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# Existence and Uniqueness Results for BVPs associated with Nonlinear Singular Interface Problems on Time Scales

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**Abstract**

In this paper we present existence and uniqueness results for BVPs associated with singular interface problems. Classical fixed point theorems are used for proving the existence and uniqueness results for these problems. We discuss two applications in the field of acoustic wave guides in ocean and transverse vibration of strings.

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**Keywords:** Regular problems, singular problems, singular interface problems, fixed point theorems.

## 1 Introduction

Interface problems are a class of problems wherein two different differential equations are defined on two adjacent intervals and the solutions satisfy matching conditions at the point of interface. We encounter these problems in the study of acoustic wave guides [12], transverse vibrations of strings [13], one dimensional scattering theory [14] etc. Some works on these problems include [1, 6–10, 16–22].

These problems for regular case have been discussed in [16–22], and the problem of having singularity at the boundary is discussed in [1]. The problem of having a singularity at the point of interface remains a challenge. The singularity at the point of interface in the domain of definition of the pair of equations could be of the following three types satisfying certain matching conditions at the singular interface:

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1. $[a, c] \cup [\sigma(c), b]$,
2. $[a, \rho(c)] \cup [c, b]$,
3. $[a, \rho(c)] \cup [\sigma(c), b]$.

To describe the singularities in the domain of definition we take help of the terminology used on time scales [11]. Dynamic equations on time scales with facilities of the two jump operators with various definitions of continuity and derivatives make one's job simple to study these singular interface problems. These dynamic equations generalize differential equations.

Some results for linear singular interface problems (SIP) can be found in [2–5]. In this paper, we intend to present existence results for these singular interface problems involving nonlinear operators.

We consider a BVP associated with these nonlinear singular interface problems on time scales. We use the classical fixed point theorem of Schauder for proving the existence of solutions for this problem. Further, the study on uniqueness of solutions using Banach's fixed point theorem has shown that under certain restricted conditions on the interface constants existence of unique solution can be proved. We finally apply the theory to two applications in the field of acoustic wave guides in ocean and transverse vibration of strings.

## 2 Mathematical Preliminaries

**Definition 2.1.** Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, then we say that $t$ is *right-scattered,* while $\rho(t) < t$, then we say that $t$ is *left-scattered.* Points that are right-scattered and left-scattered at the same time are called *isolated.* Also, if $t < \sup\mathbb{T}$ and $\sigma(t) = t$, then $t$ is called *right-dense,* and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then $t$ is called *left-dense.* Points that are right-dense and left-dense at the same time are called *dense.* Finally, the *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

**Definition 2.2.** $\mathbb{T}^{\kappa} = \left\{ \begin{array}{l} \mathbb{T} - \{m\} \text{ if } \sup\ \mathbb{T} < \infty \\ \mathbb{T} \text{ if } \sup\ \mathbb{T} = \infty \end{array} \right\}$ where $m$ is the left-scattered maximum of $\mathbb{T}$.

**Definition 2.3.** Let $f$ be a function defined on $\mathbb{T}$. We say that $f$ is *delta differentiable* at $t \in \mathbb{T}^{\kappa}$ provided there exists an $\alpha$ such that for all $\epsilon > 0$ there is a neighborhood $\mathcal{N}$ around t with

$$|f(\sigma(t)) - f(s) - \alpha(\sigma(t) - s)| \leq \epsilon\,|\sigma(t) - s| \text{ for all } s \in \mathcal{N}.$$

**Definition 2.4.** For a function $f : \mathbb{T} \to \mathbb{R}$ we shall talk about the second derivative $f^{\Delta\Delta}$ provided $f^{\Delta}$ is differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ with derivative $f^{\Delta\Delta} = (f^{\Delta})^{\Delta} : \mathbb{T}^{\kappa^2} \to \mathbb{R}$. Similarly we define the higher order derivatives $f^{\Delta^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$.

**Theorem 2.5** (Banach Contraction Mapping Theorem)**.** *If $T : X \to X$ is contractive on a complete metric space $X$, then $T$ has a unique fixed point in $X$.*

**Theorem 2.6** (Schauder's Fixed Point Theorem)**.** *Let $L$ be a convex subset of a normed linear space $E$. Then each compact map $T : L \to L$ has a fixed point.*

Let $I = [c, d]$ with $c < \rho(d)$. We define $I_c = [c, \infty)$ in case $\sup \mathbb{T} = +\infty$. By $\mathcal{C}^{B}_{TS}(I_c)$ we mean the linear space of all continuous functions $f : I_c \to \mathbb{R}$ such that $\sup\limits_{t \in I_c} |f(t)| < \infty$.

Now we quote the time scales version of the Arzela–Ascoli theorem [15].

**Theorem 2.7** (Arzela–Ascoli Theorem)**.** *Let $X$ be a subset of $\mathcal{C}^{B}_{TS}(I_c)$ having the following properties.*

(i) *X is bounded.*

(ii) *On every compact subinterval J of $[c, \infty)$ we have: For any $\epsilon > 0$ there exists $\delta > 0$ such that $t_1, t_2 \in J$, $|t_1 - t_2| < \delta$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in X$.*

(iii) *For every $\epsilon > 0$ there exists $b \in I_c$ such that $t_1, t_2 \in [b, \infty)$ implies $|f(t_1) - f(t_2)| < \epsilon$ for all $f \in X$.*

*Then $X$ is relatively compact.*

Let $\mathbb{T}_1$ and $\mathbb{T}_2$ be two time scales. Let $C(\mathbb{T})$ denote the space of all continuous functions on the time scale $\mathbb{T}$.

**Definition 2.8.** By $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we mean that $t_1 \in \mathbb{T}_1$ and $t_2 \in \mathbb{T}_2$ with the product topology on $\mathbb{T}_1 \times \mathbb{T}_2$.

