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Generalized Time Scales

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Abstract

Calculus on time scales was established in 1988 by Stefan Hilger. It includes both the classical derivative and the forward difference operator as special cases. It also includes Riemann integrals and finite sums as inverse operations. However, it does not include the Jackson q-difference operator on $\mathbb{R}$ and the Jackson q-integral on $\mathbb{R}$. Also, it does not include neither the difference operator on $\mathbb{R}$ and associated Nörlund sums nor the Hahn difference operator. This paper extends the definition of time scales in a way such that the previous difference operators and their associated sums are included. In addition, new illustrative examples will be given.

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1 The Classical Time Scales Calculus

There are several attempts to unify continuous and discrete mathematics. One of the main approaches is the time scale setting which was founded by Stefan Hilger in his Ph. D. thesis [24], see also [11]. It gives an efficient tool to unify continuous and discrete problems in one theory. Since then hundreds of papers appeared in the theory and its applications to dynamic equations, see e.g., the very interesting monographs of Bohner and Peterson [13, 14]. At the beginning a time scale is defined to be an arbitrary closed subset of the real numbers $\mathbb{R}$, with the standard inherited topology. Examples of time scales include the real numbers $\mathbb{R}$, the natural numbers $\mathbb{N}$, the integers $\mathbb{Z}$, the Cantor set, and any finite union of closed intervals of $\mathbb{R}$. Next, we review a few basics of the time scales calculus.

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For a time scale $\mathbb{T}$, the *forward jump operator* $\sigma : \mathbb{T} \longrightarrow \mathbb{T}$ is defined by $\sigma(t) := \inf\{z \in \mathbb{T} : z > t\}$ and the *backward jump operator* $\rho : \mathbb{T} \longrightarrow \mathbb{T}$ is defined by $\rho(t) := \sup\{z \in \mathbb{T} : z < t\}$. Here, we put $\inf \phi = \sup \mathbb{T}$ and $\sup \phi = \inf \mathbb{T}$. Also, the *forward stepsize function*, $\mu : \mathbb{T} \longrightarrow [0, \infty)$, is defined by $\mu(t) := \sigma(t) - t$ and the *backward stepsize function* $\nu : \mathbb{T} \longrightarrow [0, \infty)$, $\nu(t) := t - \rho(t)$, see [2, 13, 14]. In [13,14,26] the stepsize functions are called graininess functions. Here, we follow the terminology of Ahlbrandt, Bohner and Ridenhour [3].

A point $t \in \mathbb{T}$ with $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, $\rho(t) < t < \sigma(t)$ and $\rho(t) = t = \sigma(t)$ is called *right dense*, *right scattered*, *left dense*, *left scattered*, *isolated* and *dense*, respectively.

A real valued function f on $\mathbb{T}$ is called *regulated* on $\mathbb{T}$ if its right hand limits exist at all right dense points of $\mathbb{T} \setminus \{\sup \mathbb{T}\}$, and left hand limits exist at all left dense points of $\mathbb{T} \setminus \{\inf \mathbb{T}\}$. It is called *right dense continuous*, or just *rd-continuous*, on $\mathbb{T}$ if it is regulated on $\mathbb{T}$ and continuous at all right dense points of $\mathbb{T}$.

For any time scale $\mathbb{T}$, the subset $\mathbb{T}^k$ is defined by $\mathbb{T}^k := \{t \in \mathbb{T} : t \neq \sup \mathbb{T}$ or t is left dense$\}$. The *delta derivative* of $f : \mathbb{T} \longrightarrow \mathbb{R}$ at $t \in \mathbb{T}^k$ is then defined as the number $f^{\Delta}(t)$ (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(r) - f^{\Delta}(t)(\sigma(t) - r)| \leq \varepsilon |\sigma(t) - r| \quad \text{for all } r \in U,$$

cf. [13,14,25]. If $f^{\Delta}(t)$ exists for all $t \in \mathbb{T}^k$, we say that f is Δ-differentiable on $\mathbb{T}^k$.

