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Robust Exponential Stability of Neutral Systems with Nondifferentiable Interval Time-Varying Delays and Polytopic Uncertainties

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Abstract

In this paper, the problem of exponential stability for a class of polytopic neutral systems with nondifferentiable interval time-varying delays is studied. By using an improved Lyapunov–Krasovskii parameter-dependent functional and linear matrix inequality (LMI) technique, new delay-dependent sufficient conditions for the exponential stability of the systems are first established in terms of linear matrix inequalities (LMIs) conditions which allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. Numerical examples are also given to show the effectiveness of the obtain results.

AMS Subject Classifications: 34D20, 37C75, 93D20.
Keywords: Neutral systems, interval delay, polytopic systems, exponential stability.

1 Introduction

Stability analysis of linear delayed and neutral systems has received much attention in the past decades, e.g., see [4,5,8–10,13,17,18] and the references therein. Theoretically, the linear neutral system with time delays is much more complicated, especially for the case where the system matrices belong to some convex polytope [3, 6, 12, 14, 19]. By using parameter-dependent Lyapunov functionals, some less conservative results for asymptotic stability of uncertain polytopic delay systems have been proposed in [3, 12, 15, 20] via LMIs. Although these results improve the estimate of asymptotic stability domain, some conservatism still remain since common matrix variable required

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to satisfy the whole sets of LMIs and the time delays are assumed to be constants or differentiable and upper bounded. To the best of our knowledge, so far, no result on the stability for linear neutral interval state-delayed systems with polytope uncertainties is available in the literature, which is still open and remains unsolved. This motivates our present investigation.

In this paper, we develop the exponential stability problem for polytopic neutral systems with nondifferentiable interval time-varying delays. The novel feature of the results obtained in this paper is twofold. First, the system considered in this paper is neutral convex polytopic uncertain subjected to interval, nondifferentiable state delay. Second, by employing an improved parameter-dependent Lyapunov–Krasovskii functional and linear matrix inequality technology, delay-dependent sufficient conditions for the exponential stability of the system are first obtained in terms of LMI conditions. The approach also allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution.

The paper is organized as follows: Section 2 presents notations, definitions and some well-known technical propositions needed for the proof of the main result. Delay-dependent exponential stability conditions of the system is presented in Section 3. Numerical examples to show effectiveness of our conditions are given in Section 4. The paper ends with conclusions and cited references.

2 Preliminaries

The following notations will be used throughout this paper. $\mathbb{R}^+$ denotes the set of all nonnegative real numbers; $\mathbb{R}^n$ denotes the $n-$dimensional Euclidean space with the norm $\|.\|$ and scalar product $x^\mathsf{T}y$ of two vectors x, y; $\lambda_{\max}(A)$ ($\lambda_{\min}(A)$, resp.) denotes the maximal (the minimal, resp.) number of the real part of eigenvalues of A; A^T denotes the transpose of the matrix A and I denote the identity matrix; $\mathbf{0}_n$ denote the zero matrix in $\mathbb{R}^{n\times n}$. A matrix $Q \geq 0$ ($Q > 0$, resp.) means that Q is semi-positive definite (positive definite, resp.) i.e., $\langle Qx, x\rangle \geq 0$ for all $x \in \mathbb{R}^n$ (resp. $\langle Qx, x\rangle > 0$ for all $x \neq 0$); $A \geq B$ means $A - B \geq 0$; $\mathcal{C}^1([a, b], \mathbb{R}^n)$ denotes the set of all continuously differentiable functions on $[a, b]$. The segment of the trajectory $x(t)$ is denoted by $x_t = \{x(t+s) : s \in [-\bar{h}, 0]\}$. Consider a polytopic neutral system with interval time-varying delays of the form

$$\begin{cases} \dot{x}(t) - D(\xi)\dot{x}(t-\tau(t)) = A_0(\xi)x(t) + A_1(\xi)x(t-h(t)), \quad t \geq 0, \\ x(t) = \phi(t),\ t \in [-\bar{h}, 0], \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state; time-varying delays $h(t), \tau(t)$ satisfy $0 \leq h_m \leq h(t) \leq h_M$; $0 \leq \tau(t) \leq \tau_M$, $\dot{\tau}(t) \leq \mu < 1$ and $\bar{h} = \max\{\tau_M, h_M\}$. The state space data are subject to uncertainties and belong to the polytope Ω given by

$$\left\{[A_0, A_1, D](\xi) := \sum_{i=1}^{p} \xi_i [A_{0i}, A_{1i}, D_i],\ \xi_i \geq 0, \sum_{i=1}^{p} \xi_i = 1\right\},$$

where $A_{0i}, A_{1i}, D_i, i = 1, \ldots, p$, are given constant matrices with appropriate dimensions; $\phi \in \mathcal{C}^1([-\bar{h}, 0], \mathbb{R}^n)$ is initial function with its norm

$$\|\phi\| = \sup_{-\bar{h} \leq s \leq 0} \sqrt{\|\phi(s)\|^2 + \|\dot{\phi}(s)\|^2}.$$

Definition 2.1. [4,5] For given $\alpha > 0$, system (2.1) is said to be α-exponentially stable if there exists a constant $\gamma \geq 1$ such that every solution $x(t, \phi)$ of the system satisfies the following condition

$$\|x(t, \phi)\| \leq \gamma \|\phi\| e^{-\alpha t}, \quad \forall t \geq 0.$$

The following well-known inequality will be used in the proof of our results.

