Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 6, Number 2, pp. 291–314 (2011)
http://campus.mst.edu/adsa

Positive Solutions of Higher-Order Neutral Dynamic Equations on Time Scales

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Abstract

This paper is devoted to studying the existence of positive solutions for a class of higher-order neutral dynamic equations on time scales. First, all various types of the eventually positive solutions are given by means of their asymptotic behaviors. Then, by means of Kranoselskii's fixed point theorem, the existence criteria are provided for these type solutions. Some examples in the last section are also included to illustrate our results.

AMS Subject Classifications: 34N05, 34C10, 34D05.
Keywords: Time scales, higher order, neutral dynamic equations, positive solutions, fixed point.

1 Introduction

Recall that the existence of positive solutions for first-order neutral dynamic equations on time scales has been studied in [18], where the authors answer an open problem in [13]. Subsequently the present author pursued the study for the second-order case in [16]. In this paper we will follow the trend and consider the classification and existence of nonoscillatory solutions to higher-order neutral functional dynamic equation

$$[x(t) + p(t)x(g(t))]^{\Delta^n} + f(t, x(q(t))) = 0, t \in \mathbb{T}, \tag{1.1}$$

where n is a positive integer with $n \geq 2$, $\mathbb{T}$ is a time scale with $\inf \mathbb{T} = t_0$ and $\sup \mathbb{T} = \infty$. Here we assume that the reader is familiar with the basic calculus on time scales. For the details we refer to [1, 3–7].

Received September 3, 2010; Accepted January 25, 2011
Communicated by Elena Braverman

We remark that there have been numerous results dealing with the neutral dynamic equations for $\mathbb{T} = [t_0, \infty)$, a half-axis, or $\mathbb{T} = \{0, 1, 2, \ldots\}$, a set of nonnegative integers. For example, when $\mathbb{T} = [t_0, \infty)$ and both $c \neq 1$ and τ are constant, Chen [8] investigated (1.1) for $p(t) = c$, $g(t) = t - \tau$ and $q(t) = t - \delta(t) \to \infty$ (as $t \to \infty$) and obtained some sufficient criteria to guarantee the existence of nonoscillatory solutions of (1.1). Kong et al [11] considered a critical and special case, i.e., the following odd order equation

$$(x(t) - x(t - \tau))^{(n)} + p(t)x(t - \delta) = 0, \ \ t \geq t_0,$$

where δ is a constant, and gave a complete classification of nonoscillatory solutions and found conditions for each type solution to exist. The relevant topic is also included in the monograph [2, Chapter 6]. For the classification and existence of nonoscillatory solutions of neutral difference equations of the form

$$\Delta^n(x_k + c_k x_{k-\tau}) + f(k, x_{k-\delta}) = 0, \ k = 0, 1, 2, \ldots,$$

Zhu [17] and Li [12] et al had given, respectively, the relevant and no overlapping results. Precisely speaking, the results in [17] were based on the assumption $\{c_k\} \subseteq (-1, 0]$ and those in [12] based on the assumption that c_k is nonnegative constant and different from 1 for all k. The present paper extends the methods in [8] and the results to obtain will unify the ones in [8, 12, 17] when $\lim_{t\to\infty} p(t) = p_0$ with $|p_0| < 1$ (see the next assumption (H3)).

We remark also that there have been recent results which studied the existence of nonoscillatory solutions to higher-order neutral dynamic equations on time scales, see, e.g., [9, 14, 15]. It is worthy to say that the results in the present paper are different from those in [9, 14, 15]. In fact, the three papers mentioned above are concerned with the existence of bounded nonoscillatory solutions while ours will include the cases of bound and unboundedness.

Let $\mathbb{R}$ be the set of real numbers and $C_{rd}(\mathbb{T}, \mathbb{R})$ denote the set of all rd-continuous functions mapping $\mathbb{T}$ to $\mathbb{R}$. Besides the assumptions as above, we will assume further that

(H1) the forward jump operator on $\mathbb{T}$ defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$$

satisfies that $\lim_{t\to\infty} \dfrac{\sigma(t)}{t}$ exists;

(H2) $g, q \in C_{rd}(\mathbb{T}, \mathbb{T})$, $g(t) \leq t$ with $\lim_{t\to\infty} \dfrac{g(t)}{t} = 1$, $\lim_{t\to\infty} q(t) = \infty$, and there exists a sequence $\{c_m\}_{m\geq 0} \subset \mathbb{T}$ such that $\lim_{m\to\infty} c_m = \infty$ and $g(c_{m+1}) = c_m$;

(H3) $p \in C_{rd}(\mathbb{T}, \mathbb{R})$ and there exists a constant p_0 with $|p_0| < 1$ such that $\lim_{t\to\infty} p(t) = p_0$;

(H4) $f \in C(\mathbb{T} \times \mathbb{R}, \mathbb{R})$ and $xf(t,x) > 0$ for $t \in \mathbb{T}$ and $x \neq 0$. In addition, we assume that f is superlinear or sublinear. More precisely, f is said to be superlinear if

$$\frac{f(t,x)}{x} \geq \frac{f(t,y)}{y} \text{ for } x \geq y > 0 \text{ or } x \leq y < 0,$$

and sublinear if

$$\frac{f(t,x)}{x} \leq \frac{f(t,y)}{y} \text{ for } x \geq y > 0 \text{ or } x \leq y < 0.$$

Let $\mathbb{N}$ denote the set of positive integers, $h_0(t,s) \equiv 1$ on $\mathbb{T} \times \mathbb{T}$ and let functions h_k be defined recursively by

$$h_k(t,s) = \int_s^t h_{k-1}(\tau,s)\Delta\tau, k \in \mathbb{N}.$$

Then such functions h_k play a role of "polynomials" on $\mathbb{T}$ (see [3, 10]) and satisfy that

$$\begin{cases} h_k(t,s) \geq 0 \text{ for all } t \geq s \text{ and } k \in \mathbb{N}, \ h_k^{\Delta}(t,s) = h_{k-1}(t,s) \text{ for all } k \in \mathbb{N}; \\ (-1)^k h_k(t,s) \geq 0 \text{ for all } k \in \mathbb{N} \text{ and } t \leq s; \text{ and} \\ (-1)^k h_k(t,s) \text{ is decreasing in } t \text{ provided that } t \leq s, \end{cases}$$

where h_k^{Δ} is the derivative of h_k with respect to the first variable.

Now we have our first result as follows, which is similar to [16, Lemma 2.1] and [18, Theorem 7]. For completeness we supply the proof.

Lemma 1.1. *Suppose that $x(t)$ is continuous and $x(t)/h_k(t,t_0)$ is eventually positive on $\mathbb{T}$. Suppose further that $z(t) = x(t) + p(t)x(g(t))$ and $\lim_{t\to\infty}[z(t)/h_k(t,t_0)] = b$. Then $\lim_{t\to\infty}[x(t)/h_k(t,t_0)] = b/(1+p_0)$.*

Proof. First we assert that

$$\lim_{t\to\infty} \frac{h_k(t,t_0)}{t^k} \text{ exists for } k \in \mathbb{N}. \tag{1.2}$$

Indeed, it is clear that (1.2) holds for $k = 1$. Suppose that $\lim_{t\to\infty} \frac{h_k(t,t_0)}{t^k} = \mathcal{H}$ for some $k \in \mathbb{N}$ and $\lim_{t\to\infty} \frac{\sigma(t)}{t} = \mathcal{A}$ (see assumption (H1)). Then, it follows that

$$\begin{aligned} \lim_{t\to\infty} \frac{h_{k+1}(t,t_0)}{t^{k+1}} &= \lim_{t\to\infty} \frac{h_k(t,t_0)}{t^k + t^{k-1}\sigma(t) + t^{k-2}(\sigma(t))^2 + \ldots + (\sigma(t))^k} \\ &= \frac{\mathcal{H}}{1 + \mathcal{A} + \mathcal{A}^2 + \ldots + \mathcal{A}^k}, \end{aligned}$$

where we have imposed L'Hôpital's rule [1, Theorem 4] for the first step. Hence, by mathematical induction we see that the assertion (1.2) holds. Now by (1.2) we have

$$\lim_{t\to\infty}\frac{h_k(g(t),t_0)}{h_k(t,t_0)}=\lim_{t\to\infty}\left[\frac{\frac{h_k(g(t),\,t_0)}{(g(t))^k}}{\frac{h_k(t,t_0)}{t^k}}\cdot\frac{(g(t))^k}{t^k}\right]=1\ \text{ for }\ k\in\mathbb{N},$$

where $\lim_{t\to\infty}\frac{g(t)}{t}=1$ is due to the assumption (H2).

In addition, by assumptions (H2) and (H3) there exist $T_0\in\mathbb{T}$ and $|p_0|<p_1<1$ such that $x(t)>0, x(g(t))>0$ and $|p(t)|\le p_1$ for all $t\in[T_0,\infty)_{\mathbb{T}}$. Next there are two cases to consider.

