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Nontrivial Solutions for Boundary Value Problems of Nonlinear Differential Equation

Jia Mu, **Yongxiang Li**, and **Pengyu Chen**
Northwest Normal University, Department of Mathematics
Lanzhou, Gansu 730000, People's Republic of China
mujia88@163.com

Abstract

The nonlinear three-point boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t)), & t \in I, \\ \beta u(0) - \gamma u'(0) = 0, & u(1) = \alpha u(\eta), \end{cases}$$

is discussed under some conditions concerning the first eigenvalue corresponding to a special linear operator, where $I = [0, 1]$, $\eta \in (0, 1)$, α, β, $\gamma \in [0, \infty)$ with $\beta + \gamma \neq 0$, $f : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is sign–changing continuous function and may be unbounded from below. By applying the topological degree of a completely continuous field, we establish some new existence criteria of nontrivial solutions. At last, two examples are given to demonstrate the application of the main results.

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1 Introduction

Over the last thirty years, boundary value problems have attracted extensive attentions due to their wide range of applications in applied mathematics, physics, biology and engineering (see, for example, [6–8, 12–18] and references therein for more details). Most of them obtained the existence of positive solutions provided $f : [0, +\infty) \to [0, +\infty)$ is nonnegative, continuous and superlinear or sublinear by employing the cone expansion or compression fixed point theorem, the method of upper and lower solutions, Schauder's fixed point theorem or the fixed point index. Recently, some authors have

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studied the existence of nontrivial solutions of boundary value problems when the nonlinear term is sign–changing, such as [1, 3, 5, 9–11, 20–23].

L. Liu, B. Liu, Y. Wu [10] discussed the existence of nontrivial solutions of the problem

$$\begin{cases} (Lu)(t) + h(t)f(t, u(t)) = 0, \quad 0 < t < 1, \\ u(0) = 0, u(1) = \sum_{i=1}^{m-2} a_i u(\xi_i), \end{cases}$$

under the following conditions:

(A_1) There exist two nonnegative functions $b(t)$, $c(t) \in C[0,1]$ with $c(t) \not\equiv 0$ and one continuous even function $B : (-\infty, +\infty) \to [0, +\infty)$ such that

$$f(t, u) \geq -b(t) - c(t)Bu, \quad \text{for all} \quad t \in I, u \in \mathbb{R}.$$

Moreover, B is nondecreasing on $[0, +\infty)$ and satisfies

$$\lim_{u \to +\infty} \frac{Bu}{u} = 0;$$

(A_2) $\liminf\limits_{u \to +\infty} \dfrac{f(t,u)}{u} > \lambda_1'$, uniformly on $t \in I$;

(A_3) $\limsup\limits_{u \to 0} \left| \dfrac{f(t,u)}{u} \right| < \lambda_1'$, uniformly on $t \in I$.

In [10], the result improved that in [3, 21].

Inspired by the above work, this paper is concerned with the nonlinear three-point boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t)), \quad t \in I, \\ \beta u(0) - \gamma u'(0) = 0, \quad u(1) = \alpha u(\eta), \end{cases} \tag{1.1}$$

where $I = [0, 1]$, $\eta \in (0, 1)$, $\alpha, \beta, \gamma \in [0, \infty)$ with $\beta + \gamma \neq 0$, $f : [0, 1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is sign–changing continuous function and may be unbounded from below. The new features of this paper mainly include the following aspects. Firstly, we give the Green function of (1.1), and it is not necessarily symmetric. Thus our work can be applied to more general problems. Secondly, to cope with the difficulties caused by the asymmetry of Green function, a special linear operator is sought and a special cone is constructed for the study of the existence of nontrivial solutions. Thirdly, our conditions are weaker than the previous papers. Fourthly, two examples are constructed to demonstrate the application of the main results. Finally, the main technique used here is the topological degree theory.

2 Preliminaries

In Banach space $C(I)$ in which the norm is defined by $\|u\| = \max\limits_{t \in I} |u(t)|$, we set $P = \{u \in C(I) | u(t) \geq 0, t \in I\}$, then P is a positive cone in $C(I)$. The open ball in $C(I)$ is denoted by $B_r = \{u \in C(I) \mid \|u\| < r, r > 0\}$.

For convenience, we give the following assumption:
(H_1) $\beta + \gamma - \alpha\beta\eta - \alpha\gamma > 0$.
Then we set

$$L(t,s) = \frac{1}{\beta+\gamma}\begin{cases} (\beta t+\gamma)(-s+1), & 0 \le t \le s \le 1, \\ (\beta s+\gamma)(-t+1), & 0 \le s \le t \le 1. \end{cases} \tag{2.1}$$

By (2.1), we easily obtain the following lemma.