Proposition 2.2. *For any symmetric positive definite matrix W, scalar $\nu > 0$ and vector function $w : [0, \nu] \to \mathbb{R}^n$ such that the concerned integrals are well defined, then*

$$\left[\int_0^{\nu} w(s) ds\right]^{\mathsf{T}} W \left[\int_0^{\nu} w(s) ds\right] \leq \nu \int_0^{\nu} w^{\mathsf{T}}(s) W w(s) ds.$$

3 Main Results

Let $U_{ki}, k = 1, \ldots, 6, i = 1, \ldots, p$, and M be $n \times n$ matrices; $P_i, Q_i, R_i, S_i, T_i, Z_i$, $i = 1, \ldots, p$, be symmetric positive definite matrices and constant $\alpha > 0$, we denote $\delta = 1 - \mu$ and

$$\Xi_i(P_j, Q_j, R_j, S_j, T_j, Z_j, \mathcal{U}_j) =$$
$$= \begin{bmatrix} \Xi_{11} & A_{0i}^{\mathsf{T}} U_{2j} & \Xi_{13} & A_{0i}^{\mathsf{T}} U_{4j} & \Xi_{15} & \Xi_{16} \\ * & -\delta e^{-2\alpha\tau_M} R_j & U_{2j}^{\mathsf{T}} A_{1i} & 0 & -U_{2j}^{\mathsf{T}} & U_{2j}^{\mathsf{T}} D_i \\ * & * & \Xi_{33} & Z_j + A_{1i}^{\mathsf{T}} U_{4j} & \Xi_{35} & \Xi_{36} \\ * & * & * & -e^{-2\alpha h_m} Q_j - Z_j & -U_{4j}^{\mathsf{T}} & U_{4j}^{\mathsf{T}} D_i \\ * & * & * & * & \Xi_{55} & U_{5j}^{\mathsf{T}} D_i - U_{6j} \\ * & * & * & * & * & \Xi_{66} \end{bmatrix},$$

where

$$\Xi_{11} = A_{0i}^{\mathsf{T}}(P_j + U_{1j}) + (P_j + U_{1j}^{\mathsf{T}}) A_{0i} + 2\alpha P_j + Q_j + R_j - T_j;$$
$$\Xi_{13} = P_j A_{1i} + U_{1j}^{\mathsf{T}} A_{1i} + A_{0i}^{\mathsf{T}} U_{3j} + T_j;$$

$$\Xi_{15} = -U_{1j}^{\mathsf{T}} + A_{0i}^{\mathsf{T}} U_{5j};$$
$$\Xi_{16} = P_j D_i + U_{1j}^{\mathsf{T}} D_i + A_{0i}^{\mathsf{T}} U_{6j};$$
$$\Xi_{33} = -T_j - Z_j + A_{1i}^{\mathsf{T}} U_{3j} + U_{3j}^{\mathsf{T}} A_{1i};$$
$$\Xi_{35} = -U_{3j}^{\mathsf{T}} + A_{1i}^{\mathsf{T}} U_{5j};$$
$$\Xi_{36} = U_{3j}^{\mathsf{T}} D_i + A_{1i}^{\mathsf{T}} U_{6j};$$
$$\Xi_{55} = -U_{5j} - U_{5j}^{\mathsf{T}} + S_j + h_M^2 e^{2\alpha h_M} T_j + (h_M - h_m)^2 e^{2\alpha h_M} Z_j;$$
$$\Xi_{66} = -\delta e^{-2\alpha\tau_M} S_j + D_i^{\mathsf{T}} U_{6j} + U_{6j}^{\mathsf{T}} D_i;$$
$$\mathbb{M} = \operatorname{diag}\{M, \mathbf{0}_{5n}\};$$
$$\lambda_1 = \min_{1\le j\le p} \lambda_{\min}(P_j), \quad \lambda_P = \max_{1\le j\le p} \lambda_{\max}(P_j),$$
$$\lambda_Q = \max_j \lambda_{\max}(Q_j), \quad \lambda_R = \max_j \lambda_{\max}(R_j),$$
$$\lambda_S = \max_j \lambda_{\max}(S_j), \quad \lambda_T = \max_j \lambda_{\max}(T_j), \quad \lambda_Z = \max_j \lambda_{\max}(Z_j),$$
$$\lambda_2 = \lambda_P + h_m\lambda_Q + \tau_M(\lambda_R + \lambda_S) + \frac{1}{2} e^{2\alpha h_M} \left(h_M^3 \lambda_T + (h_M - h_m)^2 (h_M + h_m)\lambda_Z\right).$$

The following theorem presents the α-exponential stability of system (2.1).

