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Fractional Differential Inclusions with Impulses at Variable Times

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Abstract

In this paper, we establish sufficient conditions for the existence of solutions for a class of initial value problem for impulsive fractional differential inclusions with variable times involving the Caputo fractional derivative.

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This paper is dedicated to Johnny Henderson on the occasion of his 61st birthday.

1 Introduction

This paper deals with the existence of solutions for the initial value problems (IVP for short), for fractional order $1 < \alpha \leq 2$ differential inclusions with impulsive effects

$$^{c}D^{\alpha}y(t) \in F(t, y(t)), \text{ for a.e., } t \in J = [0, T], \ t \neq \tau_k(y(t)), \ k = 1, \dots, m, \quad (1.1)$$

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$$y(t^+) = I_k(y(t)),\ t = \tau_k(y(t)),\ k = 1, \ldots, m, \tag{1.2}$$

$$y'(t^+) = \overline{I}_k(y(t)),\ t = \tau_k(y(t)),\ k = 1, \ldots, m, \tag{1.3}$$

$$y(0) = y_0,\ y'(0) = \overline{y_0}, \tag{1.4}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative, $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a compact convex valued multivalued map, ($\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, $\tau_k : \mathbb{R} \to (0, T),\ I_k,\ \overline{I}_k : \mathbb{R} \to \mathbb{R},\ k = 1, \ldots, m$ are given continuous functions and $y_0,\ \overline{y_0} \in \mathbb{R}$.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [21, 22, 26, 32, 38]). There has been a significant development in fractional differential and partial differential equations in recent years; see the monographs of Kilbas et al [28], Lakshmikantham et al [30], Miller and Ross [33], Samko et al [42] and the papers of Agarwal et al [1], Ahmad and Nieto [3], Belmekki et al [8, 9], Benchohra et al [10, 12, 14], Chang and Nieto [16], Delbosco and Rodino [18], Diethelm et al [19], Podlubny et al [41], and the references therein.

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, which contain $y(0),\ y'(0)$, etc. the same requirements of boundary conditions. Caputo's fractional derivative satisfies these demands. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann–Liouville and Caputo types see [25, 40].

Integer order Impulsive differential equations have become important in recent years as mathematical models of phenomena in both the physical and social sciences. There has a significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments; see for instance the monographs by Benchohra et al [13], Lakshmikantham et al [29], and Samoilenko and Perestyuk [43] and the references therein. Particular attention has been given to differential equations at variable moments of impulse; see for instance the papers by Bajo and Liz [6], Belarbi and Benchohra [7], Benchohra et al [11], Frigon and O'Regan [20], Graef and Ouahab [23], Nieto et al [34–37], Stamova [44], Zhano and Feng [46] and the references therein. Very recently Agarwal et al [2], and Benchohra and Slimani [15] have initiated the study of impulsive fractional differential equations at fixed moments. These results were extended to the multivalued case by Ait Dads et al [4]. The aim of this paper is to extend these studies when the impulses are variable. This paper is organized as follows. In Section 2 we present some backgrounds on fractional derivatives and integrals and multivalued analysis. Section 3 is devoted to the existence of solutions for problem (1.1)–(1.4). In Section 4 we present an example showing the applicability of our assumptions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts from multivalued analysis which are used throughout this paper.
By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

In order to define the solution of (1.1)–(1.4) we shall consider the space of functions

$$\begin{aligned}\Omega \;=\; & \{y : J \to \mathbb{R} : \text{ there exist } 0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = T \\ & \text{such that } t_k = \tau_k(y(t_k)),\ y(t_k^+) \text{ exists},\ k = 0, \ldots, m \\ & \text{and } y_k \in C((t_k, t_{k+1}], \mathbb{R}),\ k = 0, \ldots, m\},\end{aligned}$$

where y_k is the restriction of y over $(t_k, t_{k+1}],\ k = 0, \ldots, m.$

$L^1(J,\mathbb{R})$ denotes the Banach space of measurable functions $x : J \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|x\|_{L^1} = \int_0^T |x(t)|dt \quad \text{for all } x \in L^1(J,\mathbb{R}).$$