There are many different ways to define integration on a time scale. For example, we can use the Cauchy, Riemann or Lebesgue integrals, among other concepts. Of most importance to this work is the Cauchy integral. First, for a function $f : \mathbb{T} \longrightarrow \mathbb{R}$, we say F is a Δ-antiderivative of f on $\mathbb{T}$, if $F^{\Delta}(t) = f(t)$ for all $t \in \mathbb{T}^k$. Hilger, in [25], shows that any rd-continuous function f on $\mathbb{T}$ has a Δ-antiderivative F on $\mathbb{T}$. He then defined the Δ-integral of f by

$$\int_a^b f(t)\Delta t := F(b) - F(a),$$

see [13, 25]. The classical time scales calculus includes three important special cases: $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \omega\mathbb{Z} := \{k\omega : k \in \mathbb{Z}\}$, $\omega > 0$ and $\mathbb{T} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$. One can check that in these cases we have $f^{\Delta}(t) = f'(t)$, $f^{\Delta}(t) = \Delta_\omega f(t)$ and $f^{\Delta}(t) = D_q f(t)$, respectively, where $\Delta_\omega f(t) = \dfrac{f(t+\omega) - f(t)}{\omega}$ is the forward difference operator with stepsize ω [32], and $D_q f(t) = \dfrac{f(qt) - f(t)}{t(q-1)}$ (for $t \neq 0$) is the q-difference operator [10]. There are many examples and applications of time scales, see [13, 14].

2 Why is Generalization Important?

In this work, we propose a generalization of the definition of time scale given by Stefan Hilger. The natural question here is that why generalization? In fact the theory outlined above does not include some important dynamical problems. The first problem concerns Jackson q-difference operator (q stands for quantum) and the associated Jackson q-integral. Jackson q-difference operator D_q is defined to be

$$D_q f(t) = \frac{f(qt) - f(t)}{t(q-1)}, \quad t \in \mathbb{R} \setminus \{0\} \tag{2.1}$$

where q is a fixed number, normally taken to lie in $(0,1)$. The derivative at zero is normally defined to be $f'(0)$, provided that $f'(0)$ exists, see [8, 16, 27, 28]. Jackson also introduced q-integrals

$$\int_0^a f(t) d_q t = \sum_{k=0}^{\infty} a q^k (1-q) f(a q^k), \tag{2.2}$$

and

$$\int_a^{\infty} f(t) d_q t = \sum_{k=1}^{\infty} a q^{-k} (1-q) f(a q^{-k}), \quad a \in \mathbb{R}, \tag{2.3}$$

provided that the series converge [4, 29]. He then defined

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad a, b \in \mathbb{R}. \tag{2.4}$$

There is no unique canonical choice for the q-integral over $[0,\infty)$ and the q-integral on $\mathbb{R}$. Following Jackson, we define the q-integral over $[0,\infty)$ to be

$$\int_0^{\infty} f(t) d_q t = (1-q) \sum_{k=-\infty}^{\infty} q^k f(q^k), \tag{2.5}$$

[21, 34]. Matsuo [37] defined the q-integral on more general sequences by

$$\int_0^{s \cdot \infty} f(t) d_q t = s(1-q) \sum_{k=-\infty}^{\infty} q^k f(s q^k), \tag{2.6}$$

where s is a fixed positive number, see also [33, 34]. The bilateral q-integral is defined in [33] by

$$\int_{-s \cdot \infty}^{s \cdot \infty} f(t) d_q t = s(1-q) \sum_{k=-\infty}^{\infty} \Big[q^k f(s q^k) + q^k f(-s q^k) \Big], \quad s > 0, \tag{2.7}$$

provided that the sums of (2.5), (2.6) and (2.7) converge. So, the definitions in (2.6) and (2.7) are dependent on s. This calculus which is based on the Jackson q-difference operator and the associated Jackson q-integral, is the most common tongue of quantum calculus.