**Definition 2.9.** Let $Z(\mathbb{T})$ be a function space on the time scale $\mathbb{T}$. For $x \in Z(\mathbb{T})$ we define

$$\|x\| = \sup_{t \in \mathbb{T}} |x(t)|.$$

**Definition 2.10.** By $(x_1, x_2) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$, where $X$ and $Y$ are function spaces, we mean that $x_1 \in X(\mathbb{T}_1)$ and $x_2 \in Y(\mathbb{T}_2)$ with the product topology on $X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$. For $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ we define

$$(x_1, x_2)(t_1, t_2) = (x_1(t_1), x_2(t_2)).$$

**Definition 2.11.** For $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$, we define

$$\|(t_1, t_2)\| = \|t_1\| + \|t_2\| = |t_1| + |t_2|.$$

**Definition 2.12.** For $(x_1, x_2) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$, we define

$$\begin{aligned} \|(x_1, x_2)\| &= \|x_1\| + \|x_2\| \\ &= \sup_{t_1 \in \mathbb{T}_1} |x_1(t_1)| + \sup_{t_2 \in \mathbb{T}_2} |x_2(t_2)|. \end{aligned}$$

**Definition 2.13.** Let $(y_{11}, y_{12}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$. We say that $(y_{11}, y_{12})$ is continuous on $\mathbb{T}_1 \times \mathbb{T}_2$ if for $\epsilon > 0$ there exists $\delta > 0$ such that for arbitrarily fixed $(t_{01}, t_{02}) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that

$$\|(t_1, t_2) - (t_{01}, t_{02})\| < \delta$$

implies

$$\|(y_{11}, y_{12})(t_1, t_2) - (y_{11}, y_{12})(t_{01}, t_{02})\| < \epsilon.$$

**Definition 2.14.** A sequence $(y_{n1}, y_{n2}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is said to be a Cauchy sequence if for every $\epsilon > 0$ there exists $N$ such that for all $n1, n2, m1, m2 > N$, we have

$$\|(x_{n1}, x_{n2}) - (x_{m1}, x_{m2})\| < \epsilon.$$

**Definition 2.15.** A sequence $(y_{n1}, y_{n2}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ is said to be equicontinuous if for every $\epsilon > 0$ there is a $\delta > 0$, depending only on $\epsilon$, such that for all $(y_{n1}, y_{n2})$ and all $(t_1, t_2), (t_1^{'}, t_2^{'}) \in \mathbb{T}_1 \times \mathbb{T}_2$,

$$\|(t_1, t_2) - (t_1^{'}, t_2^{'})\| < \delta$$

implies

$$\|(y_{n1}, y_{n2})(t_1, t_2) - (y_{n1}, y_{n2})(t_1^{'}, t_2^{'})\| < \epsilon.$$

**Definition 2.16.** The space $X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$ is said to be convex if

$$\alpha(y_{11}, y_{12}) + (1 - \alpha)(y_{21}, y_{22}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2) \text{ for } 0 < \alpha < 1$$

holds for every $(y_{11}, y_{12}), (y_{21}, y_{22}) \in X(\mathbb{T}_1) \times Y(\mathbb{T}_2)$.

# 3 Definition of the Problem

Let $\mathbb{T}_1 = [0,a]_{\mathbb{T}}, \mathbb{T}_2 = [\sigma(a), b]_{\mathbb{T}}$, where $a, \sigma(a), b < +\infty$. Also let $(f_1, f_2)$ be a nonlinear function tuple in $\mathcal{C}(\mathbb{T}_1 \times \mathbb{R}) \times \mathcal{C}(\mathbb{T}_2 \times \mathbb{R})$. In this paper, we consider the following BVP associated with singular interface problem (BVP-SIP).

$$y_1^{\Delta\Delta}(t) = f_1(t, y_1),\ t \in \mathbb{T}_1, \tag{3.1}$$

$$y_2^{\Delta\Delta}(t) = f_2(t, y_2),\ t \in {\mathbb{T}_2}^{\kappa^2} \tag{3.2}$$

with the initial conditions

$$y_1(0) = 0, \tag{3.3}$$

$$y_2^{\Delta}(\rho(b)) = 0 \tag{3.4}$$

followed by the matching interface conditions

$$\gamma_1 y_1(a) = \gamma_2 y_2(\sigma(a)), \tag{3.5}$$

$$\gamma_3 y_1^{\Delta}(a) = \gamma_4 y_2^{\Delta}(\sigma(a)),\ \gamma_i > 0,\ i = 1,2,3,4. \tag{3.6}$$

# 4 Existence Results for the BVP-SIP

In this section, we prove the existence of solution for the BVP-SIP using Schauder's fixed point theorem.

**Theorem 4.1.** *If $(f_1, f_2)$ is continuous and bounded, then there exists at least one solution for the BVP-SIP.*

*Proof.* Case I. Let $t \in \mathbb{T}_1$. Then

$$\begin{aligned} y_1^{\Delta\Delta}(t) &= f_1(t, y_1), \\ y_1^{\Delta}(t) &= \int_0^t f_1(s, y_1)\Delta s + c_{11}, \\ y_1(t) &= \int_0^t \int_0^m f_1(s, y_1)\Delta s \Delta m + \int_0^t c_{11}\Delta s + c_{12}, \end{aligned}$$

where $c_{11}$ and $c_{12}$ are constants to be determined. By using the initial condition (3.3), we get $c_{12} = 0$. Hence

$$y_1(t) = \int_0^t \int_0^m f_1(s, y_1)\Delta s \Delta m + \int_0^t c_{11}\Delta s.$$

Case II. Let $t \in \mathbb{T}_2$. Then

$$\begin{aligned} y_2^{\Delta\Delta}(t) &= f_2(t, y_2), \\ y_2^{\Delta}(t) &= \int_{\sigma(a)}^t f_2(s, y_2)\Delta s + c_{21}, \\ y_2(t) &= \int_{\sigma(a)}^t \int_{\sigma(a)}^m f_2(s, y_2)\Delta s \Delta m + \int_{\sigma(a)}^t c_{21}\Delta s + c_{22}, \end{aligned}$$

where $c_{21}$ and $c_{22}$ are constants to be determined. Now, by (3.4), we get

$$\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s + c_{21} = 0$$

and therefore

$$c_{21} = -\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s.$$

Hence,

$$y_2(t) = \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m} f_2(s, y_2)\Delta s\Delta m - \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s\Delta m + c_{22}.$$

Now from (3.5), we have

$$\gamma_1\Big(\int_0^a\int_0^m f_1(s, y_1)\Delta s\Delta m + \int_0^a c_{11}\Delta s\Big) = \gamma_2 c_{22}$$

and thus

$$c_{22} = \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m f_1(s, y_1)\Delta s\Delta m + \frac{\gamma_1}{\gamma_2}\int_0^a c_{11}\Delta s.$$

Using (3.6), we get

$$\gamma_3\Big(\int_0^a f_1(s, y_1)\Delta s + c_{11}\Big) = -\gamma_4\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s$$

and thus

$$c_{11} = -\frac{\gamma_4}{\gamma_3}\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s - \int_0^a f_1(s, y_1)\Delta s.$$