Theorem 3.1. *Given $\alpha > 0$. System (2.1) is α-exponentially stable if there exist matrices $U_{1i}, U_{2i}, U_{3i}, U_{4i}, U_{5i}, U_{6i}, i = 1, \dots, p$, a symmetric semi-positive definite matrix M, symmetric positive definite matrices $P_i, Q_i, R_i, S_i, T_i, Z_i, i = 1, \dots, p$, such that the following linear matrix inequalities hold:*

$$\Xi_i(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) \le -\mathbb{M}, \quad i = 1, \dots, p; \tag{3.1}$$
$$\Xi_i(P_j, Q_j, R_j, S_j, T_j, Z_j, \mathcal{U}_j) + \Xi_j(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) \le \frac{2}{p-1}\mathbb{M}, \tag{3.2}$$
$$i = 1, \dots, p-1, \; j = i+1, \dots, p.$$

Proof. We denote $[P, Q, R, S, T, Z](\xi) = \sum_{j=1}^p \xi_j [P_j, Q_j, R_j, S_j, T_j, Z_j]$ and $U_k(\xi) = \sum_{i=1}^p \xi_i U_{ki}$, $k = 1, \dots, 6$. Consider the following Lyapunov–Krasovskii functional

$$V(t, x_t) = \sum_{i=1}^6 V_k, \tag{3.3}$$

where,

$$V_1 = x^{\mathsf{T}}(t) P(\xi) x(t),$$

$$
\begin{aligned}
V_2 &= \int_{t-h_m}^{t} e^{2\alpha(s-t)} x^{\mathsf{T}}(s) Q(\xi) x(s) ds \\
V_3 &= \int_{t-\tau(t)}^{t} e^{2\alpha(s-t)} x^{\mathsf{T}}(s) R(\xi) x(s) ds, \\
V_4 &= \int_{t-\tau(t)}^{t} e^{2\alpha(s-t)} \dot{x}^{\mathsf{T}}(s) S(\xi) \dot{x}(s) ds, \\
V_5 &= h_M \int_{t-h_M}^{t} \int_{s}^{t} e^{2\alpha(\theta-t+h_M)} \dot{x}^{\mathsf{T}}(\theta) T(\xi) \dot{x}(\theta) d\theta ds, \\
V_6 &= (h_M - h_m) \int_{t-h_M}^{t-h_m} \int_{s}^{t} e^{2\alpha(\theta-t+h_M)} \dot{x}^{\mathsf{T}}(\theta) Z(\xi) \dot{x}(\theta) d\theta ds.
\end{aligned}
$$

It is easy to verify from (3.3) that

$$
\lambda_1 \|x(t)\|^2 \le V(t, x_t) \le \lambda_2 \|x_t\|^2, \quad t \in \mathbb{R}^+. \tag{3.4}
$$

Taking derivative of V_1 along trajectories of system (2.1) we have

$$
\begin{aligned}
\dot{V}_1 &= 2x^{\mathsf{T}}(t) P(\xi) \dot{x}(t) \\
&= x^{\mathsf{T}}(t) \left[P(\xi) A_0(\xi) + A_0(\xi)^{\mathsf{T}} P(\xi) \right] x(t) \\
&\quad + 2x^{\mathsf{T}}(t) P(\xi) \left[A_1(\xi) x(t - h(t)) + D(\xi) \dot{x}(t - \tau(t)) \right].
\end{aligned} \tag{3.5}
$$

Next, taking derivatives of $V_k, k = 2, \ldots, 6$, along trajectories of system (2.1) we obtain

$$
\begin{aligned}
\dot{V}_2 &= x^{\mathsf{T}}(t) Q(\xi) x(t) - e^{-2\alpha h_m} x^{\mathsf{T}}(t - h_m) Q(\xi) x(t - h_m) - 2\alpha V_2; \\
\dot{V}_3 &= x^{\mathsf{T}}(t) R(\xi) x(t) - (1 - \dot{\tau}(t)) e^{-2\alpha\tau(t)} x^{\mathsf{T}}(t - \tau(t)) R(\xi) x(t - \tau(t)) - 2\alpha V_3 \\
&\le x^{\mathsf{T}}(t) R(\xi) x(t) - (1 - \mu) e^{-2\alpha\tau_M} x^{\mathsf{T}}(t - \tau(t)) R(\xi) x(t - \tau(t)) - 2\alpha V_3; \\
\dot{V}_4 &= \dot{x}^{\mathsf{T}}(t) S(\xi) \dot{x}(t) - (1 - \dot{\tau}(t)) e^{-2\alpha\tau(t)} \dot{x}^{\mathsf{T}}(t - \tau) S(\xi) \dot{x}(t - \tau) - 2\alpha V_4 \\
&\le \dot{x}^{\mathsf{T}}(t) S(\xi) \dot{x}(t) - (1 - \mu) e^{-2\alpha\tau_M} \dot{x}^{\mathsf{T}}(t - \tau(t)) S(\xi) \dot{x}(t - \tau(t)) - 2\alpha V_4;
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\dot{V}_5 &= h_M^2 e^{2\alpha h_M} \dot{x}^{\mathsf{T}}(t) T(\xi) \dot{x}(t) \\
&\quad - h_M \int_{t-h_M}^{t} e^{2\alpha(s-t+h_M)} \dot{x}^{\mathsf{T}}(s) T(\xi) \dot{x}(s) ds - 2\alpha V_5 \\
&\le h_M^2 e^{2\alpha h_M} \dot{x}^{\mathsf{T}}(t) T(\xi) \dot{x}(t) - h_M \int_{t-h_M}^{t} \dot{x}^{\mathsf{T}}(s) T(\xi) \dot{x}(s) ds - 2\alpha V_5; \\
\dot{V}_6 &= (h_M - h_m)^2 e^{2\alpha h_m} \dot{x}^{\mathsf{T}}(t) Z(\xi) \dot{x}(t) \\
&\quad - (h_M - h_m) \int_{t-h_M}^{t-h_m} e^{2\alpha(s-t+h_M)} \dot{x}^{\mathsf{T}}(s) Z(\xi) \dot{x}(s) ds - 2\alpha V_6 \\
&\le (h_M - h_m)^2 e^{2\alpha h_m} \dot{x}^{\mathsf{T}}(t) Z(\xi) \dot{x}(t) \\
&\quad - (h_M - h_m) \int_{t-h_M}^{t-h_m} \dot{x}^{\mathsf{T}}(s) Z(\xi) \dot{x}(s) ds - 2\alpha V_6.
\end{aligned} \tag{3.7}
$$