(i) If b is finite, then $x(t)/h_k(t,t_0)$ is bounded on $\mathbb{T}$. Otherwise, there exists a sequence $\{t_l\}\subset\mathbb{T}$ with $t_l\ge T_0$ and $t_l\to\infty$ as $l\to\infty$ such that

$$\frac{x(t_l)}{h_k(t_l,t_0)}=\max_{t_0\le s\le t_l}\frac{x(s)}{h_k(s,t_0)}\ \text{ and }\ \lim_{l\to\infty}\frac{x(g(t_l))}{h_k(g(t_l),t_0)}=\infty.$$

Note that $g(t)\le t$, it follows that

$$\begin{aligned}\frac{z(t_l)}{h_k(t_l,t_0)} &= \frac{x(t_l)}{h_k(t_l,t_0)}+p(t_l)\frac{x(g(t_l))}{h_k(t_l,t_0)}\\ &\ge \frac{x(g(t_l))}{h_k(g(t_l),t_0)}-p_1\frac{x(g(t_l))}{h_k(g(t_l),t_0)}\cdot\frac{h_k(g(t_l),t_0)}{h_k(t_l,t_0)}\\ &\ge (1-p_1)\frac{x(g(t_l))}{h_k(g(t_l),t_0)}\\ &\to \infty\ \text{ as }\ l\to\infty,\end{aligned}$$

which contradicts our assumption for b, where we have imposed the relation

$$\frac{h_k(g(t_l),t_0)}{h_k(t_l,t_0)}\le 1$$

for the third step. Now we may assume that

$$\limsup_{t\to\infty}\frac{x(t)}{h_k(t,t_0)}=\overline{x},\ \liminf_{t\to\infty}\frac{x(t)}{h_k(t,t_0)}=\underline{x}.$$

Note that

$$\frac{z(t)}{h_k(t,t_0)}=\frac{x(t)}{h_k(t,t_0)}+p(t)\frac{x(g(t))}{h_k(g(t),t_0)}\cdot\frac{h_k(g(t),t_0)}{h_k(t,t_0)},$$

we have when $0\le p_0<1$,

$$b\ge\overline{x}+p_0\underline{x}\ \text{ and }\ b\le\underline{x}+p_0\overline{x},$$

which implies that $\overline{x} = \underline{x}$ when $0 \le p_0 < 1$. In case $-1 < p_0 < 0$, we have

$$b \ge \overline{x} + p_0\overline{x} \text{ and } b \le \underline{x} + p_0\underline{x}\ ,$$

which implies again that $\overline{x} = \underline{x}$.

Since $\lim_{t\to\infty} x(t)/h_k(t,t_0)$ exists, it is clear that $\lim_{t\to\infty} x(t)/h_k(t,t_0) = b/(1+p_0)$.

(ii) If b is infinite, then $b = -\infty$ is impossible. Otherwise, there exists a large $T_1 \in \mathbb{T}$ with $T_1 \ge T_0$ such that $z(t) < 0$ on $[T_1,\infty)_{\mathbb{T}}$. Consequently, it follows that

$$x(t) < -p(t)x(g(t)) \le p_1 x(g(t)) \text{ for } t \in [T_1,\infty)_{\mathbb{T}}. \tag{1.3}$$

Now by the assumption (H2) we can choose some positive integer m_0 such that $c_m \ge T_1$ for all $m \ge m_0$. Then, for any $m \ge m_0 + 1$, we have from (1.3) that

$$\begin{aligned} x(c_m) &< p_1 x(g(c_m)) = p_1 x(c_{m-1}) < p_1^2 x(g(c_{m-1})) = p_1^2 x(c_{m-2}) \\ &< \dots < p_1^{m-m_0} x(g(c_{m_0+1})) = p_1^{m-m_0} x(c_{m_0}), \end{aligned}$$

this induces $\lim_{m\to\infty} x(c_m) = 0$ and then $\lim_{m\to\infty} z(c_m)/h_k(c_m,t_0) = 0$, which contradicts with $b = -\infty$.

Now from

$$\frac{z(t)}{h_k(t,t_0)} \le \frac{x(t)}{h_k(t,t_0)} + p_1\frac{x(g(t))}{h_k(t,t_0)},\ t \in [T_0,\infty)_{\mathbb{T}},$$

we see that $\lim_{n\to\infty} x(t)/h_k(t,t_0) = \infty$, which completes our proof. □

The following is due to [10, Corollary 1].

Lemma 1.2. *If* $w \in C_{rd}(\mathbb{T},\mathbb{R})$ *and* $a, t \in \mathbb{T}$*, then*

$$\int_t^a \int_{\tau_k}^a \dots \int_{\tau_1}^a w(\tau)\Delta\tau\Delta\tau_1 \dots \Delta\tau_k = (-1)^k \int_t^a h_k(t,\sigma(\tau))w(\tau)\Delta\tau. \tag{1.4}$$

Let $T_0, T_1 \in \mathbb{T}$. For the sake of convenience, we will write $[T_0,\infty)_{\mathbb{T}}$ instead of $\{t \in \mathbb{T} : t \ge T_0\}$ and $[T_0,T_1]_{\mathbb{T}}$ instead of $\{t \in \mathbb{T} : T_0 \le t \le T_1\}$. Let $r_k(t) = h_k^2(t,t_0)$ for some integer $k \in \mathbb{N}$ and $r_0(t) \equiv 1$, $T_0 > 0$ and $C_{rd}[T_0,\infty)_{\mathbb{T}}$ denote all the rd-continuous functions mapping $[T_0,\infty)_{\mathbb{T}}$ into $\mathbb{R}$, and

$$BC_{rd_k}[T_0,\infty)_{\mathbb{T}} := \left\{ x : x \in C_{rd}[T_0,\infty)_{\mathbb{T}} \text{ and } \sup_{t\in[T_0,\infty)_{\mathbb{T}}} \frac{|x(t)|}{r_k(t)} < \infty \right\}. \tag{1.5}$$

When $BC_{rd_k}[T_0,\infty)_{\mathbb{T}}$ is endowed with the usual linear structure and the norm

$$||x|| = \sup_{t\in[T_0,\infty)_{\mathbb{T}}} |x(t)|/r_k(t),$$

$(BC_{rd_k}[T_0,\infty)_{\mathbb{T}}, ||\cdot||)$ is a Banach space. A set $X \subseteq BC_{rd_k}[T_0,\infty)_{\mathbb{T}}$ is said to be uniformly Cauchy if for any given $\varepsilon > 0$, there exists a $T_1 \in [T_0,\infty)_{\mathbb{T}}$ such that for every $x \in X$,

$$\left|\frac{x(t_1)}{r_k(t_1)} - \frac{x(t_2)}{r_k(t_2)}\right| < \varepsilon \quad \text{for all } t_1, t_2 \in [T_1,\infty)_{\mathbb{T}}.$$

X is said to be equicontinuous on $[a,b]_{\mathbb{T}} \subset [T_0,\infty)_{\mathbb{T}}$ if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that for every $x \in X$,

$$|x(t_1) - x(t_2)| < \varepsilon \text{ for all } t_1, t_2 \in [a,b]_{\mathbb{T}} \text{ with } |t_1 - t_2| < \delta.$$

Following the norm $||\cdot||$ defined as above, we can obtain the next result whose proof is similar to [17, Lemma 4].

Lemma 1.3. *Suppose that $X \subseteq BC_{rd_k}[T_0,\infty)_{\mathbb{T}}$ is bounded and uniformly Cauchy. Suppose further that X is equicontinuous on $[T_0,T_1]_{\mathbb{T}}$ for any $T_1 \in [T_0,\infty)_{\mathbb{T}}$. Then X is relatively compact.*

To continue our discussions, we need a standard result as follows.

Lemma 1.4 (Kranoselskii)**.** *Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Suppose further that there exist two operators $U, S : X \to \Omega$ such that*

(i) *$Ux + Sy \in X$ for all $x, y \in X$;*

(ii) *U is a contraction mapping;*

(iii) *S is completely continuous.*

Then $U + S$ has a fixed point in X.

As a special case of Lemma 1.4, we have the following result.

Corollary 1.5. *Suppose that Ω is a Banach space and X is a bounded, convex and closed subset of Ω. Suppose further that there exists an operator $S : X \to \Omega$ such that*

(i) *$Sx \in X$ for all $x \in X$;*

(ii) *S is completely continuous.*

Then S has a fixed point in X.

Set $t_{-1} = \min_{t\in\mathbb{T}}\{g(t),\ q(t)\}$. Referring to [4, 9], by a solution of (1.1) we mean a real-valued function $x(t) \in C_{rd}[t_{-1},\infty)_{\mathbb{T}}$ with $x(t)+p(t)x(g(t)) \in C^n_{rd}[t_1,\infty)_{\mathbb{T}}$, which satisfies (1.1) for all $t \geq t_1$, where $t_1 \geq \max\{t_0, t_{-1}\}$ and $C^n_{rd}[t_1,\infty)_{\mathbb{T}}$ denotes the set of functions $\varphi : [t_1,\infty)_{\mathbb{T}} \to \mathbb{R}$ with φ^{Δ^n} is rd-continuous. Among the solutions of (1.1), one is said to be nonoscillatory if it is eventually positive or eventually negative. Our attention will be restricted to eventually positive solutions since dual statements for the eventually negative solutions can be readily stated.