The second important problem which cannot be obtained from the calculus of time scales, concerns the forward difference operator

$$\Delta_\omega f(t) = \frac{f(t+\omega) - f(t)}{\omega}, \quad t \in \mathbb{R}, \tag{2.8}$$

where ω is a fixed positive number, see [12, 30–32]. The associated integral of (2.8) is the well known Nörlund sum

$$\int f(t)\Delta_\omega t = -\omega \sum_{j=0}^{\infty} f(t + j\omega), \tag{2.9}$$

see [20, 30, 38]. This sum is called in [31] an indefinite sum and Carmichael [17] called it just a sum.

Another problem not included in Hilger's treatment is related to the Hahn difference operator $D_{q,\omega}$ which is defined by

$$D_{q,\omega} f(t) = \frac{f(tq+\omega) - f(t)}{t(q-1)+\omega}, \;\; t \in \mathbb{R} \setminus \{\omega_0\}, \tag{2.10}$$

where $q \in (0,1)$, $\omega > 0$ and $\omega_0 = \dfrac{\omega}{1-q}$. This operator was introduced by Hahn [22] in (1949), see also [7, 19, 23, 35]. Some recent literature have applied these operators to construct families of orthogonal polynomials as well as to investigate some approximation problems, cf. [7, 15, 18, 19, 35]. In [5], the author introduced the q,ω-integral of f from a to b,

$$\int_a^b f(t)d_{q,\omega}t := \int_{\omega_0}^b f(t)d_{q,\omega}t - \int_{\omega_0}^a f(t)d_{q,\omega}t, \tag{2.11}$$

where

$$\int_{\omega_0}^x f(t)d_{q,\omega}t := (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(xq^k + \omega[k]_q), \qquad x \in \mathbb{R}, \tag{2.12}$$

provided that the series converges at $x = a$ and $x = b$. Also, a q,ω-calculus based on (2.10), (2.11) and (2.12) was introduced in [9]. The $D_{q,\omega}$ includes as particular cases the q-difference operator in (2.1) and the forward difference operator in (2.8). Our aim is to define a generalized time scale $\mathbb{T}$ and a Δ-differential operator of functions defined on $\mathbb{T}$ such that all difference operators, mentioned above, are included. Afterwards we derive the corresponding calculus.

3 Generalized Time Scales

In the following, we generalize the classical definition of a time scale [24]. To achieve our goal we generalize and extend time scales calculus to include the three mentioned above problems together with the case of differential-difference equations. We use the terminology and notation of the classical time scales calculus as far as possible. Nonetheless, we opted to use “forward and backward” instead of “right and left”.

We start with some definitions and introduce notation and terminology that will be used in the sequel.

Assume that $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ and E is an equivalence relation on $\mathbb{T}$. Suppose also that $\mathbb{T}$ has the topology inherited from the standard topology on $\mathbb{R}$. $\overline{A}$, as usual, denotes the closure of A and A' the set of all limit points of A, $A \subseteq \mathbb{T}$. For $a, b \in \mathbb{T} \cup \{\inf \mathbb{T}, \sup \mathbb{T}\}$ and $c \in \mathbb{T}$, let

- $T_c := \big\{t \in \mathbb{T} : c \in \overline{[t]}\big\}$,
- $[a, b] := \big\{x \in \mathbb{T} : a \leq x \leq b$ or $b \leq x \leq a\big\}$,
- $[a, b]_c := [a, b] \cap \overline{[c]}$,
- $\mathbb{S} := \big\{s \in \mathbb{T} : T_s \neq [s]\big\}$,
- $\alpha \in \mathbb{S} \cup \{\inf \mathbb{T}, \sup \mathbb{T}\}$ be fix,

where $[t]$ denotes the equivalence class that contains $t \in \mathbb{T}$. The points of $\mathbb{S}$ are called *special* and T_t is called the *domain set* of t. We define open and half-open intervals similarly. Whenever we say a neighborhood of c in a set $A \subseteq \mathbb{T}$ we mean the set $A \cap U$, where U is a neighborhood of c in $\mathbb{R}$. In particular, if $A = \mathbb{T}$ we say a neighborhood of c without mentioning $\mathbb{T}$.

