2 \). Define

\[
r_{k,2}(t) = \tau_k(y_2(t)) - t \quad \text{for} \quad t \geq t_1.
\]

If \( r_{k,2}(t) \neq 0 \) on \( (t_1, T] \) for \( k = 1, \ldots, m \), then

\[
y(t) = \begin{cases} 
y_1(t), & \text{if } t \in [0, t_1], \\
y_2(t), & \text{if } t \in (t_1, T]
\end{cases}
\]

is solution of the problem (1.1)–(1.4). It remains to consider the case when \( r_{2,2}(t) = 0 \), for some \( t \in (t_1, T] \).

By (H_6) we have

\[
r_{2,2}(t_1^+) = \tau_2(y_2(t_1^+)) - t_1 \\
= \tau_2(I_1(y_1(t_1))) - t_1 \\
> \tau_1(y_1(t_1)) - t_1 \\
= r_{1,1}(t_1) = 0.
\]

Since \( r_{2,2} \) is continuous, there exists \( t_2 > t_1 \) such that \( r_{2,2}(t_2) = 0 \) and \( r_{2,2}(t) \neq 0 \) for all \( t \in (t_1, t_2) \). By (H_5) we have

\[
r_{k,2}(t) \neq 0 \quad \text{for all} \quad t \in (t_1, t_2) \quad \text{and} \quad k = 2, \ldots, m.
\]

Suppose now that there exists \( s \in (t_1, t_2) \) such that \( r_{1,2}(s) = 0 \). From (H_6) it follows that

\[
r_{1,2}(t_1) = \tau_1(y_2(t_1)) - t_1 \\
= \tau_1(I_1(y_1(t_1))) - t_1 \\
\leq \tau_1(y_1(t_1)) - t_1 \\
= r_{1,1}(t_1) = 0.
\]

Also, from (H_5) we have

\[
r_{1,2}(t_2) = \tau_1(y_2(t_2)) - t_2 \\
< \tau_2(y_2(t_2)) - t_2 \\
= r_{2,2}(t_2) = 0.
\]

Since \( r_{1,2}(t_1) \leq 0 \), \( r_{1,2}(t_2) < 0 \) and \( r_{1,2}(s) = 0 \) for some \( s \in (t_1, t_2) \), the function \( r_{1,2} \) attains a nonnegative maximum at some point \( \bar{s} \in (t_1, t_2) \). Since

\[
y'_2(t) = \frac{1}{\Gamma(\alpha - 1)} \int_{t_1}^{t} (t - s)^{\alpha - 2} v(s) ds,
\]
we have
\[ r'_{1,2}(\bar{s}) = r'_1(y_2(\bar{s}))y'_2(\bar{s}) - 1 = 0. \]

Therefore
\[ \tau'_1(y_2(\bar{s})) \int_{t_1}^{\bar{s}} (\bar{s} - s)^{\alpha-2} v(s) ds = \Gamma(\alpha - 1), \]

which contradicts (H7).

**Step 3:** We continue this process and taking into account that \( y_{m+1} := y|_{[t_m,T]} \) is a solution to the problem

\[ ^cD^\alpha_{t_m} y(t) \in F(t, y(t)), \text{ for a.e., } t \in (t_m, T], \quad (4.1) \]
\[ y(t^+) = I_k(y(t)), \quad t = \tau_k(y(t)), \quad k = 1, \ldots, m, \quad (4.2) \]
\[ y'(t^+) = T_k(y(t)), \quad t = \tau_k(y(t)), \quad k = 1, \ldots, m, \quad (4.3) \]
\[ y(0) = 0, \quad y'(0) = 0, \quad (4.4) \]

where
\[ \tau_k(y) = 1 - \frac{1}{2^k(1 + y^2)}, \]
\[ I_k(y) = b_ky, \quad T_k(y) = \bar{b}_k y, \quad \bar{b}_k > 0, \quad b_k \in \left( \frac{1}{\sqrt{2}}, 1 \right], \quad k = 1, 2, \ldots, m. \]

We have
\[ \tau_{k+1}(y) - \tau_k(y) = \frac{1}{2^{k+1}(1 + y^2)} > 0 \text{ for each } y \in \mathbb{R}, \quad k = 1, 2, \ldots, m. \]
Hence

\[ 0 < \tau_1(y) < \tau_2(y) < \ldots < \tau_m(y) < 1, \text{ for each } y \in \mathbb{R}. \]

Also,

\[ \tau_k(I_k(y)) - \tau_k(y) = \frac{b_k^2 - 1}{2^k(1 + y^2)(1 + b_k^2 y^2)} \leq 0, \text{ for each } y \in \mathbb{R}, \]

and

\[ \tau_{k+1}(I_k(y)) - \tau_k(y) = \frac{1 + (2b_k^2 - 1)y^2}{2^{k+1}(1 + y^2)(1 + b_k^2 y^2)} > 0, \text{ for each } y \in \mathbb{R}. \]

Finally,

\[ F(t,y) = \{ v \in \mathbb{R} : f_1(t,y) \leq v \leq f_2(t,y) \}, \]

where \( f_1, f_2 : J \times \mathbb{R} \to \mathbb{R} \). We assume that for each \( t \in J \), \( f_1(t, \cdot) \) is lower semicontinuous (i.e, the set \{ \( y \in \mathbb{R} : f_1(t,y) > \mu \) \} is open for each \( \mu \in \mathbb{R} \)), and assume that for each \( t \in J \), \( f_2(t, \cdot) \) is upper semicontinuous (i.e the set \{ \( y \in \mathbb{R} : f_2(t,y) < \mu \) \} is open for each \( \mu \in \mathbb{R} \)). Assume that there are \( p \in C(J, \mathbb{R}^+) \) and \( \psi : [0, \infty) \to (0, \infty) \) continuous and nondecreasing such that

\[ \max \{|f_1(t,y)|, |f_2(t,y)|\} \leq p(t)\psi(|y|), \quad t \in J, \text{ and all } y \in \mathbb{R}. \]

Assume there exists a constant \( M > 0 \) such that

\[ \frac{M}{M(b_k + \bar{b}_k) + \frac{p^0}{\Gamma(\alpha+1)} \psi(M)} > 1, \]

where \( p^0 = \sup_{t \in J} p(t) \). It is clear that \( F \) is compact and convex valued, and it is upper semicontinuous (see [17]). If (H_7) holds, then by Theorem (3.4) the problem (4.1)–(4.4) has at least one solution.

**References**


Impulsive Fractional Differential Inclusions at Variable Times