Applying Proposition 2.2 and the Leibniz–Newton formula, we have

$$\begin{aligned}
-h_M \int_{t-h_M}^{t} \dot{x}^{\mathsf{T}}(s) T(\xi) \dot{x}(s) ds &\le -h(t) \int_{t-h(t)}^{t} \dot{x}^{\mathsf{T}}(s) T(\xi) \dot{x}(s) ds \\
&\le -\left[\int_{t-h(t)}^{t} \dot{x}(s) ds\right]^{\mathsf{T}} T(\xi) \left[\int_{t-h(t)}^{t} \dot{x}(s) ds\right] \\
&\le -\left[x(t) - x(t-h(t))\right]^{\mathsf{T}} T(\xi) \left[x(t) - x(t-h(t))\right];
\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}
-(h_M - h_m) &\int_{t-h_M}^{t-h_m} \dot{x}^{\mathsf{T}}(s) Z(\xi) \dot{x}(s) ds \\
&\le -(h(t) - h_m) \int_{t-h(t)}^{t-h_m} \dot{x}^{\mathsf{T}}(s) Z(\xi) \dot{x}(s) ds \\
&\le -\left[\int_{t-h(t)}^{t-h_m} \dot{x}(s) ds\right]^{\mathsf{T}} Z(\xi) \left[\int_{t-h(t)}^{t-h_m} \dot{x}(s) ds\right] \\
&\le -\left[x(t-h_m) - x(t-h(t))\right]^{\mathsf{T}} Z(\xi) \left[x(t-h_m) - x(t-h(t))\right].
\end{aligned} \tag{3.9}$$

By using the identity

$$-\dot{x}(t) + D(\xi)\dot{x}(t-\tau(t)) + A_0(\xi)x(t) + A_1(\xi)x(t-h(t)) = 0,$$

we have

$$\begin{aligned}
2\Big[&x^{\mathsf{T}}(t)U_1(\xi)^{\mathsf{T}} + x^{\mathsf{T}}(t-\tau)U_2(\xi)^{\mathsf{T}} + x^{\mathsf{T}}(t-h(t))U_3(\xi)^{\mathsf{T}} \\
&+ x^{\mathsf{T}}(t-h_m)U_4(\xi)^{\mathsf{T}} + \dot{x}^{\mathsf{T}}(t)U_5(\xi)^{\mathsf{T}} + \dot{x}^{\mathsf{T}}(t-\tau(t))U_6(\xi)^{\mathsf{T}}\Big] \\
&\times \Big[-\dot{x}(t) + D(\xi)\dot{x}(t-\tau(t)) + A_0(\xi)x(t) + A_1(\xi)x(t-h(t))\Big] = 0.
\end{aligned} \tag{3.10}$$

Combining from (3.5)–(3.10), we have

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \le \eta^{\mathsf{T}}(t)\Xi(\xi)\eta(t), \tag{3.11}$$

where,

$$\eta^{\mathsf{T}}(t) = \begin{bmatrix} x^{\mathsf{T}}(t) & x^{\mathsf{T}}(t-\tau(t)) & x^{\mathsf{T}}(t-h(t)) & x^{\mathsf{T}}(t-h_m) & \dot{x}^{\mathsf{T}}(t) & \dot{x}^{\mathsf{T}}(t-\tau(t)) \end{bmatrix},$$

and

$\Xi(\xi) = \left[\Xi_{ik}(\xi)\right], i, k = 1, \ldots, 6,$ be symmetric matrix,

$$\begin{aligned}
\Xi_{11}(\xi) = &\left[A_0(\xi) + \alpha I\right]^{\mathsf{T}} P(\xi) + P(\xi)\left[A_0(\xi) + \alpha I\right] \\
&+ A_0(\xi)^{\mathsf{T}} U_1(\xi) + U_1(\xi)^{\mathsf{T}} A_0(\xi) + Q(\xi) + R(\xi) - T(\xi);
\end{aligned}$$