Lemma 2.1. *$L(t,s)$ possesses the following properties:*

(i) *$L(t,s)$ is continuous and $L(t,s) \ge 0$ over $I \times I$;*

(ii) *$L(t,s) = L(s,t)$, $L(t,s) \le L(s,s)$ for $t, s \in I$.*

Lemma 2.2. *Assume that (H_1) holds. Then for $y \in C(I)$, the problem*

$$\begin{cases} -u''(t) = y(t), & t \in I, \\ \beta u(0) - \gamma u'(0) = 0, & u(1) = \alpha u(\eta), \end{cases} \tag{2.2}$$

has a unique solution

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.3}$$

where

$$G(t,s) = L(t,s) + \frac{\alpha L(\eta,s)}{\beta+\gamma-\alpha\beta\eta-\alpha\gamma}(\beta t+\gamma). \tag{2.4}$$

Proof. If $G(t,s)$ is the same as in (2.4), it is easy to check that the function defined by (2.3) is a solution of (2.2).

Now we show that the function defined by (2.3) is a solution of (2.2) only if $G(t,s)$ is the same as in (2.4). By the method of variation of constants, we can obtain that any solution of (2.2) can be represented by

$$u(t) = \int_0^1 L(t,s)y(s)ds + M(\beta t+\gamma). \tag{2.5}$$

In view of (2.1), we have

$$u(t) = (-t+1)\int_0^t \frac{1}{\beta+\gamma}(\beta s+\gamma)y(s)ds + (\beta t+\gamma)\int_t^1 \frac{1}{\beta+\gamma}(-s+1)y(s)ds + M(\beta t+\gamma), \tag{2.6}$$

and

$$-u''(t) = y(t), \tag{2.7}$$

Observe that

$$u(0) = \gamma \int_0^1 \frac{1}{\omega}(-s+1)y(s)ds + M\gamma,$$

$$u'(0) = \beta \int_0^1 \frac{1}{\omega}(-s+1)y(s)ds + M\beta,$$

we get

$$\beta u(0) - \gamma u'(0) = 0.$$

Since $u(1) = M(\beta+\gamma)$ and $u(\eta) = \int_0^1 L(\eta,s)y(s)ds + M(\beta\eta+\gamma)$, by the second boundary condition of (2.2), we get that

$$M = \frac{\alpha \int_0^1 L(\eta,s)y(s)ds}{\beta+\gamma-\alpha\beta\eta-\alpha\gamma},$$

so by (2.5) the proof is finished. □

Lemma 2.3. *Assume* (H_1) *holds. Then* $G(t,s)$ *defined by* (2.4) *has the following properties:*

(i) $G(t,s)$ *is nonnegative and continuous on* $I \times I$*;*

(ii) $G(t,s) \le DL(s,s)$ *for* $t, s \in I$*, where* $D = \frac{\beta+\gamma-\alpha\beta\eta+\alpha\beta}{\beta+\gamma-\alpha\beta\eta-\alpha\gamma}$.

By the above discussion, we know that the problem (1.1) is equivalent to the integral equation

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds := Au(t), \quad t \in I. \tag{2.8}$$

And it is easy to see that u is a nontrivial solution of (1.1) if and only if $u \in C(I)$ is a fixed point of A and $u(t) \not\equiv 0$, $t \in I$. Define two linear operators K, T by

$$(Ku)(t) = \int_0^1 G(t,s)u(s)ds, \quad t \in I. \tag{2.9}$$

$$(Tu)(t) = \int_0^1 G(s,t)u(s)ds, \quad t \in I. \tag{2.10}$$

By virtue of Krein–Rutmann's theorem [4] and similar as the proof of [21, Lemma 5], we get the following lemma.

Lemma 2.4. *Suppose* (H_1) *is satisfied. Then for the operator* K, T *defined by* (2.9) *and* (2.10)*, respectively, we have*

(i) $K, T : C(I) \to C(I)$ *are completely continuous positive linear operators;*

(ii) the spectral radius $r(K) \neq 0$, $r(T) \neq 0$, and K, T have positive eigenfunctions corresponding to their first eigenvalues $\lambda_1 = (r(K))^{-1}$ and $\lambda_1' = (r(T))^{-1}$ respectively.