Hence we have

$$\begin{aligned}
y_1(t) &= \int_0^t\int_0^m f_1(s, y_1)\Delta s\Delta m - \frac{\gamma_4}{\gamma_3}\int_0^t\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s\Delta m \\
&\quad - \int_0^t\int_0^a f_1(s, y_1)\Delta s\Delta m, \\
y_2(t) &= \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{m} f_2(s, y_2)\Delta s\Delta m - \int_{\sigma(a)}^{t}\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s\Delta m \\
&\quad + \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^m f_1(s, y_1)\Delta s\Delta m - \frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^a\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s\Delta m \\
&\quad - \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^a f_1(s, y_1)\Delta s\Delta m.
\end{aligned}$$

We now define the integral operator $T : C(\mathbb{T}_1) \times C(\mathbb{T}_2) \to C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ by

$$\begin{aligned} T(y_1, y_2) = & \Bigg( \int_0^{t_1} \int_0^{m} f_1(s, y_1) \Delta s \Delta m - \frac{\gamma_4}{\gamma_3} \int_0^{t_1} \int_{\sigma(a)}^{\rho(b)} f_2(s, y_2) \Delta s \Delta m \\ - & \int_0^{t_1} \int_0^{a} f_1(s, y_1) \Delta s \Delta m, \int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{m'} f_2(s, y_2) \Delta s \Delta m' \\ - & \int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{\rho(b)} f_2(s, y_2) \Delta s \Delta m' + \frac{\gamma_1}{\gamma_2} \int_0^{a} \int_0^{m'} f_1(s, y_1) \Delta s \Delta m' \\ - & \frac{\gamma_1 \gamma_4}{\gamma_2 \gamma_3} \int_0^{a} \int_{\sigma(a)}^{\rho(b)} f_2(s, y_2) \Delta s \Delta m' - \frac{\gamma_1}{\gamma_2} \int_0^{a} \int_0^{a} f_1(s, y_1) \Delta s \Delta m' \Bigg), \end{aligned}$$

where $t_1, m \in \mathbb{T}_1$ and $t_2, m' \in \mathbb{T}_2$.

*Claim* 4.2. The space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is convex.

Let $(y_{11}, y_{12}), (y_{21}, y_{22}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$. For the space to be convex, we need to show that

$$\alpha(y_{11}, y_{12}) + (1-\alpha)(y_{21}, y_{22}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2) \text{ for } \alpha < 1.$$

Since $(y_{11}, y_{12}) \in C(\mathbb{T}_1) \times C(\mathbb{T}_2)$, for fixed $(t_{01}, t_{02}) \in \mathbb{T}_1 \times \mathbb{T}_2$ and $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$ such that $\forall \epsilon > 0 \; \exists \; \delta > 0$ such that whenever (see Definition 2.13)

$$\|(t_1, t_2) - (t_{01}, t_{02})\| < \delta (> 0),$$

i.e.,

$$|t_1 - t_{01}| + |t_2 - t_{02}| < \delta,$$

we have

$$\|(y_{11}(t_1), y_{12}(t_2)) - (y_{11}(t_{01}), y_{12}(t_{02}))\| < \frac{\epsilon}{2\alpha},$$

i.e.,

$$\sup_{t_1 \in \mathbb{T}_1} |y_{11}(t_1) - y_{11}(t_{01})| + \sup_{t_2 \in \mathbb{T}_2} |y_{12}(t_2) - y_{12}(t_{02})| < \frac{\epsilon}{2\alpha}.$$

Similarly we can show that

$$\sup_{t_1 \in \mathbb{T}_1} |y_{21}(t_1) - y_{21}(t_{01})| + \sup_{t_2 \in \mathbb{T}_2} |y_{22}(t_2) - y_{22}(t_{02})| < \frac{\epsilon}{2(1-\alpha)} \tag{4.1}$$

whenever $\|(t_1, t_2) - (t_{01}, t_{02})\| < \delta$. Now, let us consider

$$\begin{aligned} \Big[\alpha(y_{11}, y_{12}) + (1-\alpha)(y_{21}, y_{22})\Big] &= \Big[(\alpha y_{11}, \alpha y_{12}) + ((1-\alpha) y_{21}, (1-\alpha) y_{22})\Big] \\ &= \Big[(\alpha y_{11} + (1-\alpha) y_{21}, \alpha y_{12} + (1-\alpha) y_{22})\Big]. \end{aligned}$$

We see that

$$\begin{aligned}
&\|\{(\alpha y_{11}+(1-\alpha)y_{21})(t_1),(\alpha y_{12}+(1-\alpha)y_{22})(t_2)\}\\
&\qquad -\{(\alpha y_{11}+(1-\alpha)y_{21})(t_{01}),(\alpha y_{12}+(1-\alpha)y_{22})(t_{02})\}\|\\
\leq\ & \sup_{t_1\in\mathbb{T}_1}\Big|(\alpha y_{11}+(1-\alpha)y_{21})(t_1)-(\alpha y_{11}+(1-\alpha)y_{21})(t_{01})\Big|\\
&+\sup_{t_2\in\mathbb{T}_2}\Big|(\alpha y_{12}+(1-\alpha)y_{22})(t_2)-(\alpha y_{12}+(1-\alpha)y_{22})(t_{02})\Big|\\
\leq\ & \sup_{t_1\in\mathbb{T}_1}\alpha|y_{11}(t_1)-y_{11}(t_{01})|+\sup_{t_2\in\mathbb{T}_2}\alpha|y_{12}(t_2)-y_{12}(t_{02})|\\
&+\sup_{t_1\in\mathbb{T}_1}(1-\alpha)|y_{21}(t_1)-y_{21}(t_{01})|+\sup_{t_2\in\mathbb{T}_2}(1-\alpha)|y_{22}(t_2)-y_{22}(t_{02})|\\
<\ & \alpha\frac{\epsilon}{2\alpha}+(1-\alpha)\frac{\epsilon}{2(1-\alpha)}=\epsilon
\end{aligned}$$

whenever $|t_1-t_{01}|+|t_2-t_{02}|<\delta$. Hence $C(\mathbb{T}_1)\times C(\mathbb{T}_2)$ is convex.

*Claim* 4.3. $T$ is a completely continuous map.