$$\begin{aligned}
&\Xi_{12}(\xi) = A_0(\xi)^{\mathsf{T}} U_2(\xi);\\
&\Xi_{13}(\xi) = P(\xi)A_1(\xi) + U_1(\xi)^{\mathsf{T}} A_1(\xi) + A_0(\xi)^{\mathsf{T}} U_3(\xi) + T(\xi);\\
&\Xi_{14}(\xi) = A_0(\xi)^{\mathsf{T}} U_4(\xi); \quad \Xi_{15}(\xi) = -U_1(\xi)^{\mathsf{T}} + A_0(\xi)^{\mathsf{T}} U_5(\xi);\\
&\Xi_{16}(\xi) = P(\xi)D(\xi) + U_1(\xi)^{\mathsf{T}} D(\xi) + A_0(\xi)^{\mathsf{T}} U_6(\xi);\\
&\Xi_{22}(\xi) = -(1-\mu)e^{-2\alpha\tau_M} R(\xi); \quad \Xi_{23}(\xi) = U_2(\xi)^{\mathsf{T}} A_1(\xi);\\
&\Xi_{24}(\xi) = 0; \quad \Xi_{25}(\xi) = -U_2(\xi)^{\mathsf{T}}; \quad \Xi_{26}(\xi) = U_2(\xi)^{\mathsf{T}} D(\xi);\\
&\Xi_{33}(\xi) = -T(\xi) - Z(\xi) + A_1(\xi)^{\mathsf{T}} U_3(\xi) + U_3(\xi)^{\mathsf{T}} A_1(\xi);\\
&\Xi_{34}(\xi) = Z(\xi) + A_1(\xi)^{\mathsf{T}} U_4(\xi); \quad \Xi_{35}(\xi) = -U_3(\xi)^{\mathsf{T}} + A_1(\xi)^{\mathsf{T}} U_5(\xi);\\
&\Xi_{36}(\xi) = U_3(\xi)^{\mathsf{T}} D(\xi) + A_1(\xi)^{\mathsf{T}} U_6(\xi);\\
&\Xi_{44}(\xi) = -e^{-2\alpha h_m} Q(\xi) - Z(\xi);\\
&\Xi_{45}(\xi) = -U_4(\xi)^{\mathsf{T}}; \quad \Xi_{46}(\xi) = U_4(\xi)^{\mathsf{T}} D(\xi);\\
&\Xi_{55}(\xi) = S(\xi) + h_M^2 e^{2\alpha h_M} T(\xi) + (h_M - h_m)^2 e^{2\alpha h_M} Z(\xi) - U_5(\xi) - U_5(\xi)^{\mathsf{T}};\\
&\Xi_{56}(\xi) = U_5(\xi)^{\mathsf{T}} D(\xi) - U_6(\xi);\\
&\Xi_{66}(\xi) = -(1-\mu)e^{-2\alpha\tau_M} S(\xi) + U_6(\xi)^{\mathsf{T}} D(\xi) + D(\xi)^{\mathsf{T}} U_6(\xi).
\end{aligned}$$

Using the property $\sum_{i=1}^{p} \xi_i = 1$, we have

$$\begin{aligned}
\dot{V}(t, x_t) + 2\alpha V(t, x_t) \le \eta^{\mathsf{T}}(t)\Big[&\sum_{i=1}^{p} \xi_i^2 \Xi_i(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i)\\
&+ \sum_{i=1}^{p-1}\sum_{j=i+1}^{p} \xi_i\xi_j(\Xi_i(P_j, Q_j, R_j, S_j, T_j, Z_j, \mathcal{U}_j)\\
&+ \Xi_j(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i))\Big]\eta(t).
\end{aligned} \tag{3.12}$$

Therefore, it follows from conditions (3.1) and (3.2) that

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \le \eta^{\mathsf{T}}(t)\left(-\sum_{i=1}^{p} \xi_i^2 + \frac{2}{p-1}\sum_{i=1}^{p-1}\sum_{j=i+1}^{p} \xi_i\xi_j\right)\mathbb{M}\eta(t). \tag{3.13}$$

Observe that,

$$(p-1)\sum_{i=1}^{p} \xi_i^2 - 2\sum_{i=1}^{p-1}\sum_{j=i+1}^{p} \xi_i\xi_j = \sum_{i=1}^{p-1}\sum_{j=i+1}^{p} (\xi_i - \xi_j)^2 \ge 0,$$

then from (3.13) we have

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \le -\frac{1}{p-1}\sum_{i=1}^{p-1}\sum_{j=i+1}^{p} (\xi_i - \xi_j)^2 \eta^{\mathsf{T}}(t)\mathbb{M}\eta(t) \le 0, \ t \ge 0.$$

which implies $V(t, x_t) \leq V(0, x_0)e^{-2\alpha t}, t \geq 0$. Taking (3.4) into account, we obtain

$$\|x(t,\phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|e^{-\alpha t}, \quad t \geq 0$$

which concludes the proof. □

Remark 3.2. Theorem 3.1 gives conditions in terms of linear matrix inequalities for the α-exponential stability problem for each given decay rate $\alpha > 0$ as considered in [4–6, 12, 15]. However, the decay rate α for the global exponential stability problem can be determined by LMIs and a nonlinear scalar equation as stated in the following theorem.

Theorem 3.3. *Assume that, for system* (2.1)*, there exist matrices* $U_{1i}, U_{2i}, U_{3i}, U_{4i}, U_{5i}, U_{6i}, i = 1, \ldots, p$*, a symmetric semi-positive definite matrix* M*, symmetric positive definite matrices* $P_i, Q_i, R_i, S_i, T_i, Z_i, i = 1, \ldots, p$*, and a positive number* λ_0 *satisfy the following linear matrix inequalities:*

$$\Phi_i(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) \leq -\mathbb{M} - \lambda_0 I, \quad i = 1, \ldots, p; \tag{3.14}$$

$$\Phi_i(P_j, Q_j, R_j, S_j, T_j, Z_j, \mathcal{U}_j) + \Phi_j(P_i, Q_i, R_i, S_i, T_i, Z_i, \mathcal{U}_i) \leq \frac{2}{p-1}\mathbb{M}, \tag{3.15}$$

