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Partial Averaging of Fuzzy Differential Equations with Maxima

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Abstract

In this paper, a scheme of partial averaging of fuzzy differential equations with maxima is considered.

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1 Introduction

The study of fuzzy differential equations (FDEs) forms a suitable setting for the mathematical modelling of real world problems in which uncertainty or vagueness pervades. Fuzzy differential equations were first formulated by Kaleva [4,5]. He used the concept of H-differentiability which was introduced by Puri and Ralescu [13], and obtained the existence and uniqueness theorem for a solution of FDE under the Lipschitz condition. Since then there appeared a lot of papers concerning the theory and applications of fuzzy differential equations, fuzzy dynamics and fuzzy differential inclusions [2,9,10,12].

In this paper, a scheme of partial averaging of fuzzy differential equations with maxima is considered that continues researches devoted to the fuzzy differential equations with delay [7,8].

2 Main Definitions

Let $conv(\mathbb{R}^n)$ be a family of all nonempty compact convex subsets of $\mathbb{R}^n$ with Hausdorff metric

$$h(A,B) = \max\{\max_{a\in A}\min_{b\in B}\|a-b\|,\ \max_{b\in B}\min_{a\in A}\|a-b\|\},$$

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where $\|\cdot\|$ denotes the usual Euclidean norm in $\mathbb{R}^n$. Denote by $|A| = h(A, 0)$.

Let E^n be a family of mappings $x : \mathbb{R}^n \to [0, 1]$ satisfying the following conditions:

1) x is normal, i.e., there exists $y_0 \in \mathbb{R}^n$ such that $x(y_0) = 1$;

2) x is fuzzy convex, i.e., $x(\lambda y + (1-\lambda)z) \geq \min\{x(y), x(z)\}$ whenever $y, z \in \mathbb{R}^n$ and $\lambda \in [0, 1]$;

3) x is upper semicontinuous, i.e., for any $y_0 \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists $\delta(y_0, \varepsilon) > 0$ such that $x(y) < x(y_0) + \varepsilon$ whenever $||y - y_0|| < \delta,\ y \in \mathbb{R}^n$;

4) the closure of the set $\{y \in \mathbb{R}^n\ :\ x(y) > 0\}$ is compact.

Let $\widehat{0}$ be a fuzzy mapping defined by $\widehat{0}(y) = \begin{cases} 0\ if\ y \neq 0, \\ 1\ if\ y = 0. \end{cases}$

Definition 2.1 (See [11])**.** The set $\{y \in \mathbb{R}^n : x(y) \geq \alpha\}$ is called an α – level $[x]^\alpha$ of a mapping $x \in \mathrm{E}^n$ for $\alpha \in (0, 1]$. A closure of the set $\{y \in \mathbb{R}^n : x(y) > 0\}$ is called a 0 - level $[x]^0$ of a mapping $x \in \mathrm{E}^\mathrm{n}$.

Define the metric $D : \mathrm{E}^n \times \mathrm{E}^n \to \mathbb{R}_+$ by the equation

$$D(x, y) = \sup_{\alpha \in [0,1]} h([x]^\alpha, [y]^\alpha).$$

Let I be an interval in $\mathbb{R}$.

Definition 2.2. A mapping $f : I \to \mathrm{E}^n$ is called weakly continuous at point $t_0 \in I$ if a multivalued mapping $f_\alpha(t) = [f(t)]^\alpha$ is continuous for any $\alpha \in [0, 1]$.

Definition 2.3 (See [11])**.** A mapping $f : I \times \mathrm{E}^\mathrm{n} \to \mathrm{E}^\mathrm{n}$ is called weakly continuous at point $(t_0, x_0) \in I \times \mathrm{E}^\mathrm{n}$ if for any $\alpha \in [0, 1]$ and $\varepsilon > 0$ there exists $\delta(\varepsilon, \alpha) > 0$ such that $h(f_\alpha(t, x), f_\alpha(t_0, x_0)) < \varepsilon$ for all $(t, x) \in I \times \mathrm{E}^\mathrm{n}$ satisfying the condition $|t - t_0| < \delta(\varepsilon, \alpha),\ h([x]^\alpha, [x_0]^\alpha) < \delta(\varepsilon, \alpha)$.

Definition 2.4 (See [11])**.** A mapping $f : I \to \mathrm{E}^n$ is called measurable on I if a multivalued mapping $f_\alpha(t)$ is Lebesgue measurable for any $\alpha \in [0, 1]$.