Let φ_1 and φ_2 be the positive eigenfunctions of K and T respectively, i.e.,

$$\lambda_1 K\varphi_1 = \varphi_1, \lambda_1' T\varphi_2 = \varphi_2. \tag{2.11}$$

Lemma 2.5. *Assume that (H_1) holds. Then there exist δ_1, $\delta_2 > 0$ such that*

$$\delta_1 G(t,s) \leq \varphi_2(s) \leq \delta_2 G(s,s), \quad t,s \in I.$$

Proof. If $\gamma = 0$, it follows from $\beta + \gamma \neq 0$, $\beta, \gamma \in [0, +\infty)$ and (2.4) that $\beta > 0$, $G(t,0) = G(t,1) = 0$ for $t \in I$. Then we obtain from (2.10) and (2.11) that $\varphi_2(0) = \varphi_2(1) = 0$, which implies that $\varphi_2'(0) > 0$, $\varphi_2'(1) < 0$ (see [19]). Since

$$\lim_{s\to 0^+} \frac{\varphi_2(s)}{\beta s + \gamma} = \frac{\varphi_2'(0)}{\beta} > 0,$$

$$\lim_{s\to 1^-} \frac{\varphi_2(s)}{-s + 1} = \frac{\varphi_2'(1)}{-1} > 0,$$

we can define

$$\Phi(s) = \frac{(\beta+\gamma)\varphi_2(s)}{(\beta s+\gamma)(-s+1)}, \quad 0 < s < 1,$$

$$\Phi(0) = \frac{(\beta+\gamma)\varphi_2'(0)}{\beta}, \quad \Phi(1) = \frac{(\beta+\gamma)\varphi_2'(1)}{-(\beta+\gamma)}.$$

Then $\Phi(s)$ is continuous on I and $\Phi(s) > 0$, $\forall s \in (0,1)$. So there exist ρ, $\delta_2 > 0$ such that $\rho \leq \Phi(s) \leq \delta_2$, $\forall s \in (0,1)$, i.e.,

$$\rho \frac{1}{\beta+\gamma}(\beta s+\gamma)(-s+1) \leq \varphi_2(s) \leq \delta_2 \frac{1}{\beta+\gamma}(\beta s+\gamma)(-s+1), \quad s \in I.$$

Then

$$\rho L(s,s) \leq \varphi_2(s) \leq \delta_2 L(s,s).$$

By Lemma 2.3 and (2.4), we obtain

$$\frac{1}{D}\rho G(t,s) \leq \varphi_2(s) \leq \delta_2 G(s,s), \quad t,s \in I.$$

Set $\delta_1 = \frac{1}{D}\rho$, we can find that

$$\delta_1 G(t,s) \leq \varphi_2(s) \leq \delta_2 G(s,s), \quad t,s \in I.$$

In the case $\gamma > 0$, we can prove the same conclusions analogically. □

Lemma 2.6. *$K(P) \subset P_1$, where* $P_1 = \left\{ u \in P \middle| \int_0^1 \varphi_2(t)u(t)dt \geq \lambda_1'^{-1}\delta_1 \|u\| \right\}$.

Proof. For all $u \in P$, by (2.11) and Lemma 2.5, we have

$$\begin{aligned} \int_0^1 \varphi_2(t)(Ku)(t)dt &= \int_0^1 \varphi_2(t) \left(\int_0^1 G(t,s)u(s)ds \right) dt \\ &= \int_0^1 \left(\int_0^1 G(t,s)\varphi_2(t)dt \right) u(s)ds \\ &= \int_0^1 (T\varphi_2)(s)u(s)ds \\ &\geq \lambda_1'^{-1}\delta_1 (Ku)(t), \quad t \in I. \end{aligned}$$

Thus $\int_0^1 \varphi_2(t)(Ku)(t)dt \geq \lambda_1'^{-1}\delta_1\|Ku\|$, i.e., $K(P) \subset P_1$. □

Lemma 2.7. *[2] Let E be a Banach space and Ω be a bounded open set in E. Suppose that $A : \overline{\Omega} \to E$ is a completely continuous operator. If there exists $u_0 \neq \theta$ such that*

$$u - Au \neq \mu u_0 \text{ for } u \in \partial\Omega, \mu \geq 0,$$

then the topological degree $\deg(I - A, \Omega, \theta) = 0$.

Lemma 2.8. *[2] Let E be a Banach space and Ω be a bounded open set in E with $\theta \in \Omega$. Suppose that $A : \overline{\Omega} \to E$ is a completely continuous operator. If*

$$Au \neq \mu u \text{ for } u \in \partial\Omega, \mu \geq 1,$$

then the topological degree $\deg(I - A, \Omega, \theta) = 1$.