$$i = 1, \ldots, p-1,\ j = i+1, \ldots, p,$$

where,

$$\begin{bmatrix} \Phi_{11} & A_{0i}^{\mathsf{T}}U_{2j} & \Phi_{13} & A_{0i}^{\mathsf{T}}U_{4j} & -U_{1j}^{\mathsf{T}} + A_{0i}^{\mathsf{T}}U_{5j} & \Phi_{16} \\ * & -\delta R_j & U_{2j}^{\mathsf{T}}A_{1i} & 0 & -U_{2j}^{\mathsf{T}} & U_{2j}^{\mathsf{T}}D_i \\ * & * & \Phi_{33} & Z_j + A_{1i}^{\mathsf{T}}U_{4j} & -U_{3j}^{\mathsf{T}} + A_{1i}^{\mathsf{T}}U_{5j} & U_{3j}^{\mathsf{T}}D_i + A_{1i}^{\mathsf{T}}U_{6j} \\ * & * & * & -Q_j - Z_j & -U_{4j}^{\mathsf{T}} & U_{4j}^{\mathsf{T}}D_i \\ * & * & * & * & \Phi_{55} & U_{5j}^{\mathsf{T}}D_i - U_{6j} \\ * & * & * & * & * & \Phi_{66} \end{bmatrix},$$

and

$$\begin{aligned}
\Phi_{11} &= A_{0i}^{\mathsf{T}}(P_j + U_{1j}) + (P_j + U_{1j}^{\mathsf{T}})A_{0i} + Q_j + R_j - T_j;\\
\Phi_{13} &= P_jA_{1i} + U_{1j}^{\mathsf{T}}A_{1i} + A_{0i}^{\mathsf{T}}U_{3j} + T_j;\\
\Phi_{16} &= P_jD_i + U_{1j}^{\mathsf{T}}D_i + A_{0i}^{\mathsf{T}}U_{6j};\\
\Phi_{33} &= -T_j - Z_j + A_{1i}^{\mathsf{T}}U_{3j} + U_{3j}^{\mathsf{T}}A_{1i};\\
\Phi_{55} &= -U_{5j} - U_{5j}^{\mathsf{T}} + S_j + h_M^2T_j + (h_M - h_m)^2Z_j;\\
\Phi_{66} &= -\delta S_j + D_i^{\mathsf{T}}U_{6j} + U_{6j}^{\mathsf{T}}D_i.
\end{aligned}$$

Then exists a positive number α_* *such that system* (2.1) *is exponentially stable with any decay rate* $\alpha \in (0, \alpha_*]$*. Moreover, every solution* $x(t, \phi)$ *of the system satisfies*

$$\|x(t,\phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}}\|\phi\|e^{-\alpha t}, \quad t \geq 0.$$

Proof. By the same argument used in the proof of Theorem 3.1, from (3.11) we have

$$\begin{aligned}\dot V(t,x_t)+2\alpha V(t,x_t)\le \eta^{\mathsf T}(t)\Big[&\sum_{i=1}^{p}\xi_i^2\Phi_i(P_i,Q_i,R_i,S_i,T_i,Z_i,\mathcal{U}_i)\\ &+\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j(\Phi_i(P_j,Q_j,R_j,S_j,T_j,Z_j,\mathcal{U}_j)\\ &+\Phi_j(P_i,Q_i,R_i,S_i,T_i,Z_i,\mathcal{U}_i))\Big]\eta(t)+\sum_{i=1}^{p}\xi_i\eta^{\mathsf T}(t)\Psi_i\eta(t),\end{aligned}\tag{3.16}$$

where,

$$\begin{aligned}\Psi_i=\mathrm{diag}\Big\{&2\alpha P_i,\delta(1-e^{-2\alpha\tau_M})R_i,\ 0,\ (1-e^{-2\alpha h_m})Q_i,\\ &h_M^2(e^{2\alpha h_M}-1)T_i+(h_M-h_m)^2(e^{2\alpha h_M}-1)Z_i,\ \delta(1-e^{-2\alpha\tau_M})S_i,\ 0\Big\}.\end{aligned}$$

Therefore,

$$\begin{aligned}\dot V(t,x_t)+2\alpha V(t,x_t)&\le \eta^{\mathsf T}(t)\left(-\sum_{i=1}^{p}\xi_i^2+\frac{2}{p-1}\sum_{i=1}^{p-1}\sum_{j=i+1}^{p}\xi_i\xi_j\right)\mathbb{M}\eta(t)\\ &\quad-\lambda_0\left(\sum_{i=1}^{p}\xi_i^2\right)\|\eta(t)\|^2+\varphi(\alpha)\|\eta(t)\|^2,\\ &\le\left[\varphi(\alpha)-\lambda_0\left(\sum_{i=1}^{p}\xi_i^2\right)\right]\|\eta(t)\|^2,\end{aligned}\tag{3.17}$$

where,

$$\begin{aligned}\varphi(\alpha)=2\alpha\lambda_P&+\delta\left(1-e^{-2\alpha\tau_M}\right)(\lambda_R+\lambda_S)+\left(1-e^{-2\alpha h_m}\right)\lambda_Q\\ &+h_M^2\left(e^{2\alpha h_M}-1\right)\lambda_T+(h_M-h_m)^2\left(e^{2\alpha h_M}-1\right)\lambda_Z.\end{aligned}$$

Observe that, $p\left(\sum_{i=1}^{p}\xi_i^2\right)\ge\left(\sum_{i=1}^{p}\xi_i\right)^2=1$, then from (3.17) we have

$$\dot V(t,x_t)+2\alpha V(t,x_t)\le\left[\varphi(\alpha)-\frac{\lambda_0}{p}\right]\|\eta(t)\|^2,\quad t\ge 0.\tag{3.18}$$

Noting that, $\varphi(\alpha)$ is continuous and strictly increasing function in $\alpha\in[0,\infty)$, $\varphi(0)=0$, $\varphi(\alpha)\to\infty$ as $\alpha\to\infty$. Hence, there is a unique positive solution α_* of the equation $\varphi(\alpha)=\dfrac{\lambda_0}{p}$ and $\varphi(\alpha)<\dfrac{\lambda_0}{p}$ for all $\alpha\in(0,\alpha_*)$.