2 Types of Positive Solutions

For simplicity, we assume throughout that

$$z(t) = x(t) + p(t)x(g(t)).$$

Then, (1.1) reads as

$$z^{\Delta^n}(t) + f(t, x(q(t))) = 0,\ t \in \mathbb{T}. \tag{2.1}$$

Let S^+ denote the set of all eventually positive solutions of (1.1) and

$$A_k(\alpha, \beta) = \left\{ x \in S^+ : \lim_{t\to\infty} \frac{x(t)}{h_{k-1}(t, t_0)} = \alpha, \lim_{t\to\infty} \frac{x(t)}{h_k(t, t_0)} = \beta \right\}, k \in \mathbb{N},$$

$$A_0(\alpha) = \left\{ x \in S^+ : \lim_{t\to\infty} x(t) = \alpha \right\}.$$

Furthermore, let $\mathbb{N}[a, b]$ denote the integer set of the form $\{a, a+1, a+2, \dots, b\}$.

Theorem 2.1. *Suppose that $x(t)$ is an eventually positive solution of* (1.1).

(i) If n is even, then either $x \in A_0(0)$ or there exists a real number $a > 0$ and some integer $k \in \mathbb{N}[1, n/2]$ such that x belongs to $A_{2k-1}(\infty, a)$, $A_{2k-1}(\infty, 0)$ or $A_{2k-1}(a, 0)$.

(ii) If n is odd, then either $x \in A_0(\alpha)$ for some $\alpha \geq 0$ or, there exists a real number $a > 0$ and some integer $k \in \mathbb{N}[1, (n-1)/2]$ such that x belongs to $A_{2k}(\infty, a)$, $A_{2k}(\infty, 0)$ or $A_{2k}(a, 0)$.

Proof. Let n be even. By the assumption (H3), the assumption for $x(t)$ and (2.1), there exists a real number p_1 with $|p_0| < p_1 < 1$ and a sufficient large $T_0 \in \mathbb{T}$ such that $z^{\Delta^n}(t) < 0$ and $|p(t)| \leq p_1$ for all $t \in [T_0, \infty)_{\mathbb{T}}$. Then, $z(t)$ is eventually of fixed sign. Without loss of generality, we may assume that $z(t)$ is of fixed sign on $[T_0, \infty)_{\mathbb{T}}$.

If we assume that $z(t) < 0$ on $[T_0, \infty)_{\mathbb{T}}$, then

$$x(t) < -p(t)x(g(t)) \leq p_1 x(g(t)),\ t \in [T_0, \infty)_{\mathbb{T}}.$$

Note that the assumption (H2) means that there exists an integer m_0 such that $c_m \geq T_0$ for all $m \geq m_0$. Consequently we have for $m \geq m_0 + 1$,

$$\begin{aligned} x(c_m) &< p_1 x(g(c_m)) = p_1 x(c_{m-1}) < p_1^2 x(g(c_{m-1})) = p_1^2 x(c_{m-2}) \\ &< \dots < p_1^{m-m_0} x(g(c_{m_0+1})) = p_1^{m-m_0} x(c_{m_0}), \end{aligned}$$

which induces that

$$\lim_{m\to\infty} z(c_m) = 0. \tag{2.2}$$

Now we invoke Kneser's theorem on time scales [1, Theorem 5] and learn that there exists some even integer $m^* \in [0, n]$ such that eventually

$$\begin{cases} z^{\Delta^j}(t) < 0 \text{ for } j \in \mathbb{N}[0, m^*], \\ (-1)^{m^*+j} z^{\Delta^j}(t) < 0 \text{ for } j \in \mathbb{N}[m^*+1, n]. \end{cases}$$

In case $m^* = 0$, we have from (2.2) that $\lim_{t\to\infty} z(t) = 0$. In case $m^* \geq 2$, it is easy to see that $\lim_{t\to\infty} z(t) < 0$, which contradicts (2.2). That says $\lim_{t\to\infty} z(t) = 0$ when $z(t) < 0$ on $\mathbb{T}$ eventually. Hence, Lemma 1.1 implies that x belongs to $A_0(0)$.

Suppose that $z(t) > 0$ on $[T_0, \infty)_{\mathbb{T}}$. Then, by Kneser's theorem, there exists an odd integer $2k-1$ for $k \in \mathbb{N}[1, n/2]$ such that $z^{\Delta^{2k-2}}(t) > 0, z^{\Delta^{2k-1}}(t) > 0, z^{\Delta^{2k}}(t) < 0$. Thus we may assume that

$$\lim_{t\to\infty} z^{\Delta^{2k-2}}(t) = L_{2k-2}, \ \lim_{t\to\infty} z^{\Delta^{2k-1}}(t) = L_{2k-1},$$

where $0 < L_{2k-2} \leq \infty$ and $0 \leq L_{2k-1} < \infty$.

If $0 < L_{2k-1} < \infty$, then $L_{2k-2} = \infty$. By L'Hôpital's rule [1, Theorem 4] we have

$$\lim_{t\to\infty} \frac{z(t)}{h_{2k-1}(t,t_0)} = L_{2k-1}, \ \lim_{t\to\infty} \frac{z(t)}{h_{2k-2}(t,t_0)} = \infty.$$

Hence, by means of Lemma 1.1, we have

$$\lim_{t\to\infty} \frac{x(t)}{h_{2k-1}(t,t_0)} = \frac{L_{2k-1}}{1+p_0}, \ \lim_{t\to\infty} \frac{x(t)}{h_{2k-2}(t,t_0)} = \infty.$$

So x belongs to $A_{2k-1}(\infty, a)$, where $a > 0$.

In case $L_{2k-1} = 0$ and $L_{2k-2} = \infty$, similarly we have $\lim_{t\to\infty}[z(t)/h_{2k-1}(t,t_0)] = 0$ and $\lim_{t\to\infty}[z(t)/h_{2k-2}(t,t_0)] = \infty$, which implies that x belongs to $A_{2k-1}(\infty, 0)$.

In case $L_{2k-1} = 0$ and $0 < L_{2k-2} < \infty$, by the same reason as above we obtain that $\lim_{t\to\infty}[x(t)/h_{2k-1}(t,t_0)] = 0$ and $\lim_{t\to\infty}[x(t)/h_{2k-2}(t,t_0)] = L_{2k-2}/(1+p_0)$. Hence x belongs to $A_{2k-1}(a, 0)$, where $a > 0$.

When n is odd, the proof is analogous to those above and we skip it. The proof is complete. □

3 Existence Criteria for Odd n

Eventually positive solutions of (1.1) have been classified according to Theorem 2.1. Next we will establish the existence criteria for each type of solutions in $A_k(\infty, 0)$, $A_k(\infty, a)$ and $A_k(a, 0)$, $A_0(\alpha)$ where $\alpha \geq 0$ and $a > 0$.

We set $t_{-1} = \min_{t\in\mathbb{T}}\{g(t), \ q(t)\}$ and $t_1 \geq \max\{t_0, t_{-1}\}$ whenever they are defined.

Theorem 3.1. *Suppose that n is odd and f is superlinear or sublinear. If (1.1) has a solution in $A_{2k}(\infty, a)$ for $a > 0$ and $k \in \mathbb{N}[0, (n-1)/2]$, then there exits some constant $C > 0$ such that*

$$\int_{t_1}^{\infty} h_{n-2k-1}(t_1, \sigma(\tau)) f\left(\tau, C h_{2k}(q(\tau), t_0)\right) \Delta\tau < \infty. \tag{3.1}$$

The converse is also true. Here $A_{2k}(\infty, a)$ denotes $A_0(a)$ when $k = 0$.

Proof. Suppose that $x(t) \in C_{rd}[t_1, \infty)_{\mathbb{T}}$ is an eventually positive solution of (1.1) satisfying

$$\lim_{t\to\infty} \frac{x(t)}{h_{2k}(t,t_0)} = a > 0 \text{ and } \lim_{t\to\infty} \frac{x(t)}{h_{2k-1}(t,t_0)} = \infty.$$

Then there exists a $T \in \mathbb{T}$ with $T \geq t_1$ such that

$$\frac{a}{2} h_{2k}(q(t), t_0) \leq x(q(t)) \leq \frac{3a}{2} h_{2k}(q(t), t_0), t \in [T, \infty)_{\mathbb{T}}. \tag{3.2}$$

Note that

$$\lim_{t\to\infty} \frac{z(t)}{h_{2k}(t,t_0)} = (1 + p_0)a, \tag{3.3}$$

we assert that

$$\lim_{t\to\infty} z^{\Delta^{2k}}(t) = (1 + p_0)a. \tag{3.4}$$

Indeed, noticing (2.1) and (3.2) infer that $z^{\Delta^n}(t) < 0$ on $[T, \infty)$, we see that $z^{\Delta^j}(t)$ is eventually monotonic for all $j \in \mathbb{N}[0, n-1]$, which means that

$$\lim_{t\to\infty} z^{\Delta^j}(t) = L_j, \ j \in \mathbb{N}[2k, n-1],$$

where L_j may be infinite. Now it follows by L'Hôpital's rule [1, Theorem 4] that

$$\lim_{t\to\infty} \frac{z(t)}{h_{2k}(t,t_0)} = \ldots = \lim_{t\to\infty} \frac{z^{\Delta^{2k-1}}(t)}{h_1(t,t_0)} = \lim_{t\to\infty} z^{\Delta^{2k}}(t) = L_{2k},$$

which, together with (3.3), yields (3.4) holds. subsequently, (3.4) deduces

$$\lim_{t\to\infty} z^{\Delta^j}(t) = 0, \ j \in \mathbb{N}[2k+1, n-1]. \tag{3.5}$$