We first show that $T$ is continuous. We prove it by showing that $T$ preserves convergence. Indeed let $(y_{n1},y_{n2})$ be a sequence of functions in $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$ such that

$$\lim_{n\to\infty}\|(y_{n1},y_{n2})-(y_1,y_2)\|\to 0.$$

The above equation implies that

$$\lim_{n\to\infty}\|(y_{n1}-y_1,y_{n2}-y_2)\|\to 0,$$

i.e.,

$$\lim_{n\to\infty}\sup_{t_1\in\mathbb{T}_1}|(y_{n1}-y_1)(t_1)|\to 0$$

and

$$\lim_{n\to\infty}\sup_{t_2\in\mathbb{T}_2}|(y_{n2}-y_2)(t_2)|\to 0.$$

Let us consider

$$\begin{aligned}
&\|T(y_{n1},y_{n2})-T(y_1,y_2)\|\\
\leq\ & \sup_{t_1\in\mathbb{T}_1}\left|\int_0^{t_1}\int_0^{m}f_1(s,y_{n1})\Delta s\Delta m-\int_0^{t_1}\int_0^{m}f_1(s,y_1)\Delta s\Delta m\right|\\
&+\sup_{t_1\in\mathbb{T}_1}\left|\frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)}f_2(s,y_2)\Delta s\Delta m-\frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)}f_2(s,y_{n2})\Delta s\Delta m\right|\\
&+\sup_{t_1\in\mathbb{T}_1}\left|\int_0^{t_1}\int_0^{a}f_1(s,y_1)\Delta s\Delta m-\int_0^{t_1}\int_0^{a}f_1(s,y_{n1})\Delta s\Delta m\right|
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{t_2\in\mathbb{T}_2}\left|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} f_2(s,y_{n2})\Delta s\Delta m' - \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} f_2(s,y_2)\Delta s\Delta m'\right| \\
&+\sup_{t_2\in\mathbb{T}_2}\left|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m' - \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_{n2})\Delta s\Delta m'\right| \\
&+\sup_{t_2\in\mathbb{T}_2}\left|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{m'} f_1(s,y_{n1})\Delta s\Delta m' - \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{m'} f_1(s,y_1)\Delta s\Delta m'\right| \\
&+\sup_{t_2\in\mathbb{T}_2}\left|\frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^a\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m' - \frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^a\int_{\sigma(a)}^{\rho(b)} f_2(s,y_{n2})\Delta s\Delta m'\right| \\
&+\sup_{t_2\in\mathbb{T}_2}\left|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^a f_1(s,y_1)\Delta s\Delta m' - \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^a f_1(s,y_{n1})\Delta s\Delta m'\right|.
\end{aligned}
$$

Since $(f_1, f_2)$ is continuous on $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$, we have

$$
\begin{aligned}
\lim_{n\to\infty}|f_1(s,y_{n1}) - f_1(s,y_1)| &\to 0, \\
\lim_{n\to\infty}|f_2(s,y_{n2}) - f_1(s,y_2)| &\to 0.
\end{aligned}
$$

Now

$$
\begin{aligned}
&\|T(y_{n1},y_{n2}) - T(y_1,y_2)\| \\
&\leq \sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^{m}|f_1(s,y_{n1}) - f_1(s,y_1)|\Delta s\Delta m \\
&\quad +\sup_{t_1\in\mathbb{T}_1}\frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)}|f_2(s,y_2) - f_2(s,y_{n2})|\Delta s\Delta m \\
&\quad +\sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^{a}|f_1(s,y_1) - f_1(s,y_{n1})|\Delta s\Delta m \\
&\quad +\sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'}|f_2(s,y_{n2}) - f_2(s,y_2)|\Delta s\Delta m' \\
&\quad +\sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)}|f_2(s,y_2) - f_2(s,y_{n2})|\Delta s\Delta m' \\
&\quad +\sup_{t_2\in\mathbb{T}_2}\frac{\gamma_1}{\gamma_2}\int_0^{a}\int_0^{m'}|f_1(s,y_{n1}) - f_1(s,y_1)|\Delta s\Delta m' \\
&\quad +\sup_{t_2\in\mathbb{T}_2}\frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^{a}\int_{\sigma(a)}^{\rho(b)}|f_2(s,y_2) - f_2(s,y_{n2})|\Delta s\Delta m' \\
&\quad +\sup_{t_2\in\mathbb{T}_2}\frac{\gamma_1}{\gamma_2}\int_0^{a}\int_0^{a}|f_1(s,y_1) - f_1(s,y_{n1})|\Delta s\Delta m'.
\end{aligned}
$$

Hence, $\lim_{n\to\infty} \|T(y_{n1}, y_{n2}) - T(y_1, y_2)\| \to 0$, proving that $T$ is continuous. Let

$$\begin{aligned} f_1(s, y_1) &\leq M_1, \text{ for some } M_1 > 0, \forall s \in \mathbb{T}_1, \\ f_2(s, y_2) &\leq M_2, \text{ for some } M_2 > 0, \forall s \in \mathbb{T}_2. \end{aligned}$$

We now show that $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is a bounded and equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$. Let us assume that $\|(y_1, y_2)\| \leq M$. Then

$$\begin{aligned} &\|T(y_1, y_2)\| \\ &\leq \sup_{t_1\in\mathbb{T}_1} \int_0^{t_1}\int_0^m |f_1(s, y_1)|\Delta s\Delta m + \sup_{t_1\in\mathbb{T}_1} \frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} |f_2(s, y_2)|\Delta s\Delta m \\ &\quad + \sup_{t_1\in\mathbb{T}_1} \int_0^{t_1}\int_0^a |f_1(s, y_1)|\Delta s\Delta m + \sup_{t_2\in\mathbb{T}_2} \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} |f_2(s, y_2)|\Delta s\Delta m' \\ &\quad + \sup_{t_2\in\mathbb{T}_2} \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} |f_2(s, y_2)|\Delta s\Delta m' + \sup_{t_2\in\mathbb{T}_2} \frac{\gamma_1}{\gamma_2}\int_0^{a}\int_0^{m'} |f_1(s, y_1)|\Delta s\Delta m' \\ &\quad + \sup_{t_2\in\mathbb{T}_2} \frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^{a}\int_{\sigma(a)}^{\rho(b)} |f_2(s, y_2)|\Delta s\Delta m' \\ &\quad + \sup_{t_2\in\mathbb{T}_2} \frac{\gamma_1}{\gamma_2}\int_0^{a}\int_0^{a} |f_1(s, y_1)|\Delta s\Delta m'. \end{aligned}$$

Since $(f_1, f_2)$ is bounded, we can conclude that there exists a $K > 0$ independent of choice of $(y_1, y_2)$ such that $\|T(y_1, y_2)\| \leq K$. Hence, $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is bounded.