Let $P_{cl}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ closed}\}$, $P_b(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ bounded}\}$, $P_{cp}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ compact}\}$ and $P_{cp,c}(\mathbb{R}) = \{Y \in \mathcal{P}(\mathbb{R}) : Y \text{ compact and convex}\}$. A multivalued map $G : \mathbb{R} \to P(\mathbb{R})$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in \mathbb{R}$, G is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in $\mathbb{R}$ for all $B \in P_b(\mathbb{R})$ (i.e. $\sup_{x\in B}\{\sup\{|y| : y \in G(x)\}\} < \infty$). G is called upper semicontinuous (u.s.c.) on $\mathbb{R}$ if for each $x_0 \in \mathbb{R}$, the set $G(x_0)$ is a nonempty closed subset of $\mathbb{R}$, and if for each open set N of $\mathbb{R}$ containing $G(x_0)$, there exists an open neighborhood N_0 of x_0 such that $G(N_0) \subseteq N$. G is said to be completely continuous if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in P_b(\mathbb{R})$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*,\ y_n \longrightarrow y_*,\ y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a fixed point if there is $x \in \mathbb{R}$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$. A multivalued map $G : [0,1] \to P_{cl}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function

$$t \longmapsto d(y, G(t)) = \inf\{|y - z| : z \in G(t)\}$$

is measurable. For more details on multivalued maps see the books of Aubin and Frankowska [5], Deimling [17] and Hu and Papageorgiou [27].

Definition 2.1. A multivalued map $F : J \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \longmapsto F(t,u)$ is measurable for each $u \in \mathbb{R}$;

(ii) $u \longmapsto F(t,u)$ is upper semicontinuous for almost all $t \in J$;

(iii) for each $q > 0$, there exists $\varphi_q \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,u)\| = \sup\{|v| : v \in F(t,u)\} \le \varphi_q(t) \text{ for all } |u| \le q \text{ and for a.e. } t \in J.$$

For each $x \in \Omega$, define the set of selections of F by

$$S_{F,x} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,x(t)) \text{ a.e. } t \in J\}.$$

Lemma 2.2 (See [31]). *Let $F : J \times \mathbb{R} \longrightarrow P_{cp,c}(\mathbb{R})$ be an L^1-Carathéodory multivalued map and let $\mathcal{L}$ be a linear continuous mapping from $L^1(J,\mathbb{R})$ to $C(J,\mathbb{R})$, then the operator*

$$\begin{array}{rl} \mathcal{L} \circ S_F : C(J,\mathbb{R}) & \longrightarrow P_{cp,c}(C(J,\mathbb{R})), \\ x & \longmapsto (\mathcal{L} \circ S_F)(x) := \mathcal{L}(S_{F,x}) \end{array}$$

is a closed graph operator in $C(J, \mathbb{R} \times C(J,\mathbb{R})$.

Definition 2.3 (See [28, 39]). The fractional (arbitrary) order integral of the function $h \in L^1([a,b],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by

$$I_a^\alpha h(t) = \int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} h(s)ds,$$

where Γ is the gamma function. When $a = 0$, we write $I^\alpha h(t) = [h * \varphi_\alpha](t)$, where $\varphi_\alpha(t) = \dfrac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $t > 0$, and $\varphi_\alpha(t) = 0$ for $t \le 0$.

$\varphi_\alpha(t) \to \delta(t)$ as $\alpha \to 1$, where δ is the delta function.

Definition 2.4 (See [28, 39]). For a function h given on the interval $[a,b]$, the αth Riemann–Liouville fractional-order derivative of h, is defined by

$$(D_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\alpha-1} h(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

Definition 2.5 (See [28,39]). For a function h given on the interval $[a,b]$, the αth Caputo fractional-order derivative of h, is defined by

$$({}^cD_{a+}^\alpha h)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} h^{(n)}(s)ds.$$

Here $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of α.

3 Existence of Solutions

Let us define what we mean by a solution of problem (1.1)–(1.4).

Definition 3.1. A function $y \in \Omega$ is said to be a solution of (1.1)–(1.4) if there exist a function $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t, y(t))$ a.e. $t \in J$ and satisfies the equation ${}^cD^{\alpha}_{t_k}y(t) = v(t),$ for a.e. $t \in (t_k, t_{k+1}),\ t \neq \tau_k(y(t)),\ k = 1,\ldots,m$, and conditions $y(t^+) = I_k(y(t)),\ y'(t^+) = \overline{I}_k(y(t)),\ t = \tau_k(y(t)),\ k = 1,\ldots,m$, and $y(0) = y_0,\ y'(0) = \overline{y_0}$ are satisfied.