Now integrating (2.1) $n - 2k - 1$ times successively and invoking (3.5) in each time, we obtain

$$z^{\Delta^{2k+1}}(t) = -\int_t^\infty \int_{\tau_{n-2k-2}}^\infty \cdots \int_{\tau_1}^\infty f(\tau, x(q(\tau)))\Delta\tau\Delta\tau_1 \cdots \Delta\tau_{n-2k-2}. \tag{3.6}$$

Integrating above equation again, we get

$$z^{\Delta^{2k}}(t) - z^{\Delta^{2k}}(t_1) = -\int_{t_1}^t \int_{\tau_{n-2k-1}}^\infty \ldots \int_{\tau_1}^\infty f(\tau, x(q(\tau)))\Delta\tau\Delta\tau_1 \cdots \Delta\tau_{n-2k-1}$$

and hence

$$\begin{aligned} &\int_{t_1}^\infty \int_{\tau_{n-2k-1}}^\infty \ldots \int_{\tau_1}^\infty f(\tau, x(q(\tau)))\Delta\tau\Delta\tau_1 \ldots \Delta\tau_{n-2k-1} \\ = \ &(-1)^{n-2k-1} \int_{t_1}^\infty h_{n-2k-1}(t_1, \sigma(\tau)) f(\tau, x(q(\tau)))\Delta\tau < \infty, \end{aligned} \tag{3.7}$$

where we have imposed the equation (3.4) and Lemma 1.2.

In addition, we see from (3.2) that

$$f(t, x(q(t))) \geq f\left(t, \frac{a}{2} h_{2k}(q(t), t_0)\right), \ t \geq T$$

when f is superlinear, and

$$f(t, x(q(t))) \geq \frac{1}{3} f\left(t, \frac{3a}{2} h_{2k}(q(t), t_0)\right), \ t \geq T$$

when f is sublinear. Now it is clear that (3.7) implies that (3.1) holds since we may take $C = a/2$ when f is superlinear and $C = 3a/2$ when f is sublinear.

To see the converse holds under the condition (3.1) for some constant $C > 0$, we first consider the case $0 \leq p_0 < 1$ and proceed in steps. First of all, we take $K_c = C$ when f is superlinear and $K_c = 2C$ when f is sublinear. Set $R_{2k}(t) = h_{2k}(t, t_0)$.

Take p_1 satisfying $p_0 < p_1 < (1 + 4p_0)/5 < 1$, then $p_0 > (5p_1 - 1)/4$. From $\lim_{t \to \infty} p(t) R_{2k}(g(t))/R_{2k}(t) = p_0$ and (3.1), there exists a $T_0 \in \mathbb{T}$ with $T_0 \geq t_1$ such that

$$\frac{5p_1 - 1}{4} \leq p(t) \leq p_1 < 1, \ \ t \in [T_0, \infty)_{\mathbb{T}}, \tag{3.8}$$

$$\frac{p(t) R_{2k}(g(t))}{R_{2k}(t)} \geq \frac{5p_1 - 1}{4}, \ \ t \in [T_0, \infty)_{\mathbb{T}} \tag{3.9}$$

and

$$\begin{aligned} &\int_{T_0}^{\infty} \int_{\tau_{n-2k-1}}^{\infty} \cdots \int_{\tau_1}^{\infty} f(\tau, C R_{2k}(q(\tau))) \Delta\tau \Delta\tau_1 \ldots \Delta\tau_{n-2k-1} \\ = \ &\int_{T_0}^{\infty} h_{n-2k-1}(T_0, \sigma(\tau)) f(\tau, C R_{2k}(q(\tau))) \Delta\tau \\ \leq \ &\int_{t_1}^{\infty} h_{n-2k-1}(t_1, \sigma(\tau)) f(\tau, C R_{2k}(q(\tau))) \Delta\tau \\ \leq \ &\frac{(1 - p_1) K_c}{16}. \end{aligned} \tag{3.10}$$

Let $r_{2k}(t) = h_{2k}^2(t, t_0) = R_{2k}^2(t)$. Define the Banach space $BC_{rd_{2k}}[T_0, \infty)_{\mathbb{T}}$ as in (1.5) and let

$$X = \left\{ x \in BC_{rd_{2k}}[T_0, \infty)_{\mathbb{T}} : \frac{K_c R_{2k}(t)}{2} \leq x(t) \leq K_c R_{2k}(t) \right\}. \tag{3.11}$$

It is obvious that X is a bounded, convex and closed subset of $BC_{rd_{2k}}[T_0, \infty)_{\mathbb{T}}$. Note that the assumption (H2) implies that there exists a $T_1 \in \mathbb{T}$ with $T_1 > T_0$ such that

$$g(T_1) = T_0, \ g(t) \geq T_0 \text{ and } q(t) \geq T_0 \text{ for } t \in [T_1, \infty)_{\mathbb{T}}. \tag{3.12}$$

So, for any $x \in X$ we have

$$f(t, x(q(t))) \le 2f(t, CR_{2k}(q(t))), \quad t \in [T_1, \infty)_{\mathbb{T}}. \tag{3.13}$$

Define two operators U and $S : X \to BC_{rd_{2k}}[T_0, \infty)_{\mathbb{T}}$ by

$$(Ux)(t) = \begin{cases} \dfrac{3K_c p_1 R_{2k}(t)}{4} - \dfrac{p(T_1)x(g(T_1))R_{2k}(t)}{R_{2k}(T_1)}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3K_c p_1 R_{2k}(t)}{4} - p(t)x(g(t)), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases} \tag{3.14}$$

and

$$(Sx)(t) = \begin{cases} \dfrac{3K_c R_{2k}(t)}{4}, & t \in [T_0, T_1]_{\mathbb{T}}, \\ \dfrac{3K_c R_{2k}(t)}{4} + F_x(t), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases} \tag{3.15}$$

where

$$F_x(t) \;=\; \int_{T_1}^{t} \int_{T_1}^{\tau_{n-1}} \cdots \int_{T_1}^{\tau_{n-2k+1}} \int_{\tau_{n-2k}}^{\infty} \cdots \int_{\tau_1}^{\infty} f(\tau, x(q(\tau)))\Delta\tau\Delta\tau_1 \cdots \Delta\tau_{n-1},$$

which, combining with (3.10) and (3.13), implies that

$$F_x(t) \le \frac{(1-p_1)K_c}{8} h_{2k}(t, T_1) \le \frac{(1-p_1)K_c}{8} R_{2k}(t), \ t \in [T_1, \infty)_{\mathbb{T}}. \tag{3.16}$$

(i) We first prove that $Ux + Sy \in X$ for all $x, y \in X$. Indeed, on $[T_0, T_1]_{\mathbb{T}}$, using $g(t) \le t$ and (3.8)–(3.9) we have

$$\begin{aligned} (Ux)(t) + (Sy)(t) &= \frac{3(1+p_1)K_c R_{2k}(t)}{4} - p(T_1)\frac{x(g(T_1))R_{2k}(t)}{R_{2k}(T_1)} \\ &\ge \frac{3(1+p_1)K_c R_{2k}(t)}{4} - p_1 \frac{K_c R_{2k}(T_1) R_{2k}(t)}{R_{2k}(T_1)} \\ &\ge \frac{K_c R_{2k}(t)}{2} \end{aligned}$$

and

$$\begin{aligned} (Ux)(t) + (Sy)(t) &\le \frac{3(1+p_1)K_c R_{2k}(t)}{4} - p(T_1)\frac{K_c R_{2k}(g(T_1))R_{2k}(t)}{2R_{2k}(T_1)} \\ &= \frac{3(1+p_1)K_c R_{2k}(t)}{4} - \frac{p(T_1)R_{2k}(g(T_1))}{R_{2k}(T_1)} \cdot \frac{K_c R_{2k}(t)}{2} \\ &\le \frac{3(1+p_1)K_c R_{2k}(t)}{4} - \frac{5p_1 - 1}{4} \cdot \frac{K_c R_{2k}(t)}{2} \\ &\le K_c R_{2k}(t). \end{aligned}$$

When $t \in [T_1, \infty)_{\mathbb{T}}$ we have from (3.8)–(3.10) and (3.16) that

$$\begin{aligned}(Ux)(t) + (Sy)(t) &= \frac{3(1+p_1)K_c R_{2k}(t)}{4} - p(t)x(g(t)) + F_y(t) \\ &\geq \frac{3(1+p_1)K_c R_{2k}(t)}{4} - p_1 K_c R_{2k}(t) \\ &= \frac{(3-p_1)K_c R_{2k}(t)}{4} \geq \frac{K_c R_{2k}(t)}{2},\end{aligned}$$

$$\begin{aligned}(Ux)(t) + (Sy)(t) &\leq \frac{3(1+p_1)K_c R_{2k}(t)}{4} - \frac{p(t)K_c R_{2k}(t)}{2} + \frac{(1-p_1)K_c R_{2k}(t)}{8} \\ &\leq \frac{3(1+p_1)K_c R_{2k}(t)}{4} - \frac{5p_1 - 1}{4} \times \frac{K_c R_{2k}(t)}{2} \\ &\quad + \frac{(1-p_1)K_c R_{2k}(t)}{8} \\ &= K_c R_{2k}(t).\end{aligned}$$

That is, $Ux + Sy \in X$ for any $x, y \in X$.

