The $L^p$-Version of the Generalized Bohl–Perron Principle for Vector Equations with Infinite Delay

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Abstract

We consider the vector equation

$$\dot{y}(t) = \int_0^\infty d\tau R(t, \tau) y(t - \tau),$$

where $R(t, \tau)$ is an $n \times n$-matrix-valued function. It is proved that, if the non-homogeneous equation

$$\dot{x}(t) = \int_0^\infty d\tau R(t, \tau) x(t - \tau) + f(t) \quad (t \geq 0)$$

with $f \in L^p([0, \infty), \mathbb{C}^n)$ ($p \geq 1$) and the zero initial condition, has a solution $x \in L^p([0, \infty), \mathbb{C}^n)$, then the considered homogeneous equation is exponentially stable.

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1 Introduction

Recall that the Bohl–Perron principle means that the homogeneous ordinary differential equation (ODE) $\dot{y} = A(t)y \ (t \geq 0)$ with a variable $n \times n$-matrix $A(t)$, bounded on $[0, \infty)$ is exponentially stable, provided the nonhomogeneous ODE $\dot{x} = A(t)x + f(t)$ with the zero initial condition has a bounded solution for any bounded vector valued function $f$, cf. [4]. In [7, Theorem 4.15] the Bohl–Perron principle was generalized to a class of retarded systems with finite delays; besides the asymptotic (not exponential) stability was proved. The result from [7] was afterwards considerably developed, cf.
the books [1, 10] and very interesting papers [2, 3], in which the generalized Bohl–Perron principle was effectively used for the stability analysis of the first and second order scalar equations. In particular, in [2] the scalar non-autonomous linear functional differential equation \( \dot{x}(t) + a(t)x(h(t)) = 0 \) is considered. The authors give sharp conditions for exponential stability, which are suitable in the case that the coefficient function \( a(t) \) is periodic, almost periodic or asymptotically almost periodic, as often encountered in applications. In the paper [3], the authors provide sufficient conditions for the stability of rather general second-order delay differential equations.

In the present paper we derive a result similar to the Bohl–Perron principle in the terms of the norm of the space \( L^p \), which we will call the \( L^p \)-version of the generalized Bohl–Perron principle. In the case \( L^\infty \) our result is deeply connected with the Bohl–Perron principle. In Section 3 below, we show that the \( L^p \)-version can be effectively used for the stability analysis. In particular, in the case \( p = 2 \) the stability analysis can be reduced to the estimates for the characteristic matrix-valued functions. Such estimates can be found, for instance, in [5,6]. As it is well-known, the basic method for the stability analysis of vector time-variant equations is the direct Lyapunov method. By that method many very strong results are obtained. But finding Lyapunov’s type functionals for equations with infinite delay is usually difficult. At the same time, in Section 3 we suggest explicit stability conditions. Besides, we improve and generalize [5, Theorem 9.2.2] and [8, Theorem 5.2.2] in the case of linear equations with continuous delay.

Let \( \mathbb{C}^n \) be a complex Euclidean space with the scalar product \( \langle \cdot, \cdot \rangle \) and the unit matrix \( I \); for an \( n \times n \)-matrix \( A \), \( \|A\|_n = \sup_{v \in \mathbb{C}^n} \|Av\|_n \) is the spectral norm. Denote by \( L^p(\omega) \equiv L^p(\omega, \mathbb{C}^n) (p \geq 1) \) the space of functions \( u \) defined on a set \( \omega \subseteq \mathbb{R} \) with values in \( \mathbb{C}^n \) and the finite norm

\[
\|u\|_{L^p} = \|u\|_{L^p(\omega)} := \left( \int_\omega \|u(t)\|_n^p \, dt \right)^{1/p} \quad (1 \leq p < \infty),
\]

and \( \|u\|_{L^\infty(\omega)} = \text{vrai} \sup_{x \in \omega} \|u(x)\|_n \). In addition, \( C(\omega) \equiv C(\omega, \mathbb{C}^n) \) is the space of continuous functions defined on \( \omega \) with values in \( \mathbb{C}^n \) and the finite sup-norm \( \| \cdot \|_C \). For a linear operator \( M \) acting from \( L^p(\omega) \) into \( L^p(\omega_1) \), put

\[
\|M\|_{L^p} = \frac{\sup_{u \in L^p(\omega)} \|Mu\|_{L^p(\omega_1)}}{\|u\|_{L^p(\omega)}}.
\]