For the existence of solutions for the problem (1.1)–(1.4), we need the following auxiliary lemmas.

Lemma 3.2 (See [45]). *Let $\alpha > 0$, then the linear differential equation*

$$ {}^cD^{\alpha}h(t) = 0 $$

has solutions $h(t) = c_0 + c_1t + c_2t^2 + \ldots + c_{n-1}t^{n-1}, c_i \in \mathbb{R},\ i = 0,1,2,\ldots,n-1,\ n = [\alpha]+1.$

Lemma 3.3 (See [45]). *Let $\alpha > 0$, then*

$$ I^{\alpha}\, {}^cD^{\alpha}h(t) = h(t) + c_0 + c_1t + c_2t^2 + \ldots + c_{n-1}t^{n-1} $$

for some $c_i \in \mathbb{R},\ i = 0,1,2,\ldots,n-1,\ n = [\alpha]+1.$

Let us introduce the following hypotheses which are assumed hereafter:

(H_1) The function $F : J \times \mathbb{R} \to P_{cp,c}(\mathbb{R})$ is L^1-Carathéodory;

(H_2) There exist a continuous non-decreasing function $\psi : [0,\infty) \longrightarrow (0,\infty)$ and a function $p \in C(J,\mathbb{R}_+)$ such that

$$ \|F(t,u)\| \le p(t)\psi(|u|) \text{ for each } (t,u) \in J \times \mathbb{R}; $$

(H_3) The functions $I_k, \overline{I}_k : \mathbb{R} \to \mathbb{R},\ k = 1,\ldots,m$ are continuous and nondecreasing;

(H_4) There exists a number $M > 0$ such that

$$ \min\left\{ \frac{M}{|y_0| + T|\overline{y}_1| + \frac{p^0T^{\alpha}}{\Gamma(\alpha+1)}\psi(M)}, \frac{M}{|I_k(M)| + T|\overline{I}_k(M)| + \frac{p^0T^{\alpha}}{\Gamma(\alpha+1)}\psi(M)} \right\} > 1, $$

where

$$ p^0 = \sup\{p(t) : t \in J\}; $$

(H_5) The functions $\tau_k \in C^1(\mathbb{R},\mathbb{R})$ for $k = 1,\ldots,m$. Moreover

$$0 < \tau_1(y) < \ldots < \tau_m(y) < T \text{ for all } y \in \mathbb{R};$$

(H_6) For all $y \in \mathbb{R}$

$$\tau_k(I_k(y)) \leq \tau_k(y) < \tau_{k+1}(I_k(y)) \text{ for } k = 1,\ldots,m;$$

(H_7) For $a \in J$ fixed, $y \in \Omega, v \in S_{F,y}$ and for each $t \in J$ we have

$$\tau_k'(y(t))\int_a^t (t-s)^{\alpha-2}v(s)ds \neq \Gamma(\alpha-1) \text{ for } k = 1,\ldots,m.$$

Theorem 3.4. *Assume that hypotheses* (H_1)–(H_7) *hold. Then the IVP* (1.1)–(1.4) *has at least one solution.*

Proof. The proof will be given in several steps.

Step 1: Consider the problem

$$^cD^\alpha y(t) \in F(t,y(t)), \text{ for a.e., } t \in J \tag{3.1}$$

$$y(0) = y_0,\ y'(0) = \overline{y_0}. \tag{3.2}$$

Transform the problem (3.1)–(3.2) into a fixed point problem. Consider the operator $N : C(J,\mathbb{R}) \to \mathcal{P}(C(J,\mathbb{R}))$ defined by

$$N(y) = \{h \in C(J,\mathbb{R}) : h(t) = y_0 + \overline{y_0}t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds,\ v \in S_{F,y}\}.$$

We shall show that N satisfies the assumptions of nonlinear alternative of Leray–Schauder type.