Definition 2.5 (See [11])**.** An element $g \in \mathrm{E}^n$ is called an integral of $f : I \to \mathrm{E}^n$ over I if $[g]^\alpha = (A) \int\limits_I f_\alpha(t)dt$ for any $\alpha \in (0, 1]$, where $(A) \int\limits_I f_\alpha(t)dt$ is the Aumann integral [1].

Definition 2.6 (See [11])**.** A mapping $f : I \to \mathrm{E}^n$ is called differentiable at point $t_0 \in I$ if the multivalued mapping $f_\alpha(t)$ is Hukuhara differentiable at point t_0 [3] for any $\alpha \in [0, 1]$ and the family $\{D_H f_\alpha(t_0)\ :\ \alpha \in [0, 1]\}$ defines a fuzzy number $f'(t_0) \in \mathrm{E}^n$ (which is called a fuzzy derivative of $f(t_0)$ at point t_0).

Consider a fuzzy differential equation with delay

$$x'(t) = f(t, x(t), x(\alpha(t))), \quad x(t_0) = x_0, \tag{2.1}$$

where $t \in I$ is time; $x \in S \subset \mathrm{E}^n$ is a phase variable; the initial conditions $t_0 \in I, x_0 \in S$; a fuzzy mapping $f : I \times S \times S \to \mathrm{E}^{\mathrm{n}}$; a delay function $\alpha(t) \in [t_0, t]$.

Definition 2.7. A fuzzy mapping $x : I_0 \to \mathrm{E}^{\mathrm{n}},\ \mathrm{t}_0 \in \mathrm{I}_0 \subset \mathrm{I}$, is called a solution of equation (2.1) if it is weakly continuous and for all $t \in I_0$ satisfies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s), x(\alpha(s)))ds.$$

Theorem 2.8 (See [8]). *Let $f(t, x, y)$ be a weakly continuous function in the neighborhood of the point (t_0, x_0, x_0) and satisfy the Lipschitz condition in x, y with constant λ. Then there exists a unique solution $x(t)$ of equation* (2.1) *for* $\mathrm{t} \in [t_0, t_0 + \sigma]$*, where σ is small enough.*

3 Main Results

Consider the fuzzy differential equation with maxima

$$x'(t) = \varepsilon f\left(t, x(t), \max_{\tau \in [\gamma(t), g(t)]} |x(\tau)|\right), \quad x(0) = x_0, \tag{3.1}$$

where $t \geq 0$ is time; $x \in S \subset \mathrm{E}^n$ is a phase variable; $\varepsilon > 0$ is a small parameter; the initial condition $x_0 \in S$; a fuzzy mapping $f : \mathbb{R}_+ \times S \times \mathbb{R}_+ \to \mathrm{E}^{\mathrm{n}}$; the functions $0 \leq \gamma(t) \leq g(t) \leq t$.

If there exists a fuzzy mapping $\overline{f}(t, x, z)$ such that for any $t, z \geq 0, x \in S \subset \mathrm{E}^{\mathrm{n}}$

$$\lim_{T \to \infty} \frac{1}{T} D\left(\int_{t}^{t+T} f(s, x, z)ds, \int_{t}^{t+T} \overline{f}(s, x, z)ds\right) = 0, \tag{3.2}$$

then in the correspondence to equation (3.1) we will set the following partially averaged equation

$$y'(t) = \varepsilon \overline{f}\left(t, y(t), \max_{\tau \in [\gamma(t), g(t)]} |y(\tau)|\right), \quad y(0) = x_0. \tag{3.3}$$

Theorem 3.1. *Let in the domain $Q = \{t, z \geq 0,\ x \in S \subset E^n\}$ the following hold:*

1. *the fuzzy mappings $f(t, x, z), \overline{f}(t, x, z)$ are uniformly bounded by M, weakly continuous in* t *and satisfy the Lipschitz condition in x, z with constant λ, i.e.,*

$$D(f(t, x, z), \widehat{0}) \leq M,\ D(\overline{f}(t, x, z), \widehat{0}) \leq M,$$
$$D\left(f(t, x, z), f(t, x', z')\right) \leq \lambda \left[D(x, x') + |z - z'|\right],$$
$$D\left(\overline{f}(t, x, z), \overline{f}(t, x', z')\right) \leq \lambda \left[D(x, x') + |z - z'|\right];$$