Consider in \( \mathbb{C}^n \) the equation

\[
\dot{y}(t) = \int_0^\infty d_\tau R(t, \tau)y(t - \tau) \quad (t \geq 0; \dot{y}(t) = dy/dt), \tag{1.1}
\]

where \( R(t, \tau) \) is an \( n \times n \)-matrix-valued function defined on \( [0, \infty)^2 \), which is continuous in \( t \) for each \( \tau \geq 0 \) and right-continuous in \( \tau \) for each \( t \geq 0 \). The integral in (1.1) is understood as the improper vector Riemann–Stieltjes integral. For the details see for instance [8, p. 136–138].
Everywhere below it is assumed that the variation of $R(t, \tau)$ in $\tau$ is uniformly bounded on $[0, \infty)$:

$$V(R) := \sup_{t \geq 0} \int_0^\infty \| d_\tau R(t, \tau) \|_n < \infty,$$

where

$$\int_0^\infty \| d_\tau R(t, \tau) \|_n = \lim_{\eta \to \infty} \int_0^\eta \| d_\tau R(t, \tau) \|_n,$$

and the latter integral on a finite segment $[0, \eta]$ means the limit, if it exists, of the sums

$$\sum_{k=1}^{m-1} \| R(t, s_k^{(m)}) - R(t, s^{(m)}_k) \|_n$$

$(0 = s_0^{(m)} < s_1^{(m)} < \ldots < s_m^{(m)} = \eta)$ as $\max_k |s_{k+1}^{(m)} - s_k^{(m)}| \to 0$.

Take the initial condition

$$y(t) = \phi(t) \quad (-\infty < t \leq 0) \quad \text{(1.2)}$$

for a given $\phi \in C(-\infty, 0) \cap L^1(-\infty, 0)$. A solution of problem (1.1), (1.2) is an absolutely continuous function, which satisfies that equation on $(0, \infty)$ almost everywhere and condition (1.2). The existence results and stability definitions can be found for instance in [8, p. 138].

Consider also the nonhomogeneous equation

$$\dot{x}(t) = \int_0^\infty d_\tau R(t, \tau)x(t - \tau) + f(t) \quad (t \geq 0) \quad \text{(1.3)}$$

with a given vector function $f(t)$ and the zero initial condition

$$x(t) = 0 \quad (-\infty \leq t \leq 0). \quad \text{(1.4)}$$

So problem (1.3), (1.4) can be written as

$$\dot{x}(t) = \int_0^t d_\tau R(t, \tau)x(t - \tau) + f(t) \quad (t \geq 0). \quad \text{(1.5)}$$

## 2 Main Result

If $f \in L^p(0, \infty)$, $p \geq 1$, then a solution of problem (1.3), (1.4) is defined as a locally absolutely continuous function $x(t)$, with $\dot{x}(t) \in L^p(0, t_1)$ for any positive finite $t_1$, which satisfies condition (1.4) and equation (1.3) on $(0, \infty)$ almost everywhere.
We will say that (1.1) has the $\epsilon$-property, if
\[
\sup_{t \geq 0} \int_0^\infty (e^{\epsilon \tau} - 1) \| d_\tau R(t, \tau) \|_n \to 0 \text{ as } \epsilon \to 0 \ (\epsilon > 0).
\]
Equation (1.1) is said to be exponentially stable, if there are positive constants $\nu$ and $m_0$, such that
\[
\| y(t) \|_n \leq m_0 \| \phi \|_{C(-\infty, 0)} e^{-\nu t} \quad (t \geq 0)
\]
for any solution $y(t)$ of problem (1.1), (1.2).

Now we are in a position to formulate our main result.

**Theorem 2.1.** For a $p \geq 1$ and any $f \in L^p(0, \infty)$, let the nonhomogeneous problem (1.3), (1.4) has a solution $x \in L^p(0, \infty)$. If, in addition, equation (1.1) has the $\epsilon$-property, then it is exponentially stable.