We next prove that $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is an equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$. We need to show that for all $\epsilon > 0$ there exists $\delta > 0$ such that whenever

$$\|(t_1, t_2) - (t_1', t_2')\| < \delta,$$

we have

$$\|T(y_1(t_1), y_2(t_2)) - T(y_1(t_1'), y_2(t_2'))\| < \epsilon.$$

Let us assume that $|t_1 - t_1'| + |t_2 - t_2'| < \delta$. We see that

$$\begin{aligned} &T(y_1(t_1), y_2(t_2)) - T(y_1(t_1'), y_2(t_2')) \\ &= \Bigg( \int_0^{t_1}\int_0^m f_1(s, y_1)\Delta s\Delta m - \int_0^{t_1'}\int_0^m f_1(s, y_1)\Delta s\Delta m \\ &\quad + \frac{\gamma_4}{\gamma_3}\int_0^{t_1'}\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s\Delta m - \frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} f_2(s, y_2)\Delta s\Delta m \\ &\quad + \int_0^{t_1'}\int_0^a f_1(s, y_1)\Delta s\Delta m - \int_0^{t_1}\int_0^a f_1(s, y_1)\Delta s\Delta m, \end{aligned}$$

$$
\begin{aligned}
&+ \int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{m'} f_2(s, y_2) \Delta s \Delta m' - \int_{\sigma(a)}^{t_2{}'} \int_{\sigma(a)}^{m'} f_2(s, y_2) \Delta s \Delta m' \\
&+ \int_{\sigma(a)}^{t_2{}'} \int_{\sigma(a)}^{\rho(b)} f_2(s, y_2) \Delta s \Delta m' - \int_{\sigma(a)}^{t_2} \int_{\sigma(a)}^{\rho(b)} f_2(s, y_2) \Delta s \Delta m' \Bigg).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\int_0^{t_1} \int_0^m f_1(s, y_1) \Delta s \Delta m - \int_0^{t_1'} \int_0^m f_1(s, y_1) \Delta s \Delta m \\
&\leq \int_0^{t_1} \int_0^m M_1 \Delta s \Delta m - \int_0^{t_1'} \int_0^m M_1 \Delta s \Delta m \\
&= \int_0^{t_1} M_1 m \Delta m - \int_0^{t_1'} M_1 m \Delta m \\
&\leq M_1 \left( \frac{t_1^2}{2} - \frac{t_1'^{\,2}}{2} \right) \\
&= \frac{M_1}{2} (t_1^2 - t_1'^{\,2}) \\
&= \frac{M_1}{2} (t_1 + t_1')(t_1 - t_1').
\end{aligned}
$$

Hence, whenever $|t_1 - t_1'| < \delta$, we have

$$
\left| \int_0^{t_1} \int_0^m f_1(s, y_1) \Delta s \Delta m - \int_0^{t_1'} \int_0^m f_1(s, y_1) \Delta s \Delta m \right| < \frac{\epsilon}{5}.
$$

Also,

$$
\begin{aligned}
&\frac{\gamma_4}{\gamma_3} \int_0^{t_1{}'} \int_{\sigma(a)}^{\rho(b)} f_2(s, y_2) \Delta s \Delta m - \frac{\gamma_4}{\gamma_3} \int_0^{t_1} \int_{\sigma(a)}^{\rho(b)} f_2(s, y_2) \Delta s \Delta m \\
&\leq \frac{\gamma_4}{\gamma_3} \int_0^{t_1{}'} \int_{\sigma(a)}^{\rho(b)} M_2 \Delta s \Delta m - \frac{\gamma_4}{\gamma_3} \int_0^{t_1} \int_{\sigma(a)}^{\rho(b)} M_2 \Delta s \Delta m \\
&= \frac{\gamma_4}{\gamma_3} M_2 \left( \int_0^{t_1{}'} \int_{\sigma(a)}^{\rho(b)} \Delta s \Delta m - \int_0^{t_1} \int_{\sigma(a)}^{\rho(b)} \Delta s \Delta m \right) \\
&= M_2 \frac{\gamma_4}{\gamma_3} (\rho(b) - \sigma(a)) \left( \int_0^{t_1{}'} \Delta m - \int_0^{t_1} \Delta m \right) \\
&= M_2 (\rho(b) - \sigma(a)) \frac{\gamma_4}{\gamma_3} (t_1{}' - t_1).
\end{aligned}
$$

Hence, whenever $|t_1 - t_1^{'}| < \delta$, we have

$$\left|\frac{\gamma_4}{\gamma_3}\int_0^{{t_1}'}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m - \frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m\right| < \frac{\epsilon}{5}.$$

Now,

$$\begin{aligned}
&\int_0^{{t_1}'}\int_0^a f_1(s,y_1)\Delta s\Delta m - \int_0^{t_1}\int_0^a f_1(s,y_1)\Delta s\Delta m \\
&\leq\ M_1\Big(\int_0^{{t_1}'}\int_0^a \Delta s\Delta m - \int_0^{t_1}\int_0^a \Delta s\Delta m\Big) \\
&=\ M_1 a\Big(\int_0^{{t_1}'}\Delta m - \int_0^{t_1}\Delta m\Big) \\
&=\ M_1 a({t_1}' - t_1).
\end{aligned}$$

Hence, whenever $|t_1 - t_1^{'}| < \delta$, we have

$$\left|\int_0^{{t_1}'}\int_0^a f_1(s,y_1)\Delta s\Delta m - \int_0^{t_1}\int_0^a f_1(s,y_1)\Delta s\Delta m\right| < \frac{\epsilon}{5}.$$

Also,

$$\begin{aligned}
&\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} f_2(s,y_2)\Delta s\Delta m' - \int_{\sigma(a)}^{{t_2}'}\int_{\sigma(a)}^{m'} f_2(s,y_2)\Delta s\Delta m' \\
&\leq\ \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} M_2\Delta s\Delta m' - \int_{\sigma(a)}^{t_2'}\int_{\sigma(a)}^{m'} M_2\Delta s\Delta m' \\
&=\ \int_{\sigma(a)}^{t_2} M_2(m' - \sigma(a))\Delta m' - \int_{\sigma(a)}^{t_2'} M_2(m' - \sigma(a))\Delta m' \\
&=\ M_2\left[\left(\frac{t_2^2}{2} - \sigma(a)t_2 + \frac{\sigma(a)^2}{2}\right) - \left(\frac{{t_2}'^2}{2} - \sigma(a)t_2' + \frac{\sigma(a)^2}{2}\right)\right] \\
&=\ M_2\left(\frac{(t_2^2 - {t_2'}^2)}{2} - \sigma(a)(t_2 - t_2')\right) \\
&=\ M_2\left(\frac{(t_2 + t_2')(t_2 - t_2')}{2} - \sigma(a)(t_2 - t_2')\right) \\
&=\ M_2\left[(t_2 - t_2')\left(\frac{t_2 + t_2'}{2} - \sigma(a)\right)\right].
\end{aligned}$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$, we have

$$\left|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} f_2(s,y_2)\Delta s\Delta m' - \int_{\sigma(a)}^{t_2^{'}}\int_{\sigma(a)}^{m'} f_2(s,y_2)\Delta s\Delta m'\right| < \frac{\epsilon}{5}$$

and

$$\begin{aligned}
&\int_{\sigma(a)}^{t_2{'}}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m' - \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m' \\
&\leq M_2\left[\int_{\sigma(a)}^{t_2{'}}(\rho(b)-\sigma(a))\Delta m' - \int_{\sigma(a)}^{t_2}(\rho(b)-\sigma(a))\Delta m'\right] \\
&= M_2(\rho(b)-\sigma(a))[t_2{'} - \sigma(a) - (t_2 - \sigma(a))] \\
&= M_2(\rho(b)-\sigma(a))(t_2{'} - t_2).
\end{aligned}$$