2. *the limit* (3.2) *exists uniformly with respect to* $t, z \geq 0$ *and* $x \in S$*;*
3. *the functions* $\gamma(t), g(t)$ *are uniformly continuous and* $0 \leq \gamma(t) \leq g(t) \leq t$*;*
4. *the solution* $y(t)$ *of equation* (3.3)*,* $y(0) = x_0 \in S' \subset S$ *belongs to the domain* S *together with a* ρ*-neighborhood for all* $t \geq 0$*.*

Then for any $\eta \in (0, \rho]$ *and* $L > 0$ *there exists* $\varepsilon^0(\eta, L) > 0$ *such that for all* $\varepsilon \in (0, \varepsilon^0]$ *and* $t \in [0, L\varepsilon^{-1}]$ *the following estimate holds:*

$$D\left(x(t), y(t)\right) \leq \eta, \tag{3.4}$$

where $x(t), y(t)$ *are solutions of equations* (3.1) *and* (3.3) *such that* $x(0) = y(0) = x_0$*.*

Proof. According to Definition 2.7, the solutions of equations (3.1) and (3.3) are weakly continuous fuzzy mappings that satisfy the integral equations

$$x(t) = x_0 + \varepsilon \int_0^t f\left(s, x(s), \max_{\tau \in [\gamma(s), g(s)]} |x(\tau)|\right) ds,$$

$$y(t) = x_0 + \varepsilon \int_0^t \overline{f}\left(s, y(s), \max_{\tau \in [\gamma(s), g(s)]} |y(\tau)|\right) ds.$$

Then

$$\begin{aligned}
D(x(t), y(t)) = D\Bigg(& x_0 + \varepsilon \int_0^t f\left(s, x(s), \max_{\tau \in [\gamma(s), g(s)]} |x(\tau)|\right) ds, \\
& x_0 + \varepsilon \int_0^t \overline{f}\left(s, y(s), \max_{\tau \in [\gamma(s), g(s)]} |y(\tau)|\right) ds \Bigg)
\end{aligned}$$

$$\begin{aligned}
&= \varepsilon D\left(\int_0^t f\left(s, x(s), \max_{\tau \in [\gamma(s), g(s)]} |x(\tau)|\right) ds, \int_0^t \overline{f}\left(s, y(s), \max_{\tau \in [\gamma(s), g(s)]} |y(\tau)|\right) ds\right) \\
&\leq \varepsilon D\left(\int_0^t f\left(s, x(s), \max_{\tau \in [\gamma(s), g(s)]} |x(\tau)|\right) ds, \int_0^t f\left(s, y(s), \max_{\tau \in [\gamma(s), g(s)]} |y(\tau)|\right) ds\right) \\
&+ \varepsilon D\left(\int_0^t f\left(s, y(s), \max_{\tau \in [\gamma(s), g(s)]} |y(\tau)|\right) ds, \int_0^t \overline{f}\left(s, y(s), \max_{\tau \in [\gamma(s), g(s)]} |y(\tau)|\right) ds\right) \\
&\leq \varepsilon \lambda \int_0^t \left[D(x(s), y(s)) + \left|\max_{\tau \in [\gamma(s), g(s)]} |x(\tau)| - \max_{\tau \in [\gamma(s), g(s)]} |y(\tau)|\right|\right] ds + \sigma(\varepsilon),
\end{aligned}$$

where

$$\sigma(\varepsilon) = \max_{t\in[0,L\varepsilon^{-1}]} \beta(t,\varepsilon),$$

$$\beta(t,\varepsilon) = \varepsilon D\left(\int_0^t f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds, \int_0^t \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds\right).$$

Let $\delta(t) = \max_{s\in[0,t]} D(x(s), y(s))$. As

$$\left|\max_{\tau\in[\gamma(s),g(s)]} |x(\tau)| - \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right|$$
$$\leq \max_{\tau\in[\gamma(s),g(s)]} ||x(\tau)| - |y(\tau)|| \leq \max_{\tau\in[\gamma(s),g(s)]} D(x(\tau), y(\tau)) \leq \delta(s),$$

we have

$$\delta(t) \leq 2\lambda\varepsilon \int_0^t \delta(s)ds + \sigma(\varepsilon). \tag{3.5}$$

Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = \dfrac{Li}{m\varepsilon}$, $i = \overline{0,m}$, $m \in \mathrm{N}$. Let us estimate $\beta(t,\varepsilon)$, using the properties of the metric D for $t \in [t_k, t_{k+1}]$:

$$\beta(t,\varepsilon)$$
$$= \varepsilon D\left(\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds + \int_{t_k}^{t} f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds,\right.$$
$$\left.\sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds + \int_{t_k}^{t} \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds\right)$$
$$\leq \varepsilon\left[\sum_{i=0}^{k-1} D\left(\int_{t_i}^{t_{i+1}} f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds, \int_{t_i}^{t_{i+1}} \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right)\right)\right.$$
$$\left.+ D\left(\int_{t_k}^{t} f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds, \int_{t_k}^{t} \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds\right)\right]$$
$$\leq \varepsilon \sum_{i=0}^{k-1} \left[\int_{t_i}^{t_{i+1}} D\left(f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right), f\left(s, y(t_i), \max_{\tau\in[\gamma(t_i),g(t_i)]} |y(\tau)|\right)\right) ds\right.$$

$$
\begin{aligned}
&+\int_{t_i}^{t_{i+1}} D\left(\overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]}|y(\tau)|\right), \overline{f}\left(s, y(t_i), \max_{\tau\in[\gamma(t_i),g(t_i)]}|y(\tau)|\right)\right) ds \\
&+ D\left(\int_{t_i}^{t_{i+1}} f\left(s, y(t_i), \max_{\tau\in[\gamma(t_i),g(t_i)]}|y(\tau)|\right) ds, \int_{t_i}^{t_{i+1}} \overline{f}\left(s, y(t_i), \max_{\tau\in[\gamma(t_i),g(t_i)]}|y(\tau)|\right) ds\right)\Bigg] \\
&+\varepsilon D\left(\int_{t_k}^{t} f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]}|y(\tau)|\right) ds, \int_{t_k}^{t} \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]}|y(\tau)|\right) ds\right) \\
&\le \varepsilon \sum_{i=0}^{k-1}\left[2\lambda \int_{t_i}^{t_{i+1}}\left[D(y(s), y(t_i))) + \left|\max_{\tau\in[\gamma(s),g(s)]}|y(\tau)| - \max_{\tau\in[\gamma(t_i),g(t_i)]}|y(\tau)|\right|\right] ds\right. \\
&+ D\left(\int_{t_i}^{t_{i+1}} f\left(s, y(t_i), \max_{\tau\in[\gamma(t_i),g(t_i)]}|y(\tau)|\right) ds, \int_{t_i}^{t_{i+1}} \overline{f}\left(s, y(t_i), \max_{\tau\in[\gamma(t_i),g(t_i)]}|y(\tau)|\right) ds\right)\Bigg] \\
&+\varepsilon D\left(\int_{t_k}^{t} f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]}|y(\tau)|\right) ds, \int_{t_k}^{t} \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]}|y(\tau)|\right) ds\right).
\end{aligned}
\tag{3.6}
$$

Let us estimate every summand in (3.6) separately:

$$
\begin{aligned}
&\varepsilon \int_{t_i}^{t_{i+1}} D(y(s), y(t_i)) ds \\
&= \varepsilon \int_{t_i}^{t_{i+1}} D\left(y(t_i) + \varepsilon \int_{t_i}^{s} \overline{f}\left(\tau, y(\tau), \max_{\chi\in[\gamma(\tau),g(\tau)]}|y(\chi)|\right) d\tau,\ y(t_i)\right) ds \\
&= \varepsilon^2 \int_{t_i}^{t_{i+1}} D\left(\int_{t_i}^{s} \overline{f}\left(\tau, y(\tau), \max_{\chi\in[\gamma(\tau),g(\tau)]}|y(\chi)|\right) d\tau, \widehat{0}\right) ds \\
&\le \varepsilon^2 \int_{t_i}^{t_{i+1}}\int_{t_i}^{s} D\left(\overline{f}\left(\tau, y(\tau), \max_{\chi\in[\gamma(\tau),g(\tau)]}|y(\chi)|\right), \widehat{0}\right) d\tau ds \\
&\le \frac{\varepsilon^2 M}{2}\left(\frac{L}{\varepsilon m}\right)^2 = \frac{ML^2}{2m^2}.
\end{aligned}
\tag{3.7}
$$