The proof of this theorem is divided into a series of lemmas presented in this section. Introduce the operator $E : L^p(-\infty, \infty) \to L^p(0, \infty)$ by
\[
Eu(t) = \int_0^\infty d_\tau R(t, \tau) u(t - \tau) \quad (t \geq 0; \ u \in L^p(-\infty, \infty)).
\]

**Lemma 2.2.** For any $p \geq 1$, the inequality $\| E \|_{L^p} \leq V(R)$ is true.

**Proof.** For a $u \in L^1(-\infty, \infty)$ we have
\[
\| Eu \|_{L^1(0, \infty)} = \int_0^\infty \| d_\tau R(t, s) u(t - s) \|_n dt \leq 
\]
\[
\int_0^\infty \int_0^\infty \| d_\tau R(t, s) \|_n \| u(t - s) \|_n dt.
\]
Hence,
\[
\| Eu \|_{L^1(0, \infty)} \leq \sup_{\tau \geq 0} \int_0^\infty \| d_\tau R(\tau, s) \|_n \sup_{s \in [0, \infty)} \int_0^\infty \| u(t - s) \|_n dt = V(R) \| u \|_{L^1(-\infty, \infty)}.
\]
This proves the lemma in the case $p = 1$. Furthermore, for a $v \in L^\infty(-\infty, \infty)$, we have
\[
\| Ev \|_{L^\infty(0, \infty)} = \sup_{t \geq 0} \| \int_0^\infty d_\tau R(t, s) v(t - s) \|_n \leq 
\]
\[
\sup_{\tau} \int_0^\infty \| d_\tau R(\tau, s) \|_n \sup_{s \in [0, \infty)} \| v(t - s) \|_{L^\infty(0, \infty)} = V(R) \| v \|_{L^\infty(-\infty, \infty)}.
\]
Hence, by the Riesz–Torino interpolation theorem [9] we get the required result. \qed
Lemma 2.3. Under the hypothesis of Theorem 2.1, any solution of the homogeneous problem (1.1), (1.2) is in $L^p(0, \infty)$.

Proof. With a $\mu > 0$, put

$$\tilde{\phi}_\mu(t) = \begin{cases} e^{-\mu t}\phi(0) & \text{if } t \geq 0, \\ \phi(t) & \text{if } -\infty \leq t < 0. \end{cases}$$

Clearly, $\|\phi\|_{L^p(-\infty,0)}^p \leq \|\phi\|_{L^1(-\infty,0)} \|\phi\|_{C(-\infty,0)}^{p-1}$. So $\tilde{\phi}_\mu \in L^p(-\infty, \infty)$. By the previous lemma $E\tilde{\phi}_\mu \in L^p(0, \infty)$. Substitute $y = \tilde{\phi}_\mu + x$ unto (1.1). Then we have problem (1.3), (1.4) with $f = \mu e^{-\mu \phi(0)} + E\tilde{\phi}_\mu$. By the condition of the lemma $x \in L^p(0, \infty)$. We thus get the required result. \qed

Lemma 2.4. If a solution $y(t)$ of problem (1.1), (1.2) is in $L^p(0, \infty)$ ($p \geq 1$), then it is bounded on $[0, \infty)$. Moreover, if $p < \infty$, then

$$\|y\|_{C(0, \infty)}^p \leq pV(R)\|y\|_{L^p(0, \infty)}^{p-1}\|\dot{y}\|_{L^p(-\infty, \infty)} \leq pV(R)\|y\|_{L^p(-\infty, \infty)}^p.$$

Proof. By (1.1) and Lemma 2.2, $\|y\|_{L^p(0, \infty)} \leq V(R)\|y\|_{L^p(-\infty, \infty)}$. For simplicity, in this proof we put $\|y(t)\|_n = \|y(t)\|$. The case $p = \infty$ is obvious. In the case $p = 1$ we have

$$\|y(t)\| = -\int_t^\infty \frac{d|y(t_1)|}{dt_1}dt_1 \leq \int_t^\infty |\dot{y}(t_1)|dt_1 < \infty \quad (t \geq 0),$$

since $\dot{y} \in L^1$. Assume now that $1 < p < \infty$. Then by the Gōdel inequality