3 Existence of Nontrivial Solutions

Theorem 3.1. *Suppose (H_1) holds. In addition, f satisfies the following conditions:*

(W_1) *There exist $\varepsilon_1 > 0$ and $d \in C(I)$, such that*

$$f(t,u) \geq (\lambda_1' + \varepsilon_1)u - d(t), \quad \forall t \in I, u \in \mathbb{R};$$

(W_2) *There exist $0 < \varepsilon_2 < \lambda_1'$, $(\lambda_1' - \varepsilon_2)\lambda_1^{-1} < 1$ and $e \in C(I)$ such that*

$$f(t,u) \geq (\lambda_1' - \varepsilon_2)u - e(t), \quad \forall t \in I, u \in \mathbb{R};$$

(W_3) *There exist $0 < \varepsilon_3 < \lambda_1'$ and $r > 0$, such that*

$$|f(t,u(t))| \leq (\lambda_1' - \varepsilon_3)|u(t)|, \quad \forall t \in I, u \in \mathbb{R}, |u| \leq r,$$

where λ_1, λ_1' are defined by (2.11). *Then the problem* (1.1) *has at least one nontrivial solution.*

Proof. Since $(\lambda_1' - \varepsilon_2)\lambda_1^{-1} < 1$, the operator $I - (\lambda_1' - \varepsilon_2)K$ has the bounded inverse operator $(I - (\lambda_1' - \varepsilon_2)K)^{-1}$. We set

$$R > \max \Big\{ r, \quad \|(I - (\lambda_1' - \varepsilon_2)K)^{-1}\| \times \left(\delta_1^{-1}\varepsilon_1^{-1}\varepsilon_2 \int_0^1 d(s)\varphi_2(s)ds + \delta_1^{-1} \int_0^1 e(s)\varphi_2(s)ds + \|Ke\| \right) \Big\}. \quad (3.1)$$

Now we show that

$$u - Au \neq \mu\varphi_1, \quad u \in \partial B_R, \mu \geq 0, \quad (3.2)$$

where φ_1 is defined in (2.11). In fact, if (3.2) is not true, then there exist $u_1 \in \partial B_R, \mu_1 \geq 0$ satisfying

$$u_1 - Au_1 = \mu_1\varphi_1. \quad (3.3)$$

Combining (W_1), (2.11) and (3.3) yields

$$\begin{aligned} \int_0^1 u_1(t)\varphi_2(t)dt &\geq \int_0^1 (Au_1)(t)\varphi_2(t)dt \\ &\geq \int_0^1 \left(\int_0^1 G(t,s)[(\lambda_1' + \varepsilon_1)u_1(s) - d(s)]ds \right) \varphi_2(t)dt \\ &= (\lambda_1' + \varepsilon_1) \int_0^1 \left(\int_0^1 G(t,s)\varphi_2(t)dt \right) u_1(s)ds \\ &\quad - \int_0^1 \left(\int_0^1 G(t,s)\varphi_2(t)dt \right) d(s)ds \\ &= (\lambda_1' + \varepsilon_1)\lambda_1'^{-1} \int_0^1 u_1(s)\varphi_2(s)ds - \lambda_1'^{-1} \int_0^1 d(s)\varphi_2(s)ds. \end{aligned}$$

So

$$\int_0^1 u_1(s)\varphi_2(s)ds \leq \varepsilon_1^{-1} \int_0^1 d(s)\varphi_2(s)ds. \quad (3.4)$$

By (2.11), (3.3), (3.4), (W_2) and Lemma 2.6, we get $u_1 - (\lambda_1' - \varepsilon_2)(Ku_1) + (Ke) \in P_1$ and

$$\begin{aligned} &\|u_1 - (\lambda_1' - \varepsilon_2)(Ku_1) + (Ke)\| \\ &\leq \lambda_1'\delta_1^{-1} \int_0^1 \big[u_1(t) - (\lambda_1' - \varepsilon_2)(Ku_1)(t) + (Ke)(t)\big]\varphi_2(t)dt \\ &= \delta_1^{-1}\varepsilon_2 \int_0^1 u_1(s)\varphi_2(s)ds + \delta_1^{-1} \int_0^1 e(s)\varphi_2(s)ds \\ &\leq \delta_1^{-1}\varepsilon_1^{-1}\varepsilon_2 \int_0^1 d(s)\varphi_2(s)ds + \delta_1^{-1} \int_0^1 e(s)\varphi_2(s)ds. \end{aligned}$$

So $R = \|u_1\|$, which is less than or equal to the right-hand side of (3.1), hence contradicting (3.1). Then (3.2) holds. By Lemma 2.7,

$$\deg(I - A, B_R, \theta) = 0. \quad (3.5)$$

In the following, we prove that

$$Au \neq \tau u, \quad u \in \partial B_r, \tau \geq 1. \tag{3.6}$$

Otherwise, there exist $u_2 \in \partial B_r$ and $\tau_1 \geq 1$ such that $Au_2 = \tau_1 u_2$. We may suppose $\tau_1 > 1$. By (2.8),(W_3) and (2.11), then