Hence, whenever $|t_2 - t_2^{'}| < \delta$, we have

$$\left|\int_{\sigma(a)}^{t_2{'}}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m' - \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s,y_2)\Delta s\Delta m'\right| < \frac{\epsilon}{5}.$$

So, we see that

$$\|T(y_1(t_1),y_2(t_2)) - T(y_1(t_1{'}),y_2(t_2^{'}))\| < \frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5}+\frac{\epsilon}{5} = \epsilon$$

whenever

$$\|(t_1,t_2) - (t_1{'},t_2{'})\| < \delta,$$

i.e.,

$$|t_1 - t_1{'}| + |t_2 - t_2{'}| < \delta.$$

So, $T(\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2))$ is equicontinuous subset of $\mathcal{C}(\mathbb{T}_1)\times\mathcal{C}(\mathbb{T}_2)$.

Condition (iii) of the Arzela–Ascoli theorem can be seen from the fact that any $g_i(s,z_i) \in \mathcal{C}(\mathbb{T}_i \times \mathcal{C}(\mathbb{T}_i)),\ i = 1,2$, is uniformly continuous on $\mathbb{T}_i$ as $\mathbb{T}_i$ is compact (since it is closed and bounded).

Thus $T$ is compact by the Arzela–Ascoli theorem. So from Schauder's fixed point theorem (Theorem 2.6), a fixed point exists for the operator equation $(y_1,y_2) = Ty$. Hence a solution exists for the BVP-SIP. □

# 5 Existence and Uniqueness Results for the BVP-SIP

In this section, we prove the existence of a unique solution of BVP-SIP with a restricted interface condition. We use the Banach contraction principle.

**Theorem 5.1.** *Let* $\int_{\mathbb{T}_i} f_i \Delta t < \infty$ *and* $f_i,\ i = 1, 2$ *satisfy*

$$\begin{aligned} |f_1(t, y_1) - f_1(t, z_1)| &\leq |y_1 - z_1| \textit{ for all } t \in \mathbb{T}_1,\ y_1(t), z_1(t) \in \mathbb{R}, && (5.1) \\ |f_2(t, y_2) - f_2(t, z_2)| &\leq |y_2 - z_2| \textit{ for all } t \in \mathbb{T}_2,\ y_2(t), z_2(t) \in \mathbb{R} && (5.2) \end{aligned}$$

*and*

$$\frac{(\rho^2(b))^2}{2} + a\rho^2(b) + \frac{\gamma_1}{\gamma_2}\frac{3a^2}{2} < 1, \tag{5.3}$$

$$\begin{aligned} &\frac{\gamma_4}{\gamma_3}(\rho(b) - \sigma(a))\rho^2(b) + \frac{(\rho(b))^2}{2} - \frac{3}{2}(\sigma(a))^2 \\ &\qquad + (\rho(b) - \sigma(a))^2 + \frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}(\rho(b) - \sigma(a))a < 1. \end{aligned} \tag{5.4}$$

*Then the BVP-SIP has a unique solution.*

*Proof.* We use the Banach contraction mapping theorem (Theorem 2.5) to prove the existence of a unique fixed point for the operator equation.

*Claim* 5.2. The space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is complete.

Let $(x_{n1}, x_{n2})$ be a Cauchy sequence in $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ and let $\epsilon > 0$ be given. Let $(t_{01}, t_{02}) \in \mathbb{T}_1 \times \mathbb{T}_2$ be fixed. From Definition 2.13 for $(t_1, t_2) \in \mathbb{T}_1 \times \mathbb{T}_2$,

$$\|(t_1, t_2) - (t_{01}, t_{02})\| < \delta$$

implies that

$$\|(x_{n1}(t_1), x_{n2}(t_2)) - (x_{n1}(t_{01}), x_{n2}(t_{02}))\| < \frac{\epsilon}{3},$$

i.e.,

$$\|(t_1 - t_{01}, t_2 - t_{02})\| = |t_1 - t_{01}| + |t_2 - t_{02}| < \delta.$$

$$\begin{aligned} &\|(x_{n1}(t_1) - x_{n1}(t_{01}), x_{n2}(t_2) - x_{n2}(t_{02}))\| \\ &\qquad = \sup_{t_1 \in \mathbb{T}_1}|x_{n1}(t_1) - x_{n1}(t_{01})| + \sup_{t_2 \in \mathbb{T}_2}|x_{n2}(t_2) - x_{n2}(t_{02})| < \frac{\epsilon}{3} \end{aligned} \tag{5.5}$$

Now we let $\lim_{n\to\infty}(x_{n1}, x_{n2}) \to (x_1, x_2)$. Then there exists $\mathcal{N} > 0$ such that

$$\begin{aligned} &\|(x_{n1}, x_{n2}) - (x_1, x_2)\| = \|(x_{n1} - x_1, x_{n2} - x_{n2})\| \\ &\qquad = \sup_{t_1 \in \mathbb{T}_1}|(x_{n1} - x_1)(t_1)| + \sup_{t_2 \in \mathbb{T}_2}|(x_{n2} - x_2)(t_2)| < \frac{\epsilon}{3} \end{aligned} \tag{5.6}$$