One can see that $T_c = [c] \cup \big\{t \in \mathbb{T} : c \in \overline{[t]} \setminus [t]\big\} = \cup\big\{[t] \subseteq \mathbb{T} : c \in \overline{[t]}\big\}$ for $c \in \mathbb{T}$ and $s \in \mathbb{S}$ iff there is $t \in \mathbb{T}$ such that $s \in \overline{[t]} \setminus [t]$. Notice also that, in this setting, if $a = \infty$ or $a = -\infty$, then $[a, b] = (a, b] = [b, a)$.

Definition 3.1. A class $[t]$ is called *singular* if $[t] = \{t\}$, otherwise it is called *nonsingular*. An interval I of $\mathbb{T}$ is called *nonsingular* if I is the union of some nonsingular classes which have the same infimum and supremum. Finally, we denote by $\mathscr{I}$ the family of all nonsingular intervals of $\mathbb{T}$.

Note that $\mathscr{I}$ is a family of pairwise disjoint sets. Indeed, if $I, J \in \mathscr{I}$ and $x \in I \cap J$, then $[x] \neq \{x\}$, $[x] \subseteq I$ and $[x] \subseteq J$ such that $\inf[x] = \inf I = \inf J$ and $\sup[x] = \sup I = \sup J$ which implies that $I = J$.

Definition 3.2. Let A, B be subsets of $\mathbb{T}$. We say that:

- A is *right contiguous* to B, and denote it by $A \gg B$, if $A \neq B$ and $\inf A = \sup B$.

- A is *to the right* of B, and denote it by $A > B$, if $A \neq B$ and $\inf A \geq \sup B$.
- A is *alternating* with B, and denote it by $A \approx B$, if for every $x \in A$ and $x' \in B$, there exist $y, z \in B$ and $y', z' \in A$, such that $(y, z) \cap (A \cup B) = \{x\}$ and $(y', z') \cap (A \cup B) = \{x'\}$. We write $A \not\approx B$, if A is not alternating with B.

We note that in the previous definition the relations $>$ and $\gg$ are neither symmetric nor reflexive, $>$ is transitive, $\gg$ is not transitive, while $\approx$ is symmetric and transitive. For $A \neq B$ and $A \approx B$, there are no $a_1, a_2 \in A$ such that $(a_1, a_2) \cap B = \phi$. One can check that $\mathbb{R}^+ \gg \mathbb{R}^-$, $\mathbb{Z}^{>0} > \mathbb{Z}^{<0}$, $\mathbb{Q} \not\approx \mathbb{Q}$, $\mathbb{N} \not\approx \mathbb{N}$, $\mathbb{Z} \approx \mathbb{Z}$ and $2\mathbb{Z} \approx 2\mathbb{Z} + 1$.

Definition 3.3. A family $\mathfrak{F}$ of subsets of $\mathbb{T}$ is called *arranged* if:

(A1) For every $A \in \mathfrak{F}$ with $\sup A \neq \sup \cup \mathfrak{F}$ ($\inf A \neq \inf \cup \mathfrak{F}$) there exists $B \in \mathfrak{F}$ such that $B \gg A$ ($A \gg B$).

(A2) For every $A, B \in \mathfrak{F}, A \neq B$ either $A > B,\ B > A$ or $A \approx B$.