$$\begin{aligned}
\tau_1 \int_0^1 |u_2(t)|\varphi_2(t)dt &\leq \int_0^1 \left(\int_0^1 G(t,s)\big|f(s,u_2(s))\big|ds \right) \varphi_2(t)dt \\
&\leq (\lambda_1' - \varepsilon_3) \int_0^1 \left(\int_0^1 G(t,s)|u_2(s)|ds \right) \varphi_2(t)dt \\
&= (\lambda_1' - \varepsilon_3) \int_0^1 \left(\int_0^1 G(t,s)\varphi_2(t)dt \right) |u_2(s)|ds \\
&= (\lambda_1' - \varepsilon_3) \int_0^1 (T\varphi_2)(s)|u_2(s)|ds \\
&= (\lambda_1' - \varepsilon_3) \int_0^1 \lambda_1'^{-1}\varphi_2(s)|u_2(s)|ds \\
&= (1 - \varepsilon_3 \lambda_1'^{-1}) \int_0^1 |u_2(t)|\varphi_2(t)dt,
\end{aligned}$$

which implies

$$\int_0^1 |u_2(t)|\varphi_2(t)dt \leq 0. \tag{3.7}$$

On the other hand, $\varphi_2(t) > 0$ for all $0 < t < 1$ by the maximum principle and $u_2(t)$ attains zero on isolated point by Sturm's theorem. Hence

$$\int_0^1 |u_2(t)|\varphi_2(t)dt > 0,$$

which contradicts (3.7). Thus (3.6) holds. By Lemma 2.8.

$$\deg(I - A, B_r, \theta) = 1. \tag{3.8}$$

By (3.5), (3.8) and the additivity of degree, we have

$$\deg(I - A, B_R \backslash \overline{B}_r, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta) = -1 \neq 0.$$

Thus A has at least one fixed point in $B_R \backslash \overline{B}_r$. This means that the problem (1.1) has at least one nontrivial solution. □

Corollary 3.2. *Assume that* (H_1), (A_1), (A_2), (A_3) *from Section 1 hold. Then the problem* (1.1) *has at least one nontrivial solution.*

Proof. From (A_1), for $0 < \varepsilon_2 < \lambda_1'$, $(\lambda_1' - \varepsilon_2)\lambda_1^{-1} < 1$, there exists $l_1 < 0$ such that

$$f(t,u) \geq (\lambda_1' - \varepsilon_2)u - b(t), \quad t \in I, u < l_1. \tag{3.9}$$

From (A_2), there exist $\varepsilon_1 > 0$ and a sufficiently large number $l_2 > 0$ such that

$$f(t,u) \geq (\lambda_1' + \varepsilon_1)u, \quad t \in I, u > l_2. \tag{3.10}$$

Since $f : [0,1] \times (-\infty, +\infty) \to (-\infty, +\infty)$ is continuous and $b(t) \in C(I)$, then there exists a constant l_3 that

$$f(t,u) \geq (\lambda_1' + \varepsilon_1)u - l_3 \geq (\lambda_1' - \varepsilon_2)u - l_3, \quad t \in I, u \geq 0,$$

$$f(t,u) \geq (\lambda_1' - \varepsilon_2)u - l_3 \geq (\lambda_1' + \varepsilon_1)u - l_3, \quad t \in I, u \leq 0.$$

Thus, (W_1) and (W_2) hold. It is easy to see that $(A_3) \Rightarrow (W_3)$. So, by Theorem 3.1, the problem (1.1) has at least one nontrivial solution. □

Remark 3.3. In our paper, the green function is asymmetric. In fact, our results can be applicable to the case that the green function is symmetric, and then $K = T$, $\lambda_1 = \lambda_1'$, $\varphi_1 = \varphi_2$.

Theorem 3.4. *Suppose that (H_1) holds, and f satisfies the following conditions:*
(W_4) *There exist ε_4, $r > 0$ such that*

$$f(t,u) \geq (\lambda_1' + \varepsilon_4)u, \quad \forall t \in I, u \in C(I), |u| \leq r;$$

(W_5) *There exist $0 < \varepsilon_5 < \lambda_1'$, $(\lambda_1' - \varepsilon_5)\lambda_1^{-1} < 1$ and $r > 0$ such that*

$$f(t,u) \geq (\lambda_1' - \varepsilon_5)u, \quad \forall t \in I, u \in C(I), |u| \leq r;$$

(W_6) *There exist $0 < \varepsilon_6 < \lambda_1'$ and $g \in C(I)$ such that*

$$|f(t,u(t))| \leq (\lambda_1' - \varepsilon_6)|u(t)| + |g(t)|, \quad \forall t \in I, u \in \mathbb{R},$$

where λ_1, λ_1' are defined by (2.11)*. Then the problem* (1.1) *has at least one nontrivial solution.*