(ii) Next we show that U is a contraction mapping. First we note that $\lim_{t\to\infty} |p(t)| = p_0$. For the convenience, we may deem that $|p(t)| \leq p_1$ for $t \geq T_0$. Then, for any $x, y \in X$, we have when $t \in [T_0, T_1]_{\mathbb{T}}$,

$$\begin{aligned}\frac{|(Ux)(t) - (Uy)(t)|}{R_{2k}^2(t)} &= |p(T_1)| \frac{|x(g(T_1)) - y(g(T_1))|}{R_{2k}(T_1)R_{2k}(t)} \\ &\leq \frac{|x(g(T_1)) - y(g(T_1))|}{R_{2k}^2(g(T_1))} \cdot \frac{|p(T_1)|R_{2k}^2(g(T_1))}{R_{2k}(T_1)R_{2k}(T_0)} \\ &\leq p_1 \sup_{t\geq T_0} \frac{|x(t) - y(t)|}{R_{2k}^2(t)},\end{aligned}$$

where we have used the relations $g(T_1) = T_0$ in (3.12) for the last step.

When $t \in [T_1, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}\frac{|(Ux)(t) - (Uy)(t)|}{R_{2k}^2(t)} &= |p(t)| \frac{|x(g(t)) - y(g(t))|}{R_{2k}^2(t)} \\ &\leq p_1 \frac{|x(g(t)) - y(g(t))|}{R_{2k}^2(g(t))} \\ &\leq p_1 \sup_{t\geq T_0} \frac{|x(t) - y(t)|}{R_{2k}^2(t)}.\end{aligned}$$

In a word, $||Ux - Uy|| \leq p_1 ||x - y||$ for all $x, y \in X$. Since $0 < p_1 < 1$, we see that U is contracting.

(iii) In this section we prove that S is a completely continuous mapping. From the proofs in (i), we learn immediately that $Sx \in X$ for all $x \in X$.

The continuity for S is similar to those in [18, Theorem 8]. For the completeness, we supply the proof. Let $x_n \in X$ and $||x_n - x|| \to 0$ as $n \to \infty$. Then $x \in X$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [T_0, \infty)_{\mathbb{T}}$ and hence, for any $t \in [T_1, \infty)_{\mathbb{T}}$ we have

$$|f(t, x_n(q(t))) - f(t, x(q(t)))| \to 0 \quad \text{as} \quad n \to \infty. \tag{3.17}$$

Invoking (3.13), it follows that

$$|f(t, x_n(q(t))) - f(t, x(q(t)))| \le 4f(t, CR_{2k}(q(t))). \tag{3.18}$$

By the definition S we have

$$(Sx_n)(t) - (Sx)(t) = F_{x_n} - F_x,$$

which implies that

$$\begin{aligned}
& |(Sx_n)(t) - (Sx)(t)| \\
\le\ & \int_{T_1}^{t} h_{2k-1}(t, \sigma(\tau_1))\Delta\tau_1 \times \\
& \int_{T_1}^{\infty} h_{n-2k-1}(T_1, \sigma(\tau))\, |f(\tau, x_n(q(\tau))) - f(\tau, x(q(\tau)))|\, \Delta\tau \\
\le\ & h_{2k}(t, t_0) \int_{T_1}^{\infty} h_{n-2k-1}(T_1, \sigma(\tau)) |f(\tau, x_n(q(\tau))) - f(\tau, x(q(\tau)))| \Delta\tau.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& ||(Sx_n)(t) - (Sx)(t)|| \\
\le\ & \sup_{t \ge T_0} \frac{1}{R_{2k}(t)} \int_{T_1}^{\infty} h_{n-2k-1}(T_1, \sigma(\tau)) |f(\tau, x_n(q(\tau))) - f(\tau, x(q(\tau)))| \Delta\tau,
\end{aligned}$$

which, together with (3.17) and (3.18), yields that

$$||(Sx_n)(t) - (Sx)(t)|| \to 0 \text{ as } n \to \infty,$$

where we have imposed Lebesgue's dominated convergence theorem [6, Chapter 5]. Thus, the continuity of S is complete.

Next we verify that SX satisfies all the conditions in Lemma 1.3. Since S maps X into X, we have $||Sx|| = \sup_{t \ge T_0} |(Sx)(t)/R_{2k}^2(t)| \le K_c/R_{2k}(T_0)$ for all $x \in X$ and hence SX is bounded.

For any given $\varepsilon > 0$, take $T_2 \in [T_1, \infty)_{\mathbb{T}}$ so that

$$\frac{1}{R_{2k}(t)} \le \frac{\varepsilon}{3K_c} \text{ for all } t \in [T_2, \infty)_{\mathbb{T}}.$$

Then, for any $x \in X$ and $t_1, t_2 \in [T_2, \infty)_{\mathbb{T}}$, it holds that

$$\begin{aligned} & \left| \frac{(Sx)(t_1)}{R_{2k}^2(t_1)} - \frac{(Sx)(t_2)}{R_{2k}^2(t_2)} \right| \\ \leq \; & \frac{3K_c}{4} \left(\frac{1}{R_{2k}(t_1)} + \frac{1}{R_{2k}(t_2)} \right) + \frac{(1-p_1)K_c}{8} \left(\frac{1}{R_{2k}(t_1)} + \frac{1}{R_{2k}(t_2)} \right) \\ \leq \; & \frac{\varepsilon}{2} + \frac{\varepsilon}{12} < \varepsilon. \end{aligned}$$

Thus, SX is uniformly Cauchy.

Let $T_2 \in [T_0, \infty)_{\mathbb{T}}$ be arbitrary and, without loss of generality, $T_2 > T_1$. We now show that SX is equicontinuous on $[T_0, T_2]_{\mathbb{T}}$. For any $x \in X$, we have

$$|(Sx)(t_1) - (Sx)(t_2)| = \frac{3K_c}{4} |R_{2k}(t_1) - R_{2k}(t_2)|, \; t_1, t_2 \in [T_0, T_1]_{\mathbb{T}}$$

and when $t_1, t_2 \in [T_1, T_2]_{\mathbb{T}}$,

$$\begin{aligned} & |(Sx)(t_1) - (Sx)(t_2)| \\ \leq \; & \frac{3K_c}{4} |R_{2k}(t_1) - R_{2k}(t_2)| + \frac{(1-p_1)K_c}{8} |h_{2k}(t_1, T_1) - h_{2k}(t_2, T_1)|. \end{aligned}$$

Note that since $h_{2k}(t, s)$ is continuous with respect to the first variable, we see that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that $|R_{2k}(t_1) - R_{2k}(t_2)| \leq \varepsilon$ and $|h_{2k}(t_1, T_1) - h_{2k}(t_2, T_1)| \leq \varepsilon$ when $|t_1 - t_2| \leq \delta$ on $[T_0, T_2]_{\mathbb{T}}$. This means that SX is equicontinuous on $[T_0, T_2]_{\mathbb{T}}$.

To sum up, S is a completely continuous mapping by means of Lemma 1.3. Hence, Lemma 1.4 implies that there exists $x \in X$ so that $(U + S)x = x$. Therefore, we have

$$x(t) = \frac{3(1+p_1)K_c R_{2k}(t)}{4} - p(t)x(g(t)) + F_x(t), \;\; t \in [T_1, \infty)_{\mathbb{T}}. \tag{3.19}$$

Also, we can readily verify that $x(t)$ solves (1.1) on $[T_1, \infty)_{\mathbb{T}}$. In addition, invoking (3.13) and Lemma 1.2 we have

$$\begin{aligned} F_x^{\Delta^{2k}}(t) \; & \leq \; 2 \int_t^\infty \int_{\tau_{n-2k-1}}^\infty \dots \int_{\tau_1}^\infty f(\tau, CR_{2k}(q(\tau))) \Delta\tau \Delta\tau_1 \dots \Delta\tau_{n-2k-1} \\ & \leq \; 2 \int_t^\infty h_{n-2k-1}(t, \sigma(\tau)) f(\tau, CR_{2k}(q(\tau))) \Delta\tau < \infty, \end{aligned}$$

which, together with (3.1) and L'Hôpital's rule, results in

$$\lim_{t\to\infty} \frac{F_x(t)}{R_{2k}(t)} = \lim_{t\to\infty} \frac{F_x^{\Delta}(t)}{R_{2k-1}(t)} = \dots = \lim_{t\to\infty} \frac{F_x^{\Delta^{2k}}(t)}{R_0(t)} = 0.$$

Hence, (3.19) implies that

$$\lim_{t\to\infty}\frac{z(t)}{h_{2k}(t,t_0)}=\frac{3(1+p_1)K_c}{4},\ \lim_{t\to\infty}\frac{z(t)}{h_{2k-1}(t,t_0)}=\infty.$$

This, associated with Lemma 1.1, implies that $\lim_{t\to\infty}[x(t)/h_{2k}(t,t_0)]=3(1+p_1)K_c/(4+4p_0)$ and $\lim_{t\to\infty}[x(t)/h_{2k-1}(t,t_0)]=\infty$. In summary, (1.1) has a solution in $A_{2k}(\infty,a)$ when $0\le p_0<1$.