Claim 1: *$N(y)$ is convex for each $y \in C(J,\mathbb{R})$.*

Indeed, if $h_1,\ h_2$ belong to $N(y)$, then there exist $v_1, v_2 \in S_{F,y}$ such that for each $t \in J$, we have

$$h_i(t) = y_0 + \overline{y_0}t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v_i(s)ds\ \ (i = 1,2).$$

Let $0 \leq d \leq 1$. Then for each $t \in J$, we have

$$(dh_1 + (1-d)h_2)(t) = y_0 + \overline{y_0}t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}[dv_1(s) + (1-d)v_2(s)]ds.$$

Since $S_{F,y}$ is convex (because F has convex values), then

$$dh_1 + (1-d)h_2 \in N(y).$$

Claim 2: *N maps bounded sets into bounded sets in $C(J,\mathbb{R})$.*

Indeed, it is enough to show that for any $q > 0$, there exists a positive constant ℓ such that for each $y \in B_q = \{y \in C(J,\mathbb{R}) : \|y\|_\infty \le q\}$, we have $\|N(y)\|_\infty \le \ell$. Then for each $h \in N(y)$, there exists $v \in S_{F,y}$ such that

$$h(t) = y_0 + \overline{y_0}t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}v(s)ds$$

By (H_2) we have for each $t \in J$,

$$\begin{aligned}
|h(t)| &\le |y_0| + T|\overline{y_0}| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds \\
&\le |y_0| + T|\overline{y_0}| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}p(s)\psi(|y(s)|)ds \\
&\le |y_0| + T|\overline{y_0}| + \frac{p^0\psi(q)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \\
&\le |y_0| + T|\overline{y_0}| + \frac{T^\alpha p^0\psi(q)}{\Gamma(\alpha+1)} := \ell.
\end{aligned}$$

Claim 3: *N maps bounded sets into equicontinuous sets of $C(J,\mathbb{R})$.*

Let $\tilde{t}_1, \tilde{t}_2 \in J, \tilde{t}_1 < \tilde{t}_2,\ B_q$ a bounded set of $C(J,\mathbb{R})$ as in Claim 2 and let $y \in B_q$ and $h \in N(y)$. Then

$$\begin{aligned}
|h(\tilde{t}_2) - h(\tilde{t}_1)| &\le \overline{y_0}(\tilde{t}_2 - \tilde{t}_1) + \frac{1}{\Gamma(\alpha)}\int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha-1}|v(s)|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{\tilde{t}_1} [(\tilde{t}_2 - s)^{\alpha-1} - (\tilde{t}_1 - s)^{\alpha-1}]|v(s)|ds \\
&\le \overline{y_0}(\tilde{t}_2 - \tilde{t}_1) + \frac{\psi(q)}{\Gamma(\alpha)}\int_{\tilde{t}_1}^{\tilde{t}_2} (\tilde{t}_2 - s)^{\alpha-1}p(s)ds \\
&\quad + \frac{\psi(q)}{\Gamma(\alpha)}\int_0^{\tilde{t}_1} [(\tilde{t}_2 - s)^{\alpha-1} - (\tilde{t}_1 - s)^{\alpha-1}]p(s)ds.
\end{aligned}$$

As $\tilde{t}_1 \to \tilde{t}_2$, the right-hand side of the above inequality tends to zero. As a consequence of Claims 1 to 3 together with the Arzelá–Ascoli theorem, we can conclude that $N : C(J,\mathbb{R}) \longrightarrow \mathcal{P}(C(J,\mathbb{R}))$ is completely continuous.

Claim 4: *N has a closed graph.*

Let $y_n \to y_*,\ h_n \in N(y_n)$ and $h_n \to h_*$. We need to show that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that, for each $t \in J$

$$h_n(t) = y_0 + \overline{y_0}t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_n(s).$$

We must show that there exist $v_* \in S_{F,y_*}$ such that for each $t \in J$,

$$h_*(t) = y_0 + \overline{y_0}t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_*(s).$$

Clearly $\|(h_n - \phi) - (h_* - \phi)\|_\infty \to 0$ as $n \to \infty$, where $\phi(t) = y_0 + t\overline{y_0}$.
Consider the continuous linear operator

$$\Gamma : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$$

defined by

$$v \longmapsto (\Gamma v)(t) = \int_0^t (t-s)^{\alpha-1} v(s) ds.$$

From Lemma (2.2), it follows that $\Gamma \circ S_F$ is a closed graph operator . Moreover, we have

$$\Big(h_n(t) - \phi(t)\Big) \in \Gamma(S_{F,y_n}).$$

Since $y_n \to y_*$, it follows from Lemma 2.2 that

$$h_*(t) = y_0 + \overline{y_0}t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v_*(s).$$