Proof. Now, we show that

$$u - Au \neq \mu\varphi_1, \quad u \in \partial B_r, \mu \geq 0, \tag{3.11}$$

where φ_1 is defined by (2.11). Suppose on the contrary, then there are $u_1 \in \partial B_r$, $\mu_1 \geq 0$ satisfying

$$u_1 - Au_1 = \mu_1\varphi_1. \tag{3.12}$$

Combining (W_4), (2.11) and (3.12) yields

$$\begin{aligned}\int_0^1 u_1(t)\varphi_2(t)dt &\geq (\lambda_1' + \varepsilon_4)\int_0^1 \left(\int_0^1 G(t,s)u_1(s)ds\right)\varphi_2(t)dt \\ &= (\lambda_1' + \varepsilon_4)\lambda_1'^{-1}\int_0^1 u_1(s)\varphi_2(s)ds.\end{aligned}$$

Thus

$$\int_0^1 u_1(t)\varphi_2(t)dt \leq 0. \tag{3.13}$$

On the other hand, by (2.11), (3.12), (3.13), (W_5) and Lemma 2.6, we get $u_1 - (\lambda_1' - \varepsilon_5)(Ku_1) \in P_1$ and

$$\begin{aligned}u_1(t) - (\lambda_1' - \varepsilon_5)(Ku_1)(t) &\leq \lambda_1'\delta_1^{-1}\int_0^1 \big(u_1(t) - (\lambda_1' - \varepsilon_5)(Ku_1)(t)\big)\varphi_2(t)dt \\ &= \delta_1^{-1}\varepsilon_5\int_0^1 u_1(s)\varphi_2(s)ds \\ &\leq 0.\end{aligned}$$

Then $u_1 \equiv 0$, which contradicts $u_1 \in \partial B_r$. Thus (3.11) holds. By Lemma 2.7,

$$\deg(I - A, B_r, \theta) = 0. \tag{3.14}$$

Let

$$R > \max\left\{r, \varepsilon_6^{-1}\|g\|\right\}. \tag{3.15}$$

In the following, we prove that

$$Au \neq \tau u, \quad u \in \partial B_R, \tau \geq 1. \tag{3.16}$$

Otherwise, there is $u_2 \in \partial B_R$ and $\tau_1 \geq 1$ such that $Au_2 = \tau_1 u_2$. We may suppose $\tau_1 > 1$. Then by (2.11) and (W_6),

$$\begin{aligned}\int_0^1 |u_2(t)|\varphi_2(t)dt &\leq (\lambda_1' - \varepsilon_6)\int_0^1 \left(\int_0^1 G(t,s)|u(s)|ds\right)\varphi_2(t)dt \\ &\quad + \int_0^1 \left(\int_0^1 G(t,s)|g(s)|ds\right)\varphi_2(t)dt \\ &= (1 - \varepsilon_6\lambda_1'^{-1})\int_0^1 |u_2(s)|\varphi_2(s)ds + \lambda_1'^{-1}\int_0^1 |g(s)|\varphi_2(s)ds.\end{aligned}$$

So

$$\int_0^1 (|u_2(s)| - \varepsilon_6^{-1}|g(s)|)\varphi_2(t)dt \leq 0. \tag{3.17}$$

Because $\varphi_2(t) > 0$ for $0 < t < 1$ by maximum principle, then $|u_2(s)| - \varepsilon_6^{-1}|g(s)| \le 0$ for $0 < s < 1$. Thus, $R = \|u_2\| \le \varepsilon_6^{-1}\|g\|$, which contradicts (3.15). So (3.16) holds. By Lemma 2.8,

$$\deg(I - A, B_R, \theta) = 1 \ne 0. \tag{3.18}$$

By (3.14), (3.18) and the additivity of degree, we have

$$\deg(I - A, B_R \backslash \overline{B}_r, \theta) = \deg(I - A, B_R, \theta) - \deg(I - A, B_r, \theta) = 1 \ne 0.$$

Thus A has at least one fixed point in $B_R \backslash \overline{B}_r$. This means that the problem (1.1) has at least one nontrivial solution. □

4 Examples

In this section, we give two examples to demonstrate the applications of our main results.

Example 4.1. Consider the boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t)) = h_1 u(t) + u^2(t), & t \in I, \\ u(0) - u'(0) = 0, u(1) = u(\eta), \end{cases} \tag{4.1}$$

where $\eta \in (0, 1), 0 < h_1 < \lambda_1'$ and λ_1' is the first eigenvalue of T corresponding to (4.1).