$$|y(t)|^p = -\int_t^\infty \frac{d|y(t_1)|^p}{dt_1}dt_1 = -p\int_t^\infty |y(t_1)|^{p-1} \frac{d|y(t_1)|}{dt_1}dt_1 \leq$$

$$p\int_t^\infty |y(t_1)|^{p-1} |\dot{y}(t_1)|dt_1 \leq p\left[\int_t^\infty |y(t_1)|^{q(p-1)}dt_1\right]^{1/q}\left[\int_t^\infty |\dot{y}(t_1)|^{p}dt_1\right]^{1/p},$$

where $q = p/(p - 1)$. Since $q(p - 1) = p$, we get the inequalities

$$|y(t)|^p \leq p\|y\|_{L^p(0, \infty)}^{p-1}\|\dot{y}\|_{L^p(0, \infty)} \leq p\|y\|_{L^p(0, \infty)}^{p-1}V(R)\|y\|_{L^p(-\infty, \infty)} \quad (t \geq 0).$$

As claimed. \qed

Proof of Theorem 2.1. By the variation of constants formula, $x = Wf$, where

$$Wf(t) = \int_0^t G(t, s)f(s)ds.$$ 

Here $G(t, s)$ is the fundamental solution to (1.1). For all finite $T > 0$ and $w \in L^p(0, \infty)$, let $P_T$ be the projections defined by

$$(P_T w)(t) = \begin{cases} w(t) & \text{if } t \leq T, \\ 0 & \text{if } t > T. \end{cases}$$
and $P_\infty = I$. Operator $W$ is bounded in $L^p(0,T)$, since $[0,T]$ is finite. Thus, the operators $W_T = P_T W$ are bounded on $L^p(0,\infty)$. But $W_T$ strongly converge to $W$ as $T \to \infty$. So by the Banach–Steinhaus theorem, $W$ is bounded in $L^p(0,\infty)$.

Furthermore, substituting into (1.1) the equality

$$y_\epsilon(t) = e^{\epsilon t}y(t) \quad (\epsilon > 0; \ t > 0), \quad y_\epsilon(t) = y(t) \quad (t \leq 0)$$

we obtain the equation

$$\dot{y}_\epsilon(t) = E_\epsilon y_\epsilon(t),$$

where

$$E_\epsilon w(t) = \epsilon w(t) + \int_0^\infty e^{\epsilon \tau} d_\tau R(t, d\tau) w(t - \tau) \quad (t > 0).$$

Besides, $y_\epsilon(t) = e^{\epsilon t}\phi(t) \ (t \leq 0)$. We thus get

$$y_\epsilon - y = E_\epsilon y_\epsilon - Ey = E(y_\epsilon - y) + (E_\epsilon - E)y_\epsilon.$$

Hence,

$$\|y_\epsilon - y\|_{L^p(0,T)} \leq \|W\|_{L^p(0,T)} \|(E_\epsilon - E)y_\epsilon\|_{L^p(0,T)}$$

for a finite $T$. Here $\|W\|_{L^p(0,T)} = \sup_{w \in L^p(0,T)} \|Ww\|_{L^p(0,T)}/\|w\|_{L^p(0,T)}$. But

$$\|(E_\epsilon - E)y_\epsilon\|_{L^p(0,T)} \leq \epsilon \|y_\epsilon\|_{L^p(0,T)} + \|\int_0^\infty (1 - e^{\epsilon \tau}) d_\tau R(t, \tau)y_\epsilon(t - \tau)\|_{L^p(-\infty,T)} \leq$$

$$\quad v(\epsilon) \|y_\epsilon\|_{L^p(-\infty,T)},$$

where

$$v(\epsilon) := \epsilon + \sup_{\tau \geq 0} \int_0^\infty (e^{\epsilon \tau} - 1) \|d_\tau R(t, \tau)\|_\tau.$$

So

$$\|y_\epsilon\|_{L^p(0,T)} \leq \|y\|_{L^p(0,\infty)} + v(\epsilon) \left(\|y_\epsilon\|_{L^p(0,T)} + \|\phi\|_{L^p(-\infty,0)}\right)\|W\|_{L^p(0,T)}.$$