Claim 5: *A priori bounds on solutions.*

Let $y \in C(J, \mathbb{R})$ be such that $y \in \lambda N(y)$ for some $\lambda \in (0,1)$. Then, there exists $v \in L^1(J, \mathbb{R})$ with $v \in S_{F,y}$ such that, for each $t \in J$,

$$y(t) = \lambda y_0 + \lambda \overline{y_0} t + \frac{\lambda}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} v(s) ds.$$

From (H_2)

$$\begin{aligned}
|y(t)| &\leq |y_0| + T|\overline{y_0}| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|v(s)|ds \\
&\leq |y_0| + T|\overline{y_0}| + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} p(s)\psi(|y(s)|)ds \\
&\leq |y_0| + T|\overline{y_0}| + \frac{p^0\psi(\|y\|_\infty)}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} ds \\
&\leq |y_0| + T|\overline{y_0}| + \frac{T^\alpha p^0 \psi(\|y\|_\infty)}{\Gamma(\alpha+1)}.
\end{aligned}$$

Thus

$$\frac{\|y\|_\infty}{|y_0| + T|\overline{y_0}| + \frac{p^0 T^\alpha}{\Gamma(\alpha+1)}\psi(\|y\|_\infty)} \leq 1$$

Then by (H$_4$), $\|y\|_\infty \neq M$.
Let

$$U = \{y \in C(J, \mathbb{R}) : \|y\|_\infty < M\}.$$

The operator $N : \overline{U} \to \mathcal{P}(C(J, \mathbb{R}))$ is upper semicontinuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y \in \lambda N(y)$ for some $\lambda \in (0, 1)$. As a consequence of the nonlinear alternative of Leray–Schauder type [24], we deduce that N has a fixed point in $\overline{U}$ which is a solution of the problem (3.1)–(3.2). Denote this solution y_1. Define the function

$$r_{k,1}(t) = \tau_k(y_1(t)) - t \text{ for } t \geq 0.$$

(H$_5$) implies that

$$r_{k,1}(0) \neq 0 \text{ for } k = 1, \ldots, m.$$

If

$$r_{k,1}(t) \neq 0 \text{ on } J \text{ for } k = 1, \ldots, m,$$

then y_1 is solution of the problem (1.1)–(1.4).

Now we consider the case when $r_{1,1}(t) = 0$ for some $t \in J$. Since $r_{1,1}(0) \neq 0$ and $r_{1,1}$ is continuous, there exists $t_1 > 0$ such that

$$r_{1,1}(t_1) = 0 \text{ and } r_{1,1}(t) \neq 0 \text{ for all } t \in [0, t_1).$$

Thus by (H$_5$) we have

$$r_{k,1}(t) \neq 0 \text{ for all } t \in [0, t_1) \text{ and } k = 1, \ldots, m.$$

Step 2: Consider the problem

$$^cD^\alpha_{t_1} y(t) \in F(t, y(t)), \text{ for a.e., } t \in [t_1, T] \tag{3.3}$$

$$y(t_1^+) = I_1(y_1(t_1)) \tag{3.4}$$

$$y'(t_1^+) = \overline{I}_1(y_1(t_1)). \tag{3.5}$$

Consider the operator $N_1 : C([t_1, T], \mathbb{R}) \to \mathcal{P}(C([t_1, T], \mathbb{R}))$ defined by

$$N_1(y) = \{h \in C([t_1, T], \mathbb{R})\}$$

with

$$h(t) = I_1(y_1(t_1)) + (t - t_1)\overline{I}_1(y_1(t_1)) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t - s)^{\alpha-1} v(s),$$

and $v \in S_{F,y}$. As in Step 1 we can show that N_1 satisfies the assumptions of the nonlinear alternative of Leray–Schauder type and deduce that the problem (3.3)–(3.5) has a solution denoted by y_2. Define

$$r_{k,2}(t) = \tau_k(y_2(t)) - t \text{ for } t \geq t_1.$$

If

$$r_{k,2}(t) \neq 0 \text{ on } (t_1, T] \text{ for } k = 1, \ldots, m,$$

then

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0, t_1], \\ y_2(t), & \text{if } t \in (t_1, T] \end{cases}$$

is solution of the problem (1.1)–(1.4). It remains to consider the case when

$$r_{2,2}(t) = 0, \text{ for some } t \in (t_1, T].$$

By (H_6) we have

$$\begin{aligned} r_{2,2}(t_1^+) &= \tau_2(y_2(t_1^+)) - t_1 \\ &= \tau_2(I_1(y_1(t_1))) - t_1 \\ &> \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Since $r_{2,2}$ is continuous, there exists $t_2 > t_1$ such that $r_{2,2}(t_2) = 0$ and $r_{2,2}(t) \neq 0$ for all $t \in (t_1, t_2)$. By (H_5) we have

$$r_{k,2}(t) \neq 0 \text{ for all } t \in (t_1, t_2) \text{ and } k = 2, \ldots, m.$$