The boundary value problem (4.1) can be regarded as a form of (1.1), where $\beta = \gamma = \alpha = 1$. Then (H_1) holds.
(i) For $u(t) \ge \lambda_1' + 1 - h_1$ or $u(t) \le 0$, $f(t, u(t)) \ge (\lambda_1' + 1)u(t)$; for $0 < u(t) < \lambda_1' + 1 - h_1$, $f(t, u(t)) \ge (\lambda_1' + 1)u(t) - (\lambda_1' + 1 - h_1)^2$. So,

$$f(t, u) \ge (\lambda_1' + 1)u - (\lambda_1' + 1 - h_1)^2, \quad \forall t \in I, u \in C(I).$$

This means that (W_1) holds.
(ii) We select ε_2 that $\max\{\lambda_1' - \lambda_1, 0\} < \varepsilon_2 < \lambda_1'$, where λ_1 is the first eigenvalue of K corresponding to BVP (4.1). If $\lambda_1' - \varepsilon_2 - h_1 > 0$, then for $u(t) \ge \lambda_1' - \varepsilon_2 - h_1$ or $u(t) \le 0$, $f(t, u(t)) \ge (\lambda_1' - \varepsilon_2)u(t)$; for $0 < u(t) < \lambda_1' - \varepsilon_2 - h_1$, $f(t, u(t)) \ge (\lambda_1' - \varepsilon_2)u(t) - (\lambda_1' - \varepsilon_2 - h_1)^2$. If $\lambda_1' - \varepsilon_2 - h_1 < 0$, then for $u(t) \ge 0$ or $u(t) \le \lambda_1' - \varepsilon_2 - h_1$, $f(t, u(t)) \ge (\lambda_1' - \varepsilon_2)u(t)$; for $\lambda_1' - \varepsilon_2 - h_1 < u(t) < 0$, $f(t, u(t)) \ge (\lambda_1' - \varepsilon_2)u(t) - (\lambda_1' - \varepsilon_2 - h_1)^2$. Thus

$$f(t, u) \ge (\lambda_1' - \varepsilon_2)u - (\lambda_1' - \varepsilon_2 - h_1)^2, \quad \forall t \in I, u \in C(I).$$

This means that (W_2) holds.
(iii) We select ε_3 that $0 < \varepsilon_3 < \lambda_1' - h_1$, $0 \le u(t) \le \lambda_1' - \varepsilon_3 - h_1$, $f(t, u(t)) \le (\lambda_1' - \varepsilon_3)u(t)$; for $-(\lambda_1' - \varepsilon_3) - h_1 \le u(t) < 0$, $(\lambda_1' - \varepsilon_3 - h_1)u(t) \le u^2(t) \le (-(\lambda_1' - \varepsilon_3) - h_1)u(t)$, and then $(\lambda_1' - \varepsilon_3)u(t) \le f(t, u(t)) \le -(\lambda_1' - \varepsilon_3)u(t)$. So,

$$|f(t, u(t))| \le (\lambda_1' - \varepsilon_3)|u(t)| \quad \forall t \in I, u \in C(I), ||u|| \le \lambda_1' - \varepsilon_3 - h_1.$$

Thus, (W_3) holds. By Theorem 3.1, BVP (4.1) has at least one nontrivial solution.

Example 4.2. Consider the boundary value problem

$$\begin{cases} -u''(t) = f(t, u(t)) = h_2 u(t) + u^{\frac{2}{3}}(t), \quad t \in I, \\ u(0) - u'(0) = 0, u(1) = u(\eta), \end{cases} \tag{4.2}$$

where $0 < h_2 < \lambda_1'$ and λ_1' is the first eigenvalue of T corresponding to BVP (4.2).

(i) Similar as Example 4.1, we can find that (H_1) holds.
(ii) For $u(t) \leq (\lambda_1' + 1 - h_2)^{-3}$, $f(t, u(t)) \geq (\lambda_1' + 1)u(t)$. Thus

$$f(t, u) \geq (\lambda_1' + 1)u, \quad \forall t \in I, u \in C(I), ||u|| \leq (\lambda_1' + 1 - h_2)^{-3}.$$

Then (W_4) holds.
(iii) We select ε_5 that $\max\{\lambda_1' - \lambda_1, 0\} < \varepsilon_5 < \lambda_1'$. If $\lambda_1' - \varepsilon_5 - h_2 > 0$, then for $u(t) \leq (\lambda_1' - \varepsilon_5 - h_2)^{-3}$, $f(t, u(t)) \geq (\lambda_1' - \varepsilon_5)u(t)$. If $\lambda_1' - \varepsilon_5 - h_2 < 0$, then for $u(t) \geq (\lambda_1' - \varepsilon_5 - h_2)^{-3}$, $f(t, u(t)) \geq (\lambda_1' - \varepsilon_5)u(t)$. So

$$f(t, u) \geq (\lambda_1' - \varepsilon_5)u, \quad \forall t \in I, u \in C(I), ||u|| \leq |(\lambda_1' - \varepsilon_5 - h_2)|^{-3}.$$