For any $\alpha \in (0, \alpha_*]$ we have

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \leq \left[\varphi(\alpha) - \frac{\lambda_0}{p}\right] \|\eta(t)\|^2 \leq 0$$

which implies $V(t, x_t) \leq V(0, x_0)e^{-2\alpha t}, t \geq 0$. Taking (3.4) into account, we obtain

$$\|x(t, \phi)\| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} \|\phi\| e^{-\alpha t}, \quad t \geq 0$$

which concludes the proof. □

Remark 3.4. If $p = 1$, then condition (3.2) is automatically removed and Theorem 3.1 is reduced to asymptotic stability conditions for neutral systems with interval time-varying state delay as stated in the following corollary.

Corollary 3.5. *Assume that all the eigenvalues of matrix D are inside the unit circle, the system* (2.1)*, with $p = 1$, is asymptotically stable if there exist matrices U_k, $k = 1, \ldots, 6$, symmetric positive definite matrices P, Q, R, S, T, Z, such that the following linear matrix inequality hold:*

$$\begin{bmatrix} \Xi_{11} & A_0^\mathsf{T} U_2 & \Xi_{13} & A_0^\mathsf{T} U_4 & \Xi_{15} & \Xi_{16} \\ * & -(1-\mu)R & U_2^\mathsf{T} A_1 & 0 & -U_2^\mathsf{T} & U_2^\mathsf{T} D \\ * & * & \Xi_{33} & Z + A_1^\mathsf{T} U_4 & \Xi_{35} & \Xi_{36} \\ * & * & * & -Q - Z & -U_4^\mathsf{T} & U_4^\mathsf{T} D \\ * & * & * & * & \Xi_{55} & U_5^\mathsf{T} D - U_6 \\ * & * & * & * & * & \Xi_{66} \end{bmatrix} < 0, \tag{3.19}$$

where

$$\begin{aligned}
\Xi_{11} &= A_0^\mathsf{T}(P + U_1) + (P + U_1^\mathsf{T})A_0 + Q + R - T; \\
\Xi_{13} &= PA_1 + U_1^\mathsf{T} A_1 + A_0^\mathsf{T} U_3 + T; \\
\Xi_{15} &= -U_1^\mathsf{T} + A_0^\mathsf{T} U_5; \ \Xi_{16} = PD + U_1^\mathsf{T} D + A_0^\mathsf{T} U_6; \\
\Xi_{33} &= -T - Z + A_1^\mathsf{T} U_3 + U_3^\mathsf{T} A_1; \\
\Xi_{35} &= -U_3^\mathsf{T} + A_1^\mathsf{T} U_5; \ \Xi_{36} = U_3^\mathsf{T} D + A_1^\mathsf{T} U_6; \\
\Xi_{55} &= -U_5 - U_5^\mathsf{T} + S + h_M^2 T + (h_M - h_m)^2 Z; \\
\Xi_{66} &= -(1-\mu)S + D^\mathsf{T} U_6 + U_6^\mathsf{T} D.
\end{aligned}$$

4 Examples

In this section, we give some numerical examples to show the effectiveness of our conditions.

Example 4.1. Consider system (2.1) with $p = 3$, $h(t) = 1 + |\sin t|, \tau(t) = \cos^2 0.1t$ and

$$A_{01} = \begin{bmatrix} -4 & 1 \\ 0 & -1 \end{bmatrix}, \quad A_{02} = \begin{bmatrix} -2 & -1 \\ 0 & -3 \end{bmatrix}, \quad A_{03} = \begin{bmatrix} -2 & 0 \\ -1 & -1 \end{bmatrix},$$
$$A_{11} = \begin{bmatrix} 0.1 & 0 \\ 0.1 & -0.1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -0.2 & 0 \\ 0.1 & 0.1 \end{bmatrix}, \quad A_{13} = \begin{bmatrix} -0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix},$$
$$D_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_3 = \begin{bmatrix} -0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.$$