Suppose now that there exists $s \in (t_1, t_2)$ such that $r_{1,2}(s) = 0$. From (H_6) it follows that

$$\begin{aligned} r_{1,2}(t_1) &= \tau_1(y_2(t_1)) - t_1 \\ &= \tau_1(I_1(y_1(t_1))) - t_1 \\ &\leq \tau_1(y_1(t_1)) - t_1 \\ &= r_{1,1}(t_1) = 0. \end{aligned}$$

Also, from (H_5) we have

$$\begin{aligned} r_{1,2}(t_2) &= \tau_1(y_2(t_2)) - t_2 \\ &< \tau_2(y_2(t_2)) - t_2 \\ &= r_{2,2}(t_2) = 0. \end{aligned}$$

Since $r_{1,2}(t_1) \leq 0,\ r_{1,2}(t_2) < 0$ and $r_{1,2}(s) = 0$ for some $s \in (t_1, t_2)$, the function $r_{1,2}$ attains a nonnegative maximum at some point $\overline{s} \in (t_1, t_2)$. Since

$$y_2'(t) = \frac{1}{\Gamma(\alpha - 1)} \int_{t_1}^{t} (t - s)^{\alpha - 2} v(s) ds,$$

we have

$$r'_{1,2}(\overline{s}) = \tau'_1(y_2(\overline{s}))y'_2(\overline{s}) - 1 = 0.$$

Therefore

$$\tau'_1(y_2(\overline{s}))\int_{t_1}^{\overline{s}}(\overline{s}-s)^{\alpha-2}v(s)ds = \Gamma(\alpha-1),$$

which contradicts (H_7).

Step 3: We continue this process and taking into account that $y_{m+1} := y\big|_{[t_m,T]}$ is a solution to the problem

$$^cD^{\alpha}_{t_m}y(t) \in F(t,y(t)),\ \text{for a.e.},\ t \in (t_m,T], \tag{3.6}$$

$$y(t_m^+) = I_m(y_{m-1}(t_m^-)), \tag{3.7}$$

$$y'(t_m^+) = \overline{I}_m(y_{m-1}(t_m^-)). \tag{3.8}$$

The solution y of the problem (1.1)–(1.4) is then defined by

$$y(t) = \begin{cases} y_1(t), & \text{if } t \in [0,t_1], \\ y_2(t), & \text{if } t \in (t_1,t_2], \\ \dots \\ y_{m+1}(t), & \text{if } t \in (t_m,T]. \end{cases}$$

□

4 An Example

$$^cD^{\alpha}y(t) \in F(t,y(t)),\ \text{for a.e.},\ t \in J = [0,1],\ t \neq \tau_k(y(t)),\ k = 1,\dots,m,\ 1 < \alpha \leq 2, \tag{4.1}$$

$$y(t^+) = I_k(y(t)),\ t = \tau_k(y(t)),\ k = 1,\dots,m, \tag{4.2}$$

$$y'(t^+) = \overline{I}_k(y(t)),\ t = \tau_k(y(t)),\ k = 1,\dots,m, \tag{4.3}$$

$$y(0) = 0,\ y'(0) = 0, \tag{4.4}$$

where

$$\tau_k(y) = 1 - \frac{1}{2^k(1+y^2)},$$

$$I_k(y) = b_k y,\ \overline{I}_k(y) = \overline{b}_k y,\ \overline{b}_k > 0,\ b_k \in \left(\frac{1}{\sqrt{2}}, 1\right],\ k = 1,2,\dots,m.$$

We have

$$\tau_{k+1}(y) - \tau_k(y) = \frac{1}{2^{k+1}(1+y^2)} > 0 \text{ for each } y \in \mathbb{R},\ k = 1,2,\dots,m.$$

Hence

$$0 < \tau_1(y) < \tau_2(y) < \ldots < \tau_m(y) < 1, \text{ for each } y \in \mathbb{R}.$$