In case $-1<p_0<0$, take p_1 so that $-p_0<p_1<(1-4p_0)/5<1$. Then $p_0<(1-5p_1)/4$. Similarly, we can choose $T_0\in\mathbb{T}$ large enough such that (3.10) holds and

$$\frac{p(t)R_{2k}(g(t))}{R_{2k}(t)}\le-\frac{5p_1-1}{4},\ \frac{5p_1-1}{4}\le-p(t)\le p_1<1,\ t\in[T_0,\infty)_{\mathbb{T}}.$$

Let $T_1\in\mathbb{T}$ with $T_1>T_0$ such that (3.12) holds. Now we introduce the Banach space $BC_{2k}[T_0,\infty)$ and its subset X as above. Define operator S as in (3.15) and operator U on X by

$$(Ux)(t)=\begin{cases}-\dfrac{3K_cp_1R_{2k}(t)}{4}-\dfrac{p(T_1)x(g(T_1))R_{2k}(t)}{R_{2k}(T_1)},\ t\in[T_0,T_1]_{\mathbb{T}},\\ -\dfrac{3K_cp_1R_{2k}(t)}{4}-p(t)x(g(t)),\ t\in[T_1,\infty)_{\mathbb{T}}.\end{cases}$$

Then we may prove in a similar manner that (1.1) has a solution in $A_{2k}(\infty,a)$ when $-1<p_0<0$. The proof is complete. □

Similarly to the proof of Theorem 3.1, we may take $R_{2k-1}(t)=h_{2k-1}(t,t_0)$ instead of $R_{2k}(t)=h_{2k}(t,t_0)$ and show that (1.1) has a solution in $A_{2k}(a,0)$. We give the result without demonstration.

Theorem 3.2. *Suppose that n is odd and f is superlinear or sublinear. If* (1.1) *has a solution in $A_{2k}(a,0)$ for $a>0$ and $k\in\mathbb{N}[1,(n-1)/2]$, then there exits some constant $C>0$ such that*

$$\int_{t_1}^{\infty}|h_{n-2k}(t_1,\sigma(\tau))|f(\tau,Ch_{2k-1}(q(\tau),t_0))\Delta\tau<\infty.$$

The converse is also true.

Theorem 3.3. *Suppose that n is odd, f is superlinear and $k\in\mathbb{N}[1,(n-1)/2]$. If* (1.1) *has a solution in $A_{2k}(\infty,0)$, then*

$$\int_{t_1}^{\infty}h_{n-2k-1}(t_1,\sigma(\tau))f(\tau,h_{2k-1}((q(\tau),t_0)))\Delta\tau<\infty,\tag{3.20}$$

$$\int_{t_1}^{\infty} |h_{n-2k}(t_1, \sigma(\tau))| f(\tau, h_{2k}(q(\tau), t_0)) \Delta\tau = \infty. \tag{3.21}$$

Conversely, if

$$\int_{t_1}^{\infty} h_{n-2k-1}(t_1, \sigma(\tau)) f(\tau, h_{2k}((q(\tau), t_0))) \Delta\tau < \infty, \tag{3.22}$$

$$\int_{t_1}^{\infty} |h_{n-2k}(t_1, \sigma(\tau))| f(\tau, h_{2k-1}(q(\tau), t_0)) \Delta\tau = \infty \tag{3.23}$$

and $\lim_{t\to\infty}[p(t)h_{2k}(t,t_0)/h_{2k-1}(t,t_0)] = 0$ *provided* $p_0 \geq 0$*, then* (1.1) *has a solution in* $A_{2k}(\infty, 0)$.

Proof. We set $R_{2k}(t) = h_{2k}(t, t_0)$ and $L_{2k-1}(t) = h_{2k-1}(t, t_0)$ throughout the following discussions. Suppose that x is a solution of (1.1) in $A_{2k}(\infty, 0)$. Then

$$\lim_{t\to\infty}[x(t)/R_{2k}(t)] = 0 \;\text{ and }\; \lim_{t\to\infty}[x(t)/L_{2k-1}(t)] = \infty$$

and hence, there exists a $T \in \mathbb{T}$ such that

$$L_{2k-1}(t) \leq x(t) \leq R_{2k}(t), \;\; t \in [T, \infty)_{\mathbb{T}}, \tag{3.24}$$

Since $q(t) \to \infty$ as $t \to \infty$, we may deem that $q(t) \geq T$ on $[T, \infty)_{\mathbb{T}}$. Furthermore, since $\lim_{t\to\infty}[z(t)/R_{2k}(t)] = 0$ and $\lim_{t\to\infty}[z(t)/L_{2k-1}(t)] = \infty$, we have

$$\lim_{t\to\infty} z^{\Delta^j}(t) = 0, \; j \in \mathbb{N}[2k, n-1]. \tag{3.25}$$

as well as

$$\lim_{t\to\infty} z^{\Delta^{2k-1}}(t) = \infty. \tag{3.26}$$

Now integrating (2.1) $n-2k$ times successively from t_1 to t and using (3.25) each time, we have

$$z^{\Delta^{2k}}(t) - z^{\Delta^{2k}}(t_1) = -\int_{t_1}^{t}\int_{\tau_{n-2k-1}}^{\infty} \dots \int_{\tau_1}^{\infty} f(\tau, x(q(\tau))) \Delta\tau\Delta\tau_1 \dots \Delta\tau_{n-2k-1}. \tag{3.27}$$

Invoking (3.24)–(3.25), together with Lemma 1.2, we find that (3.27) fetches (3.20).

Invoking (3.27) again, we gain that

$$z^{\Delta^{2k}}(t) = \int_{t}^{\infty}\int_{\tau_{n-2k-1}}^{\infty} \dots \int_{\tau_1}^{\infty} f(\tau, x(q(\tau))) \Delta\tau\Delta\tau_1 \dots \Delta\tau_{n-2k-1},$$

which, integrating from t_1 to t, gives

$$z^{\Delta^{2k-1}}(t) - z^{\Delta^{2k-1}}(t_1) = \int_{t_1}^{t}\int_{\tau_{n-2k}}^{\infty} \dots \int_{\tau_1}^{\infty} f(\tau, x(q(\tau))) \Delta\tau\Delta\tau_1 \dots \Delta\tau_{n-2k}.$$

This, associating with (3.26), induces

$$\int_{t_1}^{\infty}\int_{\tau_{n-2k}}^{\infty}\cdots\int_{\tau_1}^{\infty} f(\tau, x(q(\tau)))\Delta\tau\Delta\tau_1\ldots\Delta\tau_{n-2k} = \infty.$$

Now by the right side of (3.24) and Lemma 1.2, it is clear that (3.21) holds.

Conversely, we note first that $h_{2k}(t,t_0) > 4h_{2k-1}(t,t_0)$ eventually. Suppose that $0 \le p_0 < 1$ and

$$\lim_{t\to\infty}\frac{p(t)R_{2k}(t)}{L_{2k-1}(t)} = 0.$$

Take p_1 so that $p_0 < p_1 < 1$. Then, by the same ways as the proof in Theorem 3.1, there exist T_0 and T_1 with $T_1 > T_0$ such that

$$g(t) \ge T_0,\ \ q(t) \ge T_0 \ \text{ for } \ t \in [T_1,\infty)_{\mathbb{T}}, \tag{3.28}$$

$$4L_{2k-1}(t) \le R_{2k}(t),\ |p(t)| \le \frac{|p(t)|R_{2k}(t)}{L_{2k-1}(t)} \le p_1,\ t \in [T_0,\infty)_{\mathbb{T}},$$

and

$$\int_{T_0}^{\infty}\int_{\tau_{n-2k-1}}^{\infty}\cdots\int_{\tau_1}^{\infty} f(\tau, R_{2k}(q(\tau)))\Delta\tau\Delta\tau_1\ldots\Delta\tau_{n-2k-1} \le \frac{1-p_1}{8}, \tag{3.29}$$

where the last relation is due to (3.22).