Using the properties of the modulus of continuity

$$
\omega(\psi, \sigma) = \sup\{|\psi(\tau_1) - \psi(\tau_2)|,\ |\tau_1 - \tau_2| < \sigma\}
$$

of functions $\gamma(t), g(t)$, we get

$$
\begin{aligned}
&\varepsilon \int_{t_i}^{t_{i+1}} \left| \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)| - \max_{\tau\in[\gamma(t_i),g(t_i)]} |y(\tau)| \right| ds \\
&\leq \varepsilon \int_{t_i}^{t_{i+1}} \varepsilon M \max\left\{\omega\left(\gamma, \frac{L}{\varepsilon m}\right), \omega\left(g, \frac{L}{\varepsilon m}\right)\right\} ds \\
&= \frac{\varepsilon LM}{m} \max\left\{\omega\left(\gamma, \frac{L}{\varepsilon m}\right), \omega\left(g, \frac{L}{\varepsilon m}\right)\right\} \\
&\leq \frac{\varepsilon LM}{m} \left(1 + \frac{1}{\varepsilon m}\right) \max\left\{\omega\left(\gamma, L\right), \omega\left(g, L\right)\right\}. \qquad (3.8)
\end{aligned}
$$

From the uniform convergence to the average in (3.2) it follows that there exists a monotone decreasing function $\Theta(t) \to 0$ as $t \to \infty$ such that

$$
\varepsilon D\left(\int_{t_i}^{t_{i+1}} f\left(s, y(t_i), \max_{\tau\in[\gamma(t_i),g(t_i)]} |y(\tau)|\right) ds, \int_{t_i}^{t_{i+1}} \overline{f}\left(s, y(t_i) \max_{\tau\in[\gamma(t_i),g(t_i)]} |y(\tau)|\right) ds\right) \leq \frac{L}{m}\Theta\left(\frac{L}{\varepsilon m}\right). \qquad (3.9)
$$

As the mappings f and $\overline{f}$ are uniformly bounded by constant M, we have

$$
\begin{aligned}
&\varepsilon D\left(\int_{t_k}^{t} f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds, \int_{t_k}^{t} \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right) ds\right) \\
&\leq \varepsilon \int_{t_k}^{t} D\left(f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right), \overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right)\right) ds \\
&\leq \varepsilon \int_{t_k}^{t} D\left(f\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right), \hat{0}\right) ds \\
&\quad + \varepsilon \int_{t_k}^{t} D\left(\overline{f}\left(s, y(s), \max_{\tau\in[\gamma(s),g(s)]} |y(\tau)|\right), \hat{0}\right) ds \leq 2M\varepsilon \int_{t_k}^{t} ds = 2\frac{ML}{m}. \qquad (3.10)
\end{aligned}
$$

Hence using (3.6)–(3.10), we have

$$
\beta(t, \varepsilon) \leq \frac{\lambda M L^2}{m} + 2\lambda LM\left(\frac{1}{m} + \varepsilon\right) \max\{\omega(\gamma, L), \omega(g, L)\}
$$

$$+\frac{2ML}{m}+L\Theta\left(\frac{L}{\varepsilon m}\right). \quad (3.11)$$

Let us choose m_0 to satisfy the inequality

$$\frac{\lambda ML^2}{m_0}+\frac{2\lambda LM}{m_0}\max\{\omega(\gamma,L),\omega(g,L)\}+\frac{2ML}{m_0}<\frac{\eta}{2e^{2\lambda L}} \quad (3.12)$$

and then choose ε_0 such that

$$2\lambda LM\max\{\omega(\gamma,L),\omega(g,L)\}\varepsilon+L\Theta\left(\frac{L}{\varepsilon m_0}\right)<\frac{\eta}{2e^{2\lambda L}}. \quad (3.13)$$

From (3.11)–(3.13) and (3.5) using the Gronwall–Bellman lemma, we get the statement of the theorem. □

Remark 3.2. From the definition of $|x(t)|$ it follows that

$$|x(t)|=D(x(t),\hat{0})=\sup_{\alpha\in[0,1]}h([x(t)]^\alpha,\{0\})=h([x(t)]^0,\{0\})=|[x(t)]^0|.$$

So the fuzzy differential equation (3.1) is equivalent to the following

$$x'(t)=\varepsilon f\left(t,x(t),\max_{\tau\in[\gamma(t),g(t)]}|[x(\tau)]^0|\right),\quad x\,(0)=x_0. \quad (3.14)$$

It is easy to show that instead of 0-level set of the fuzzy mapping $x(t)$ any α-level set, $\alpha\in[0,1]$, can be taken. The substantiation of the averaging scheme will be almost the same.

4 Conclusion

In this paper the substantiation of one scheme of averaging for fuzzy differential equations with maxima is considered. These results generalize the results of [6] for differential equations with Hukuhara derivative with maxima.

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