Also,

$$\tau_k(I_k(y)) - \tau_k(y) = \frac{b_k^2 - 1}{2^k(1+y^2)(1+b_k^2y^2)} \le 0, \text{ for each } y \in \mathbb{R},$$

and

$$\tau_{k+1}(I_k(y)) - \tau_k(y) = \frac{1+(2b_k^2-1)y^2}{2^{k+1}(1+y^2)(1+b_k^2y^2)} > 0, \text{ for each } y \in \mathbb{R}.$$

Finally,

$$F(t,y) = \{v \in \mathbb{R} : f_1(t,y) \le v \le f_2(t,y)\},$$

where $f_1, f_2 : J \times \mathbb{R} \to \mathbb{R}$. We assume that for each $t \in J$, $f_1(t,\cdot)$ is lower semicontinuous (i.e, the set $\{y \in \mathbb{R} : f_1(t,y) > \mu\}$ is open for each $\mu \in \mathbb{R}$), and assume that for each $t \in J$, $f_2(t,\cdot)$ is upper semicontinuous (i.e the set $\{y \in \mathbb{R} : f_2(t,y) < \mu\}$ is open for each $\mu \in \mathbb{R}$). Assume that there are $p \in C(J,\mathbb{R}^+)$ and $\psi : [0,\infty) \to (0,\infty)$ continuous and nondecreasing such that

$$\max\{|f_1(t,y)|,\ |f_2(t,y)|\} \le p(t)\psi(|y|), \quad t \in J, \text{ and all } y \in \mathbb{R}.$$

Assume there exists a constant $M > 0$ such that

$$\frac{M}{M(b_k+\overline{b}_k) + \frac{p^0}{\Gamma(\alpha+1)}\psi(M)} > 1,$$

where $p^0 = \sup_{t\in J} p(t)$. It is clear that F is compact and convex valued, and it is upper semicontinuous (see [17]). If (H_7) holds, then by Theorem (3.4) the problem (4.1)–(4.4) has at least one solution.

References

[1] R.P Agarwal, M. Benchohra and S. Hamani, Boundary value problems for fractional differential equations, *Adv. Stud. Contemp. Math.* **16** (2) (2008), 181–196.

[2] R. P. Agarwal, M. Benchohra and B.A. Slimani, Existence results for differential equations with fractional order and impulses, *Mem. Differential Equations Math. Phys.* **44** (2008), 1–21.

[3] B. Ahmad, J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions. *Boundary Value Problems* Volume 2009 (2009), Article ID 708576, 11 pages.

[4] E. Ait Dads, M. Benchohra and S. Hamani, Impulsive fractional differential inclusions involving the Caputo fractional derivative, *Fract. Calc. Appl. Anal.* **12** (1) (2009), 15–38.

[5] J.P. Aubin and H. Frankowska, *Set-Valued Analysis*, Birkhäuser, Boston, 1990.

[6] I. Bajo and E. Liz, Periodic boundary value problem for first order differential equations with impulses at variable times, *J. Math. Anal. Appl.* **204** (1996), 65–73.

[7] A. Belarbi and M. Benchohra, Existence theory for perturbed impulsive hyperbolic differential inclusions with variable times, *J. Math. Anal. Appl.* **327** (2007), 1116–1129.

[8] M. Belmekki and M. Benchohra, Existence results for fractional order semilinear functional differential equations, *Proc. A. Razmadze Math. Inst.* **146** (2008), 9–20.

[9] M. Belmekki, M. Benchohra and L. Gorniewicz, Semilinear functional differential equations with fractional order and infinite delay, *Fixed Point Theory* **9** (2) (2008), 423–439.

[10] M. Benchohra, J.R. Graef and S. Hamani, Existence results for boundary value problems with nonlinear fractional differential equations, *Appl. Anal.* **87** (7) (2008), 851–863.

[11] M. Benchohra, J.R. Graef, S.K. Ntouyas, and A. Ouahab, Upper and lower solutions method for impulsive differential inclusions with nonlinear boundary conditions and variable times. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **12** (3-4) (2005), 383–396.

[12] M. Benchohra, S. Hamani and S.K. Ntouyas, Boundary value problems for differential equations with fractional order, *Surv. Math. Appl.* **3** (2008), 1–12.

[13] M. Benchohra, J. Henderson and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Hindawi Publishing Corporation, Vol 2, New York, 2006.