Take $\epsilon$, such that $v(\epsilon)\|W\|_{L^p(0,T)} < 1$, for all sufficiently large $T$. This is possible due to the $\epsilon$-property. We have

$$\|y_\epsilon\|_{L^p(0,T)} \leq (\|y\|_{L^p(0,\infty)} + v(\epsilon)\|W\|_{L^p(0,T)}\|\phi\|_{L^p(-\infty,0)})(1 - v(\epsilon)\|W\|_{L^p(0,T)})^{-1}.$$

Letting $T \to \infty$, we get $y_\epsilon \in L^p(0,\infty)$. By Lemma 2.4 hence it follows that a solution $y_\epsilon$ of (2.2) is bounded on $[0,\infty)$, if $\epsilon$ is small enough. Now (2.1) proves the theorem. ☐
3 Equations with Continuous Delay

In this section we illustrate Theorem 2.1 in the case $p = 2$ and show that it enables us to apply the Laplace transform. Consider in $\mathbb{C}^n$ the equation

$$\dot{y}(t) = \int_0^\infty A(\tau)y(t-\tau)d\tau + \int_0^\infty K(t, \tau)y(t-\tau)d\tau \quad (t \geq 0),$$

(3.1)

where $A(\tau)$ is a piece-wise continuous matrix-valued function defined on $[0, \infty)$ and $K(t, \tau)$ is a piece-wise continuous matrix-valued function defined on $[0, \infty)^2$. Besides,

$$\|A(s)\|_n \leq Ce^{-\mu s} \text{ and } \|K(t, s)\|_n \leq Ce^{-\mu s} \quad (C, \mu = \text{const} > 0; \ t, s \geq 0).$$

(3.2)

Then, clearly, (3.1) has the $\epsilon$-property. It is also supposed that the operator $\tilde{K}$ defined by

$$\tilde{K}w(t) = \int_0^t K(t, \tau)w(t-\tau)d\tau$$

is bounded in $L^2 = L^2(0, \infty)$. To apply Theorem 2.1, consider the equation

$$\dot{x}(t) = \int_0^t A(\tau)x(t-\tau)d\tau + \int_0^t K(t, \tau)x(t-\tau)d\tau + f(t) \quad (t \geq 0)$$

(3.3)

with $f \in L^2$. To estimate solutions of the latter equation, we need the equation

$$\dot{u}(t) = \int_0^t A(\tau)u(t-\tau)d\tau + h(t) \quad (t \geq 0)$$

(3.4)

with $h \in L^2(0, \infty)$. Applying to (3.4) the Laplace transform we have

$$z\hat{u}(z) = \hat{A}(z)\hat{u}(z) + \hat{h}(z)$$

where are $\hat{A}(z), \hat{u}(z)$ and $\hat{h}(z)$ are the Laplace transforms of $A(t), u(t)$ and $h(t)$, respectively, and $z$ is the dual variable. Then

$$\hat{u}(z) = (zI - \hat{A}(z))^{-1}\hat{h}(z).$$

It is assumed that det $(zI - \hat{A}(z))$ is a stable function, that is all its zeros are in the open left half plane, and

$$\theta_0 := \sup_{\omega \in \mathbb{R}} \|i\omega I - \hat{A}(i\omega)\|^{-1}_n < \frac{1}{\|\tilde{K}\|_{L^2}}.$$  (3.5)

As it was above mentioned, various estimates for $\theta_0$ can be found in [5, 6]. By the Parseval equality we have $\|u\|_{L^2} \leq \theta_0\|h\|_{L^2}$. By this inequality, from (3.3) we get

$$\|x\|_{L^2} \leq \theta_0\|f + \tilde{K}x\|_{L^2} \leq \theta_0\|f\|_{L^2} + \theta_0\|\tilde{K}\|_{L^2}\|x\|_{L^2}.$$  

Hence (3.5) implies that $x \in L^2$. Now by Theorem 2.1 we get the following result.
Corollary 3.1. Let the conditions (3.2) and (3.5) hold. Then (3.1) is exponentially stable.

For example, $K$ satisfies conditions (3.2) and (3.5), if

$$\|K(t, s)\|_n \leq e^{-\nu s}a(t) \quad (t, s \geq 0)$$

with a bounded function $a \in L^2(0, \infty)$ satisfying the inequality $\theta_0\|a\|_{L^2(0, \infty)} < \sqrt{2\nu}$.

References