For the space $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ to be complete we need to show that $(x_1, x_2)$ belongs to the space. That is

$$\|(t_1, t_2) - (t_{01}, t_{02})\| < \delta$$

implies

$$\|(x_1(t_1), x_2(t_2)) - (x_1(t_{01}), x_2(t_{02}))\| < \epsilon.$$

Now let us consider

$$\begin{aligned}
&\|(x_1(t_1), x_2(t_2)) - (x_1(t_{01}), x_2(t_{02}))\| \\
&= \|(x_1(t_1) - x_1(t_{01}), x_2(t_2) - x_2(t_{02}))\| \\
&= \sup_{t_1\in\mathbb{T}_1}|x_1(t_1) - x_1(t_{01})| + \sup_{t_2\in\mathbb{T}_2}|x_2(t_2) - x_2(t_{02})| \\
&= \sup_{t_1\in\mathbb{T}_1}|x_1(t_1) - x_{n1}(t_1) + x_{n1}(t_1) - x_{n1}(t_{01}) + x_{n1}(t_{01}) - x_1(t_{01})| \\
&\quad +\sup_{t_2\in\mathbb{T}_2}|x_2(t_2) - x_{n2}(t_2) + x_{n2}(t_2) - x_{n2}(t_{02}) + x_{n2}(t_{02}) - x_2(t_{02})| \\
&\leq \sup_{t_1\in\mathbb{T}_1}|x_1(t_1) - x_{n1}(t_1)| + \sup_{t_2\in\mathbb{T}_2}|x_2(t_2) - x_{n2}(t_2)| \\
&\quad +\sup_{t_1\in\mathbb{T}_1}|x_{n1}(t_1) - x_{n1}(t_{01})| + \sup_{t_2\in\mathbb{T}_2}|x_{n2}(t_2) - x_{n2}(t_{02})| \\
&\quad +\sup_{t_1\in\mathbb{T}_1}|(x_{n1} - x_1)(t_{01})| + \sup_{t_2\in\mathbb{T}_2}|(x_{n2} - x_2)(t_{02})| \\
&< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \text{ (using (5.5), (5.6))} \\
&= \epsilon.
\end{aligned}$$

Hence, $C(\mathbb{T}_1) \times C(\mathbb{T}_2)$ is complete.

*Claim* 5.3. The map $T$ is a contraction.

We see that

$$\begin{aligned}
&\|T(x_1, x_2) - T(z_1, z_2)\| \\
&\leq \sup_{t_1\in\mathbb{T}_1}\left|\int_0^{t_1}\int_0^m f_1(s, x_1)\Delta s\Delta m - \int_0^{t_1}\int_0^m f_1(s, z_1)\Delta s\Delta m\right| \\
&\quad +\sup_{t_1\in\mathbb{T}_1}\left|\frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} f_2(s, z_2)\Delta s\Delta m - \frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} f_2(s, x_2)\Delta s\Delta m\right| \\
&\quad +\sup_{t_1\in\mathbb{T}_1}\left|\int_0^{t_1}\int_0^a f_1(s, z_1)\Delta s\Delta m - \int_0^{t_1}\int_0^a f_1(s, x_1)\Delta s\Delta m\right| \\
&\quad +\sup_{t_2\in\mathbb{T}_2}\left|\int_{\sigma(a)^{t_2}}^{t_2}\int_{\sigma(a)}^{m'} f_2(s, x_2)\Delta s\Delta m' - \int_{\sigma(a)^{t_2}}^{t_2}\int_{\sigma(a)}^{m'} f_2(s, z_2)\Delta s\Delta m'\right| \\
&\quad +\sup_{t_2\in\mathbb{T}_2}\left|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s, z_2)\Delta s\Delta m' - \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s, x_2)\Delta s\Delta m'\right| \\
&\quad +\left|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{m'} f_1(s, x_1)\Delta s\Delta m' - \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{m'} f_1(s, z_1)\Delta s\Delta m'\right| \\
&\quad +\left|\frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^a\int_{\sigma(a)}^{\rho(b)} f_2(s, z_2)\Delta s\Delta m' - \frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^a\int_{\sigma(a)}^{\rho(b)} f_2(s, x_2)\Delta s\Delta m'\right|
\end{aligned}$$

$$+\left|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^a f_1(s,z_1)\Delta s\Delta m' - \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^a f_1(s,x_1)\Delta s\Delta m'\right|.$$

Now let us consider each of the term in the above equation separately:

$$\begin{aligned}
\sup_{t_1\in\mathbb{T}_1} & \quad \left|\int_0^{t_1}\int_0^m f_1(s,x_1)\Delta s\Delta m - \int_0^{t_1}\int_0^m f_1(s,z_1)\Delta s\Delta m\right| \\
\leq & \quad \sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^m \Delta s\Delta m \text{ (use (5.1))} \\
= & \quad \sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\sup_{t_1\in\mathbb{T}_1}\frac{{t_1}^2}{2},
\end{aligned}$$

$$\begin{aligned}
\sup_{t_1\in\mathbb{T}_1} & \quad \left|\frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} f_2(s,z_2)\Delta s\Delta m - \frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} f_2(s,x_2)\Delta s\Delta m\right| \\
\leq & \quad \sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\sup_{t_1\in\mathbb{T}_1}\frac{\gamma_4}{\gamma_3}\int_0^{t_1}\int_{\sigma(a)}^{\rho(b)} \Delta s\Delta m \text{ (use (5.2))} \\
= & \quad \sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_4}{\gamma_3}(\rho(b)-\sigma(a))\sup_{t_1\in\mathbb{T}_1}(t_1),
\end{aligned}$$

$$\begin{aligned}
\sup_{t_1\in\mathbb{T}_1} & \quad \left|\int_0^{t_1}\int_0^a f_1(s,z_1)\Delta s\Delta m - \int_0^{t_1}\int_0^a f_1(s,x_1)\Delta s\Delta m\right| \\
\leq & \quad \sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\sup_{t_1\in\mathbb{T}_1}\int_0^{t_1}\int_0^a \Delta s\Delta m \text{ (use (5.1))} \\
= & \quad \sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\sup_{t_1\in\mathbb{T}_1}(at_1),
\end{aligned}$$

$$\begin{aligned}
\sup_{t_2\in\mathbb{T}_2} & \quad \left|\int_{\sigma(a)^{t_2}}^{t_2}\int_{\sigma(a)}^{m'} f_2(s,x_2)\Delta s\Delta m' - \int_{\sigma(a)^{t_2}}^{t_2}\int_{\sigma(a)}^{m'} f_2(s,z_2)\Delta s\Delta m'\right| \\
\leq & \quad \sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{m'} \Delta s\Delta m' \text{ (use (5.2))} \\
= & \quad \sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\sup_{t_2\in\mathbb{T}_2}\left(\frac{{t_2}^2}{2}-\sigma(a)t_2-\frac{(\sigma(a))^2}{2}\right),
\end{aligned}$$