[14] M. Benchohra, J. Henderson, S.K. Ntouyas and A. Ouahab, Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2008), 1340–1350.

[15] M. Benchohra and B. A. Slimani, Impulsive fractional differential equations, *Electron. J. Differential Equations* **2009** No. 10, pp. 1–11.

[16] Y.-K. Chang and J.J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions, *Math. Comput. Model.* **49** (2009), 605–609.

[17] K. Deimling, *Multivalued Differential Equations*, Walter De Gruyter, Berlin-New York, 1992.

[18] D. Delbosco and L. Rodino, Existence and uniqueness for a nonlinear fractional differential equation, *J. Math. Anal. Appl.* **204** (1996), 609–625.

[19] K. Diethelm and N. J. Ford, Analysis of fractional differential equations, *J. Math. Anal. Appl.* **265** (2002), 229–248.

[20] M. Frigon and D. O'Regan, Impulsive differential equations with variable times, *Nonlinear Anal.* **26** (1996), 1913–1922.

[21] L. Gaul, P. Klein and S. Kempfle, Damping description involving fractional operators, *Mech. Systems Signal Processing* **5** (1991), 81–88.

[22] W.G. Glockle and T.F. Nonnenmacher, A fractional calculus approach of self-similar protein dynamics, *Biophys. J.* **68** (1995), 46–53.

[23] J.R. Graef and A. Ouahab, Global existence and uniqueness results for impulsive functional differential equations with variable times and multiple delays, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **16** (2009), 27–40.

[24] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[25] N. Heymans and I. Podlubny, Physical interpretation of initial conditions for fractional differential equations with Riemann–Liouville fractional derivatives. *Rheologica Acta* **45** (5) (2006), 765–772.

[26] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[27] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Theory* **I**, Kluwer, Dordrecht, 1997.

[28] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[29] V. Lakshmikantham, D.D. Bainov and P.S. Simeonov, *Theory of Impulsive Differntial Equations*, Worlds Scientific, Singapore, 1989.

[30] V. Lakshmikantham, S. Leela and J. Vasundhara, *Theory of Fractional Dynamic Systems*, Cambridge Academic Publishers, Cambridge, 2009.

[31] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys.* **13** (1965), 781–786.

[32] F. Metzler, W. Schick, H. G. Kilian and T. F. Nonnenmacher, Relaxation in filled polymers: A fractional calculus approach, *J. Chem. Phys.* **103** (1995), 7180–7186.

[33] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[34] J.J. Nieto, Impulsive resonance periodic problems of first order, *Appl. Math. Letter.* **15** (2002), 489–493.

[35] J.J. Nieto and D. O'Regan, Variational approach to impulsive differential equations, *Nonlinear Anal. Real World Appl.* **10** (2009), 680–690.

[36] J.J. Nieto R. Rodriguez-Lopez, Hybrid metric dynamical systems with impulses, *Nonlinear Anal.* **64** (2006), 368–380.

[37] J.J. Nieto, R. Rodriguez-Lopez, Boundary value problems for a class of impulsive functional equations, *Comput. Math. Appl.* **55** (2008), 2715–2731.

[38] K.B. Oldham and J. Spanier, *The Fractional Calculus*, Academic Press, New York, London, 1974.

[39] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[40] I. Podlubny, Geometric and physical interpretation of fractional integration and fractional differentiation, *Fract. Calc. Appl. Anal.* **5** (2002), 367–386.

[41] I. Podlubny, I. Petraš, B.M. Vinagre, P. O'Leary and L. Dorčak, Analogue realizations of fractional-order controllers. Fractional order calculus and its applications, *Nonlinear Dynam.* **29** (2002), 281–296.

[42] S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[43] A.M. Samoilenko, N.A. Perestyuk, *Impulsive Differential Equations* World Scientific, Singapore, 1995.

[44] I.M. Stamova, Boundedness of impulsive functional differential equations with variable impulsive perturbations, *Bull. Aust. Math. Soc.* **77** (2008), 331–345.

[45] S. Zhang, Positive solutions for boundary-value problems of nonlinear fractional differential equations, *Electron. J. Differential Equations* 2006, No. 36, pp, 1–12.

[46] H. Zhao and E. Feng, Stability of impulsive control systems with variable times, *J. Appl. Math. Comput.* **23** (2007), 345–352.