Define the Banach space as in (1.5) with $r_{2k}(t) = R_{2k}^2(t)$ and let

$$X = \{x \in BC_{rd_{2k}}[T_0,\infty)_{\mathbb{T}} : L_{2k-1}(t) \le x(t) \le R_{2k}(t)\}.$$

Define two operators by

$$(Ux)(t) = \begin{cases} \dfrac{1}{2}L_{2k-1}(t), & t \in [T_0,T_1)_{\mathbb{T}},\\[2mm] \dfrac{3p_1}{2}L_{2k-1}(t) - p(t)x(g(t)), & t \in [T_1,\infty)_{\mathbb{T}}, \end{cases}$$

and

$$(Sx)(t) = \begin{cases} \dfrac{1}{2}L_{2k-1}(t), & t \in [T_0,T_1)_{\mathbb{T}},\\[2mm] \dfrac{3}{2}L_{2k-1}(t) + F_x(t), & t \in [T_1,\infty)_{\mathbb{T}}. \end{cases}$$

where $F_x(t)$ is defined as in (3.15). Then, when $t \in [T_1,\infty)_{\mathbb{T}}$, it is easy to see that for any $x, y \in X$,

$$\begin{aligned}(Ux)(t) + (Sy)(t) &= \frac{3+3p_1}{2}L_{2k-1}(t) - p(t)x(g(t)) + F_y(t)\\ &\le \left[\frac{3+3p_1}{2} + \frac{|p(t)|R_{2k}(t)}{L_{2k-1}(t)}\right]L_{2k-1}(t) + \frac{1-p_1}{8}R_{2k}(t)\\ &\le R_{2k}(t)\end{aligned}$$

and

$$\begin{aligned}(Ux)(t)+(Sy)(t) &\geq \left[\frac{3+3p_1}{2}-\frac{|p(t)|R_{2k}(t)}{L_{2k-1}(t)}\right]L_{2k-1}(t)\\ &\geq L_{2k-1}(t).\end{aligned}$$

Similarly to the proof in Theorem 3.1, there exists $x \in X$ such that

$$x(t)=\frac{3(1+p_1)}{4}L_{2k-1}(t)-p(t)x(g(t))+F_x(t),\ \ t\in[T_1,\infty)_{\mathbb{T}}. \tag{3.30}$$

Using $\lim_{t\to\infty}[L_{2k-1}(t)/R_{2k}(t)]=0$, together with (3.22) and (3.23), equation (3.30) implies that $\lim_{t\to\infty}[z(t)/R_{2k}(t)]=0$ and $\lim_{t\to\infty}[z(t)/L_{2k-1}(t)]=\infty$. Hence, by Lemma 1.1 we are convinced that (1.1) has a solution in $A_{2k}(\infty,0)$ when $0\leq p_0<1$.

When $-1<p_0<0$, take p_1 satisfying $-p_0<p_1<(1-4p_0)/5<1$ and $T_0<T_1$ so that (3.28) and (3.29) come into existence and

$$0<-p(t)\leq p_1,\ 4L_{2k-1}(t)\leq R_{2k}(t),\ \frac{p(t)L_{2k-1}(g(t))}{L_{2k-1}(t)}\leq-\frac{5p_1-1}{4},\ t\in[T_0,\infty)_{\mathbb{T}}.$$

Let U be replaced by

$$(Ux)(t)=\begin{cases}\frac{1}{2}L_{2k-1}(t), & t\in[T_0,T_1)_{\mathbb{T}},\\ -\frac{3p_1}{2}L_{2k-1}(t)-p(t)x(g(t)), & t\in[T_1,\infty)_{\mathbb{T}}.\end{cases}$$

The remainder is similar to those as above so we skip it. The proof is complete. □

Let $\mathbb{R}^+$ denote the set of positive real numbers. The next two results are concerned with the solutions of (1.1) in $A_0(0)$. We only give a brief proof for the first one.

Theorem 3.4. *Suppose that n is odd and f is superlinear. Suppose further there exists a function $\lambda:\mathbb{T}\to\mathbb{R}^+$ with $\lambda(t)<\frac{1}{t}$ for $t>0$ and a $T_0\in\mathbb{T}$ with $T_0>0$ such that*

$$p(t)\lambda(g(t))\leq-\lambda(t),\ \ t\in[T_0,\infty)_{\mathbb{T}}$$

and

$$\int_t^{\infty}h_{n-1}(t,\sigma(\tau))f\left(\tau,\frac{1}{q(\tau)}\right)\Delta\tau\leq\frac{1}{t}+\frac{p(t)}{g(t)},\ \ t\in[T_0,\infty)_{\mathbb{T}}.$$

Then equation (1.1) *has a solution in* $A_0(0)$.

Proof. We may take $T_1>T_0$ in $\mathbb{T}$ so that $g(t)\geq T_0$ and $q(t)\geq T_0$ on $[T_1,\infty)_{\mathbb{T}}$. Define the Banach space as in (1.5) with $r_0(t)=1$ and then let

$$\begin{aligned}X \ =\ &\Big\{x\in BC_{rd_0}[T_0,\infty)_{\mathbb{T}}:\lambda(v(t))\leq x(v(t))\leq\frac{1}{v(t)}\text{ for }t\in[T_1,\infty)_{\mathbb{T}}\text{ and}\\ &\lambda(T_1)\leq x(t)\leq\frac{1}{t}\ \text{ for }\ t\in[T_0,T_1]_{\mathbb{T}}\Big\},\end{aligned}$$

where $v(t)$ is one of $g(t)$ and $q(t)$.

Define an operator S on X by

$$(Sx)(t) = \begin{cases} -p(T_1)x(g(T_1)) + F_x(T_1), & t \in [T_0, T_1]_{\mathbb{T}}, \\ -p(t)x(g(t)) + F_x(t), & t \in [T_1, \infty)_{\mathbb{T}}, \end{cases}$$

where $F_x(t)$ defined by

$$F_x(t) = \int_t^\infty h_{n-1}(t, \sigma(\tau)) f(\tau, x(q(\tau)))\Delta\tau.$$

The straightforward verifications indicate that all the conditions in Corollary 1.5 are satisfied and thus, there exists $x \in X$ such that

$$x(t) + p(t)x(g(t)) = \int_t^\infty h_{n-1}(t, \sigma(\tau)) f(\tau, x(q(\tau)))\Delta\tau, \ \ t \in [T_1, \infty)_{\mathbb{T}}.$$

By Lemma 1.2, we see that x is an eventually positive solutions of (1.1). Furthermore, by $\lim\limits_{t\to\infty} z(t) = 0$ we have $\lim\limits_{t\to\infty} x(t) = 0$. The proof is complete. □

Quoting the same ways as the proof of Theorem 3.4, it follows that

Theorem 3.5. *Suppose that n is odd and f is superlinear. Suppose further there exists a function $\lambda : \mathbb{T} \to \mathbb{R}^+$ with $\lambda(t) < \dfrac{1}{t}$ for $t > 0$, a $T_0 \in \mathbb{T}$ with $T_0 > 0$ and a constant $K > 0$ such that*

$$0 \le p(t) \le Kg(t)\lambda(t), \ \ t \in [T_0, \infty)_{\mathbb{T}},$$

$$\int_t^\infty h_{n-1}(t, \sigma(\tau)) f\left(\tau, \lambda(q(\tau))\right)\Delta\tau \ge (K+1)\lambda(t), \ \ t \in [T_0, \infty)_{\mathbb{T}},$$

and

$$\int_t^\infty h_{n-1}(t, \sigma(\tau)) f\left(\tau, \frac{1}{q(\tau)}\right)\Delta\tau \le \frac{1}{t}, \ \ t \in [T_0, \infty)_{\mathbb{T}}.$$

Then equation (1.1) *has a solution in* $A_0(0)$.

We remark that the conclusions similar to Theorem 3.3–3.5 can been established when f is sublinear.

4 Existence Criteria for Even n

When n is even and $k \in \mathbb{N}[1, n/2]$, similar manners as above section can be used to prove that (1.1) has a solution in $A_{2k-1}(\infty, a)$, $A_{2k-1}(a, 0)$, $A_{2k-1}(\infty, 0)$ or $A_0(0)$. We will only sketch out the proof of Theorem 4.4 and ignore the others.

Theorem 4.1. *Suppose that n is even and f is superlinear or sublinear. If* (1.1) *has a solution in $A_{2k-1}(\infty, a)$ for $a > 0$ and $k \in \mathbb{N}[1, n/2]$, then there exits some constant $C > 0$ such that*

$$\int_{t_1}^{\infty} h_{n-2k}(t_1, \sigma(\tau)) f(\tau, C h_{2k-1}(q(\tau), t_0)) \Delta\tau < \infty.$$

The converse is also true.

Theorem 4.2. *Suppose that n is even and f is superlinear or sublinear. If* (1.1) *has a solution in $A_{2k-1}(a, 0)$ for $a > 0$ and $k \in \mathbb{N}[1, n/2]$, then there exits some constant $C > 0$ such that*

$$\int_{t_1}^{\infty} |h_{n-2k+1}(t_1, \sigma(\tau))| f(\tau, C h_{2k-2}(q(\tau), t_0)) \Delta\tau < \infty.$$

The converse is also true.