$$\begin{aligned}
\sup_{t_2\in\mathbb{T}_2} & \quad \left|\int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s,z_2)\Delta s\Delta m' - \int_{\sigma(a)}^{t_2}\int_{\sigma(a)}^{\rho(b)} f_2(s,x_2)\Delta s\Delta m'\right| \\
\leq & \quad \sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\sup_{t_2\in\mathbb{T}_2}\int_{\sigma(a)}^{t_2}(\sigma(a)-\rho(b))\Delta m' \text{ (use (5.2))} \\
= & \quad \sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|(\rho(b)-\sigma(a))\sup_{t_2\in\mathbb{T}_2}(t_2-\sigma(a)),
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{m'} f_1(s,x_1)\Delta s\Delta m' - \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{m'} f_1(s,z_1)\Delta s\Delta m'\right| \\
&\le \sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{m'}\Delta s\Delta m' \text{ (use (5.1))} \\
&= \sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_1}{\gamma_2}\frac{a^2}{2},
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^a\int_{\sigma(a)}^{\rho(b)} f_2(s,z_2)\Delta s\Delta m' - \frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}\int_0^a\int_{\sigma(a)}^{\rho(b)} f_2(s,x_2)\Delta s\Delta m'\right| \\
&\le \sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}a(\rho(b)-\sigma(a)) \text{ (use (5.2))},
\end{aligned}$$

$$\begin{aligned}
&\left|\frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{a} f_1(s,z_1)\Delta s\Delta m' - \frac{\gamma_1}{\gamma_2}\int_0^a\int_0^{a} f_1(s,x_1)\Delta s\Delta m'\right| \\
&\le \sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\frac{\gamma_1}{\gamma_2}a^2 \text{ (use (5.1))}.
\end{aligned}$$

From the above we can conclude that

$$\|T(x_1,x_2)-T(z_1,z_2)\| \le K_1\Big(\sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|\Big)+K_2\Big(\sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\Big),$$

where

$$\begin{aligned}
K_1 &= \sup_{t_1\in\mathbb{T}_1}(\frac{t_1^2}{2})+\sup_{t_1\in\mathbb{T}_1}(at_1)+\frac{\gamma_1}{\gamma_2}\frac{a^2}{2}+\frac{\gamma_1}{\gamma_2}a^2, \\
K_2 &= \frac{\gamma_4}{\gamma_3}(\rho(b)-\sigma(a))\sup_{t_1\in\mathbb{T}_1}(t_1)+\sup_{t_2\in\mathbb{T}_2}\Big(\frac{{t_2}^2}{2}-\sigma(a)t_2-\frac{\sigma(a)^2}{2}\Big) \\
&\quad +(\rho(b)-\sigma(a))\sup_{t_2\in\mathbb{T}_2}(t_2-\sigma(a))+\frac{\gamma_1\gamma_4}{\gamma_2\gamma_3}a(\rho(b)-\sigma(a)).
\end{aligned}$$

Let $K=\max\{K_1,K_2\}$. Then $K<1$ (use (5.3)–(5.4)). We now have

$$\begin{aligned}
&\|T(x_1,x_2)-T(z_1,z_2)\| \\
&\le K\Big(\sup_{s\in\mathbb{T}_1}|(x_1-z_1)(s)|+\sup_{s\in\mathbb{T}_2}|(x_2-z_2)(s)|\Big) \\
&= K\|(x_1-z_1,x_2-z_2)\| \\
&= K\|(x_1,x_2)-(z_1,z_2)\|.
\end{aligned}$$

Since $K<1$, by the Banach contraction mapping theorem (Theorem 2.5), we have a unique fixed point for $(y_1,y_2)=T(y_1,y_2)$. Hence a unique solution exists for the BVP-SIP. □

*Remark* 5.4. The above theorems can be proved for BVPs for the Interface II and Interface III with suitable changes in the notations.

# 6 Applications

The results presented here are generalization for the nonlinear problems of corresponding linear problems studied in [1,6–10,16–22]. A pair of nonlinear ordinary differential equations with matching interface conditions is a special case of the problem considered here, and our results hold true by considering $\rho(c) = \sigma(c) = c$ and the delta derivative becomes the ordinary derivative.

Here we discuss the theory developed to few regular interface problems in the field of acoustic wave guides in ocean and transverse vibrations in strings.

**1)** In the study of acoustic wave guides in ocean [12], where the ocean is considered to be consisting of two homogeneous layers, we encounter the following problem given by

$$\begin{aligned} L_1u_1 &= \frac{d}{dx}\left(\frac{1}{\rho_1}\frac{du_1}{dx}\right) + \left(\frac{{k_1}^2}{\rho_1} - \frac{\lambda}{\rho_1}\right)u_1 = 0,\ 0 \le x \le d_1, \\ L_2u_2 &= \frac{d}{dx}\left(\frac{1}{\rho_2}\frac{du_2}{dx}\right) + \left(\frac{{k_2}^2}{\rho_2} - \frac{\lambda}{d_2}\right)u_2 = 0,\ d_1 \le x \le d_2 \end{aligned}$$

together with the mixed boundary conditions given by

$$u_1(0) = {u_2}'(d_2) = 0, \quad u_1(d_1) = u_2(d_1), \quad \frac{1}{\rho_1}{u_1}'(d_1) = \frac{1}{\rho_2}{u_2}'(d_1),$$

where $\rho_1$ and $\rho_2$ are constant densities of the two layers, $k_1$,$k_2$ are constants which depend upon the frequency constant $w$ and the constant sound velocities $c_1, c_2$ of the two layers, respectively, $\lambda$ is an unknown constant, $[0, d_1]$ and [$d_1$,$d_2$] denote the two layers and $u_1$ and $u_2$ denote the depth eigenfunctions.

Here we see that $a = d_1 = \sigma(a)$ is the regular interface and $d_2 = \rho(b)$. Hence from Theorem 4.1, we see that a solution exists for the problem of acoustic wave guides in the ocean.

**2)** We encounter the following problem in the study of transverse vibrations of strings [13] consisting of two portions of lengths $d_1$ and $d_2$ and different uniform densities $\rho_1$ and $\rho_2$ respectively, having a tension T stretched between the points $x = 0$ and $x = d_1 + d_2$:

$$\begin{aligned} L_1u_1 = {c_1}^2(-{u_1}'') &= \lambda u_1,\ 0 \le x \le d_1, \\ L_2u_2 = {c_2}^2(-{u_2}'') &= \lambda u_2,\ d_1 \le x \le d_1 + d_2 \end{aligned}$$

with the mixed boundary conditions given by

$$u_1(d_1) = u_2(d_1), u_1'(d_1) = u_2'(d_1)$$

and $u_1(0) = u_2(d_1 + d_2) = 0$, where ${c_i}^2 = T|_{\rho_i}, i = 1, 2$.

Here we see that $a = d_1 = \sigma(a)$ is the regular interface. Though the theory developed was for a BVP with a different boundary condition, in similar lines existence of solution for the above BVP associated with regular interface can be shown using Schauder's fixed point theorem.

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