Note that, the delay function $h(t)$ is continuous, but nondifferentiable on $\mathbb{R}^+$. We have, $h_m = 1, h_M = 2, \tau_M = 1$ and $\mu = 0.1$. For given $\alpha = 0.25$, by using Matlab LMI toolbox, we find that conditions (3.1), (3.2) are satisfied with

$$U_{1i} = U_{2i} = U_{4i} = 0, i = 1, 2, 3, \ M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$
$$P_1 = 10^3 \times \begin{bmatrix} 1.1587 & -0.1349 \\ -0.1349 & 1.4127 \end{bmatrix}, \quad P_2 = 10^3 \times \begin{bmatrix} 1.2071 & -0.1059 \\ -0.1059 & 1.0060 \end{bmatrix},$$
$$P_3 = 10^3 \times \begin{bmatrix} 1.3456 & 0.0467 \\ 0.0467 & 1.5802 \end{bmatrix}, \quad Q_1 = 10^3 \times \begin{bmatrix} 3.1220 & -0.0156 \\ -0.0156 & 3.0962 \end{bmatrix},$$
$$Q_2 = 10^3 \times \begin{bmatrix} 3.1026 & 0.0168 \\ 0.0168 & 3.0959 \end{bmatrix}, \quad Q_3 = 10^3 \times \begin{bmatrix} 3.1002 & 0.0094 \\ 0.0094 & 3.1846 \end{bmatrix},$$
$$R_1 = 10^3 \times \begin{bmatrix} 3.2641 & -0.0229 \\ -0.0229 & 3.2396 \end{bmatrix}, \quad R_2 = 10^3 \times \begin{bmatrix} 3.3781 & 0.0293 \\ 0.0293 & 3.2252 \end{bmatrix},$$
$$R_3 = 10^3 \times \begin{bmatrix} 3.2277 & -0.0268 \\ -0.0268 & 3.3383 \end{bmatrix}, \quad S_1 = \begin{bmatrix} 80.7481 & 2.1035 \\ 2.1035 & 113.5618 \end{bmatrix},$$
$$S_2 = \begin{bmatrix} 53.0203 & -14.7176 \\ -14.7176 & 27.3034 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 45.1212 & 6.3806 \\ 6.3806 & 61.9915 \end{bmatrix},$$
$$T_1 = \begin{bmatrix} 12.0558 & 0.9591 \\ 0.9591 & 28.4793 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 25.9109 & -10.6601 \\ -10.6601 & 8.2741 \end{bmatrix},$$
$$T_3 = \begin{bmatrix} 11.0547 & 9.9937 \\ 9.9937 & 21.1726 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} 73.7291 & 2.4344 \\ 2.4344 & 113.5906 \end{bmatrix},$$
$$Z_2 = \begin{bmatrix} 98.8278 & -36.7086 \\ -36.7086 & 41.8415 \end{bmatrix}, \quad Z_3 = \begin{bmatrix} 42.1947 & 27.7165 \\ 27.7165 & 105.8770 \end{bmatrix},$$
$$U_{31} = \begin{bmatrix} 44.2623 & -9.5044 \\ -27.3436 & -19.8529 \end{bmatrix}, \quad U_{32} = \begin{bmatrix} 47.2710 & 3.4638 \\ -34.6174 & -13.3825 \end{bmatrix},$$
$$U_{33} = \begin{bmatrix} 1.7296 & -19.4483 \\ -2.0176 & -57.3286 \end{bmatrix}, \quad U_{51} = \begin{bmatrix} 378.0268 & 83.4725 \\ -9.8866 & 488.9168 \end{bmatrix},$$
$$U_{52} = \begin{bmatrix} 514.0416 & -203.5652 \\ -213.2774 & 180.2979 \end{bmatrix}, \quad U_{53} = \begin{bmatrix} 209.9268 & 148.7830 \\ 143.0793 & 387.8910 \end{bmatrix},$$

Table 1: Upper bounds of decay rate α for different values of μ

μ	0	0.1	0.3	0.5	0.7	0.8
α	0.323	0.307	0.266	0.209	0.121	0.047

Table 2: Upper bound of h_M for different values of h_m

h_m	0	1	2	3	4
He *et al.* [2]	1.34	1.74	2.43	3.22	4.06
Jiang & Han [7]	1.34	1.80	2.52	3.33	4.18
Shao [16]	1.34	1.76	2.44	3.22	4.06
Zhang *et al.* [21]	1.86	2.06	2.61	3.31	4.09
Corollary 3.5	1.86	2.11	2.68	3.34	4.12

$$U_{61} = \begin{bmatrix} 1.5949 & 2.6459 \\ -6.5208 & -24.4674 \end{bmatrix}, \quad U_{62} = \begin{bmatrix} -12.5017 & -11.9106 \\ 19.3965 & 16.4398 \end{bmatrix},$$
$$U_{63} = \begin{bmatrix} -15.0699 & 6.1030 \\ 12.9969 & 26.8754 \end{bmatrix}.$$

By Theorem 3.1, System (2.1) is exponentially stable with decay rate $\alpha = 0.25$. Moreover, every solution of the system satisfies

$$\|x(t, \phi)\| \leq 3.0754\|\phi\|e^{-0.25t}, \quad t \geq 0.$$

With $U_{1i} = U_{2i} = U_{4i} = 0,\ i = 1, 2, 3$ and $M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, Table 1 gives the upper values of decay rate α for different values of μ.

Example 4.2. Consider the system with interval time-varying delay studied in ([2,21]):

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - h(t)), \tag{4.1}$$

where

$$A_0 = \begin{bmatrix} -2 & 0 \\ 0 & -0.9 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, \quad h_m \leq h(t) \leq h_M.$$

Table 2 gives the upper bound of h_M for different values of h_m.

5 Conclusions

In this paper, new delay-dependent exponential stability conditions for polytopic neutral system with nondifferentiable interval time-varying delays are proposed. By using an improved Lyapunov–Krasovskii functional, the exponential stability conditions are derived in terms of LMIs, which can be solved by various computational tools and allows to compute simultaneously the two bounds that characterize the exponential stability rate of the solution. Numerical examples are given to show the effectiveness of our conditions.

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