Theorem 4.3. *Suppose that n is even, f is superlinear and $k \in \mathbb{N}[1, n/2]$. If* (1.1) *has a solution in $A_{2k-1}(\infty, 0)$, then*

$$\int_{t_1}^{\infty} h_{n-2k}(t_1, \sigma(\tau)) f(\tau, h_{2k-2}((q(\tau), t_0))) \Delta\tau < \infty,$$

$$\int_{t_1}^{\infty} |h_{n-2k+1}(t_1, \sigma(\tau))| f(\tau, h_{2k-1}(q(\tau), t_0)) \Delta\tau = \infty.$$

Conversely, if

$$\int_{t_1}^{\infty} h_{n-2k}(t_1, \sigma(\tau)) f(\tau, h_{2k-1}((q(\tau), t_0))) \Delta\tau < \infty,$$

$$\int_{t_1}^{\infty} |h_{n-2k+1}(t_1, \sigma(\tau))| f(\tau, h_{2k-2}(q(\tau), t_0)) \Delta\tau = \infty$$

and $\lim_{t\to\infty} [p(t) h_{2k-1}(t, t_0)/h_{2k-2}(t, t_0)] = 0$ *provided* $p_0 \geq 0$*, then* (1.1) *has a solution in* $A_{2k-1}(\infty, 0)$.

Theorem 4.4. *Suppose that n is even and f is superlinear. Suppose further there exists a function $\lambda : \mathbb{T} \to \mathbb{R}^+$ with $\lambda(t) < \dfrac{1}{t}$ for $t > 0$ and a $T_0 \in \mathbb{T}$ with $T_0 > 0$ such that*

$$p(t)t \geq -g(t), \quad t \in [T_0, \infty)_{\mathbb{T}},$$

$$p(t)\lambda(g(t)) < -\lambda(t), \quad t \in [T_0, \infty)_{\mathbb{T}}$$

and

$$\int_{t}^{\infty} |h_{n-1}(t, \sigma(\tau))| f\left(\tau, \frac{1}{q(\tau)}\right) \Delta\tau \leq \left\{-p(t)\frac{\lambda(g(t))}{\lambda(t)} - 1\right\} \lambda(t), \quad t \in [T_0, \infty)_{\mathbb{T}},$$

then equation (1.1) has a solution in $A_0(0)$.

Proof. Take X as in Theorem 3.4 and define S by

$$(Sx)(t) = \begin{cases} -p(T_1)x(g(T_1)) - F_x(T_1), & t \in [T_0, T_1]_{\mathbb{T}}, \\ -p(t)x(g(t)) - F_x(t), & t \in [T_1, \infty)_{\mathbb{T}}. \end{cases}$$

Then, with the same manners as the proof in Theorem 3.4, we can obtain an $x \in X$ solving (1.1) on $[T_1, \infty)_{\mathbb{T}}$ and satisfying $\lim_{t\to\infty} x(t) = 0$. The proof is complete. □

When $p(t)$ is eventually nonnegative and n is even, the fact is different from Theorem 3.5. Indeed, if, for the time being, (1.1) has an eventually positive solution $x(t)$ satisfying

$$\lim_{t\to\infty} x(t) = 0,$$

then, $\lim_{t\to\infty} z(t) = 0$. Moreover, $z(t) > 0$ and $z^{\Delta^n}(t) < 0$ eventually. Therefore we have

$$\lim_{t\to\infty} z^{\Delta^j}(t) = 0, \ j \in \mathbb{N}[1, n-1].$$

Now we assume that

$$x(g(t)) > 0, x(q(t)) > 0, p(t) \geq 0, \ \ t \in [T_0, \infty)_{\mathbb{T}}.$$

Then from (2.1) it comes into being that

$$x(t) = -p(t)x(g(t)) - \int_t^\infty \int_{\tau_{n-1}}^\infty \cdots \int_{\tau_1}^\infty \int_\tau^\infty f(\tau, x(q(\tau)))\Delta\tau\Delta\tau_1 \ldots \Delta\tau_{n-1}, t \geq T_0,$$

which results in $x(t) < 0$ on $[T_0, \infty)_{\mathbb{T}}$ and arrives at a contradiction. Immediately we obtain our last result as follows.

Theorem 4.5. *Suppose that n is even and f is superlinear or sublinear. If $p(t)$ is eventually nonnegative, then equation* (1.1) *has no solutions in $A_0(0)$.*

5 Examples

Example 5.1. Let $[t]$ stand for the integral part of the real number t. Further, let $\mathbb{T} = \cup_{k=0}^\infty [2k, 2k+1]$ and ρ be a backward jump operator defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

Consider equation

$$\left[x(t) + \frac{(-1)^{[t]}}{t} x(\rho(t))\right]^{\Delta^3} + \frac{h_2(\sigma(t), 0)}{\sigma^6(t)x(\sigma(t))} = 0, \ \ t \in \mathbb{T}, \tag{5.1}$$

then $f(t,x) = h_2(\sigma(t),0)/[\sigma^6(t)x]$ is sublinear, $\sigma(t)/t \to 1$ and $\rho(t)/t \to 1$ as $t \to \infty$. So the assumptions (H1)–(H4) are satisfied. Furthermore,

$$\int_1^\infty h_0(1,\sigma(\tau))f(\tau, h_2(\sigma(\tau),0))\Delta\tau \leq 1,$$

$$\int_1^\infty h_2(1,\sigma(\tau))f(\tau,1)\Delta\tau \leq \int_1^\infty \frac{(\sigma(\tau)-1)^2(\sigma(\tau)-0)^2}{\sigma^6(\tau)}\Delta\tau \leq 1,$$

as well as

$$\int_1^\infty |h_1(1,\sigma(\tau))| f(\tau, h_1(\sigma(\tau),0))\Delta\tau \leq 1.$$

Now by making use of Theorem 3.1–3.2 we see that (5.1) has a solution in $A_2(\infty,a)$, $A_0(a)$ and $A_2(a,0)$, respectively.

Example 5.2. Let $c > 0$ and $\mathbb{T} = \{kc : k \in \mathbb{N}\}$. Then $h_2(t,s) = (t-s)(t-s-c)/2$. Consider the following equation

$$[x(t) + p(t)x(\rho(t))]^{\Delta^4} + \frac{x(\sigma(t))}{t^3} = 0, \ \ t \in \mathbb{T}, \tag{5.2}$$

where $p(t)$ satisfies the assumption (H3) and $f(t,x) = x/t^3$ is superlinear. Then, referring to [6, Chapter 5] we have

$$\int_c^\infty h_2(c,\sigma(\tau))f(\tau, h_0(\sigma(\tau),c))\Delta\tau = \infty$$

as well as

$$\int_c^\infty h_0(c,\sigma(\tau))f(\tau, h_2(\sigma(\tau),c))\Delta\tau = \infty.$$

Now we see that Theorem 4.3 implies (5.2) has neither a solution in $A_1(\infty,0)$ nor in $A_3(\infty,0)$.

Example 5.3. Let $\mathbb{T} = [0,\infty)$, a half-axis, and consider the following equation

$$(x(t) + p(t)x(g(t)))^{\Delta^n} + 2e^{-\frac{3+(-1)^n}{4}t}x\left(\frac{t}{2}\right) = 0, \ \ t \geq 0, \tag{5.3}$$

where $n \in \{2,3\}$, $q(t) = t/2$ and $f(t,x) = 2xe^{-\frac{3+(-1)^n}{4}t}$. Let $p(t) = (t-1)e^{-t}$ and $g(t) = t-1$ for $n = 3$, $p(t) = -e^{-\sqrt{t}}$ and $g(t) = t - 2\sqrt{t}$ for $n = 2$. Then the assumptions (H1)-(H4) are all satisfied. In addition, we have, when $n = 3$,

$$p(t) = g(t)e^{-t},$$

$$\int_t^\infty h_2(t,\sigma(\tau))f(\tau, e^{-q(t)})\Delta\tau = 2e^{-t}$$

and

$$\int_t^\infty h_2(t,\sigma(\tau)) f\left(\tau, \frac{1}{q(\tau)}\right)\Delta\tau \le \frac{32e^{-\frac{t}{2}}}{t},\ t>0.$$

Hence, if we let $\lambda(t)=e^{-t}$, then Theorem 3.5 implies that (5.3) has a solution in $A_0(0)$ as $n=3$.

For the case $n=2$, it follows from $p(t)\to 0$ and $g(t)/t\to 1$ as $t\to\infty$ that

$$p(t)t \ge -g(t) \text{ for eventually large } t\in\mathbb{T}.$$

Further, we have

$$p(t)e^{-g(t)} < -e^{-t} \text{ for } t>0,$$

$$\int_t^\infty |h_1(t,\sigma(\tau))| f\left(\tau, \frac{1}{q(\tau)}\right)\Delta\tau \le e^{-t} \text{ for } t\ge 4$$

as well as

$$\left[-p(t)e^{t-g(t)}-1\right]e^{-t} \ge e^{-t} \text{ for } t\ge 1.$$

Thus, Theorem 4.4 implies that, when $n=2$, (5.3) has a solution in $A_0(0)$.

Acknowledgment

The author is very grateful to ADSA Associate Editor Professor Elena Braverman and the anonymous reviewers for their valuable comments and suggestions.

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