Then (W_5) holds.
(iv) We select ε_6 that $0 < \varepsilon_6 < \lambda_1' - h_2$. Then for $u(t) \geq (\lambda_1' - \varepsilon_6 - h_2)^{-3}$, $f(t, u(t)) \leq (\lambda_1' - \varepsilon_6)u(t)$; for $0 < u(t) < (\lambda_1' - \varepsilon_6 - h_2)^{-3}$, $f(t, u(t)) \leq (\lambda_1' - \varepsilon_6)u(t) + (\lambda_1' - \varepsilon_6 - h_2)^{-2}$; for $-((\lambda_1' - \varepsilon_6) - h_2)^{-3} \leq u(t) \leq 0$, $|f(t, u(t))| \leq (\lambda_1' - \varepsilon_6)|u(t)| + (\lambda_1' - \varepsilon_6 + h_2)^{-2}$; for $u(t) < (-(\lambda_1' - \varepsilon_6) - h_2)^{-3}$, $|f(t, u(t))| \leq (\lambda_1' - \varepsilon_6)|u(t)|$; So,

$$|f(t, u(t))| \leq (\lambda_1' - \varepsilon_6)|u(t)| + (\lambda_1' - \varepsilon_6 + h_2)^{-2} \quad \forall t \in I, u \in C(I).$$

Thus, (W_6) holds. By Theorem 3.4, BVP (4.2) has at least one nontrivial solution.

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References

[1] Y. Cui, Nontrivial solutions of singular superlinear m–point boundary value problems, Appl. Math. Comput. 187 (2007) 1256–1264.

[2] K. Deimling, Nonlinear Functional Analysis, Springer–Verlag, Berlin, 1985.

[3] G. Han, Y. Wu, Nontrivial solutions of singular two–point boundary value problems with sign–changing nonlinear terms, J. Math. Anal. Appl. 325 (2007) 1327–1338.

[4] M. G. Krein, M.A. Rutmann, Linear operators leaving invariant a cone in a Banach space, Amer. Math. Soc. Translations. 10 (1950) 1–128.

[5] Y. Li, Multiply sign–changing solutions for fourth–order nonlinear boundary value problems, Nonlinear Anal. 67 (2007) 601–608.

[6] Y. Li, On the existence and nonexistence of positive solutions for nonlinear Sturm–Liouville boundaryvalue problems, J. Math. Anal. Appl. 304 (2005) 74–86.

[7] Y. Li, Positive solutions of fourth–order boundary value problems with two parameters, J. Math. Anal. Appl. 281 (2003) 477–484.

[8] Y. Li, Positive solutions of second–order boundary value problems with sign–changing nonlinear terms, J. Math. Anal. Appl. 282 (2003) 232–240.

[9] B. Liu, L. Liu, Y. Wu, Existence of nontrivial periodic solutions for a nonlinear second order periodic boundary value problem, Nonlinear Anal. 72 (2010) 3337–3345.

[10] L. Liu, B. Liu, Y. Wu, Nontrivial solutions of m–point boundary value problems for nonlinear second–order differential equations with a sign–changing nonlinear term, J. Comput. Appl. Math. 224 (2009) 373–382.

[11] X. Liu, Nontrivial solutions of singular nonlinear m–point boundary value problems, J. Math. Anal. Appl. 284 (2003) 576–590.

[12] R. Ma, Existence of positive solutions of a four–order boundary value problem, Appl. Math. Comput. 168 (2005) 1219–1231.

[13] R. Ma, Existence of solutions of nonlinear m–point boundary value problem, J. Math. Anal. Appl. 256 (2001) 556–567.

[14] R. Ma, Existence theorems for a second order m–point boundary value problem, J. Math. Anal. Appl. 211 (1997) 545–555.

[15] R. Ma, Existence theorems for a second order three-point boundary value problem, J. Math. Anal. Appl. 212 (1997) 430–442.

[16] R. Ma, Positive solutions for second order three-point boundary value problems, Appl. Math. Lett. 14 (2001) 1–5.

[17] R. Ma, Positive solutions of a nonlinear three-point boundary value problems, Electron. J. Differential Equations 34 (1999) 1–8.

[18] R. Ma, H. Wang, On the existence of positive solutions of fourth–order ordinary differential equations, Appl. Anal. 59 (1995) 225–231.

[19] M. H. Protter, H. F. Weinberger, Maximum Principles in Differential Equations, Prentice Hall, New York, 1967.

[20] J. Sun, G. Zhang, Nontrivial solutions of singular sublinear Sturm–Liouville problems, J. Math. Anal. Appl. 326 (2007) 242–251.

[21] J. Sun, G. Zhang, Nontrivial solutions of singular superlinear Sturm–Liouville problem, J. Math. Anal. Appl. 313 (2006) 518–536.

[22] Z. Yang, Existence of nontrivial solutions for a nonlinear Sturm–Liouville problem with integral boundary conditions, Nonlinear Anal. 68 (2008) 216–225.

[23] X. Zhang, L. Liu, Nontrivial solutions for higher order multi–point boundary value problems, Comput. Math. Appl. 56 (2008) 861–873.