Example 3.4. Let $\mathbb{T} = \mathbb{R}$, $\mathfrak{F}_1 := \Big\{A_n := (n, n+1) : n \in \mathbb{Z}\Big\}$ and $\mathfrak{F}_2 := \Big\{B_{m,n} := \Big(m + \dfrac{n-1}{n}, m + \dfrac{n}{n+1}\Big) : m, n \in \mathbb{N}\Big\}$. Then, we can see that $A_{n+1} \gg A_n, n \in \mathbb{Z}$ and $A_n > A_m$, $n, m \in \mathbb{Z}, n > m$. That is $\mathfrak{F}_1$ is arranged, but $\mathfrak{F}_2$ is not. This is because for any $m \in \mathbb{N}$, there is no $B \in \mathfrak{F}_2$ such that $B_{m,1} \gg B$.

Definition 3.5. An equivalence relation E on $\mathbb{T}$ is called *arranged* if the family of all equivalence classes is arranged.

The universal equivalence relation $E = \mathbb{T} \times \mathbb{T}$ is arranged while the identity relation $E = \{(x, x) : x \in \mathbb{T}\}$ is not whenever $\mathbb{T}$ contains more than one element. In [5], the author gave necessary and sufficient conditions for an equivalence relation E to be arranged.

We are now in a position to give a definition that widens the scope of time scales and preserves the classical ones as special cases.

Definition 3.6. A triple $\langle \mathbb{T}, E, \alpha \rangle$ is called a ***time scale*** if $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$ and E is an arranged equivalence relation on $\mathbb{T}$.

Throughout this work, one can see that all concepts (jump operators, derivatives, Cauchy integrals, etc.) with respect to the triple $\langle \mathbb{T}, E, \alpha \rangle$ where $\mathbb{T} \subseteq \mathbb{R}$ is a nonempty closed subset, E is the universal equivalence relation and $\alpha = \sup \mathbb{T}$ coincide with the corresponding ones with respect to the classical time scale, defined by Stefan Hilger [13, 14, 24–26].

Remark 3.7. On any nonempty closed subset $\mathbb{T}$ of $\mathbb{R}$, we can obtain as many time scales as the number of arranged equivalence relations defined on $\mathbb{T}$ and the number of values of $\alpha \in \mathbb{S} \cup \{\inf \mathbb{T}, \sup \mathbb{T}\}$. α represents a point where the "time" changes its direction.

For more generality, we can replace the condition of closedness of $\mathbb{T}$ by the following condition: $\mathbb{T}$ is a nonempty subset of $\mathbb{R}$ such that every Cauchy sequence in $\mathbb{T}$ converges to a point of $\mathbb{T}$ with the possible exception of Cauchy sequences which converge to a finite infimum or finite supremum of $\mathbb{T}$. See [3].

Example 3.8. Fix $q \in \mathbb{R}$, $0 < q < 1$. Let $\mathbb{T} = \mathbb{R}$ and E be defined by

$$xEy \Longleftrightarrow \exists k \in \mathbb{Z} \text{ such that } x = q^k y, \quad x, y \in \mathbb{T}.$$

It is easily seen that E is an equivalence relation. For $t \neq 0$, we have $[t] = \{tq^k : k \in \mathbb{Z}\}$, $\overline{[t]} = [t] \cup \{0\}$, $[t]' = \{0\}$, and $t \notin \overline{[c]}$ for any $c \in \mathbb{T} \setminus \{t\}$ which implies $T_t = [t]$. While $[0] = \{0\}$, and $0 = \inf[t]$ for any $t \in (0, \infty)$, $0 = \sup[t]$ for any $t \in (-\infty, 0)$ which implies $0 \in \overline{[t]}$ for any $t \in \mathbb{T}$, and then $T_0 = \cup_{t \in \mathbb{T}} \overline{[t]} = \mathbb{R}$. Thus, $\mathbb{S} = \{0\}$ and all classes are nonsingular except the class of zero. For $[t] \subset \mathbb{T}$, we have either $[t] = \{0\}$, $[t] \subset (0, \infty)$ or $[t] \subset (-\infty, 0)$. If $[t] \subset [0, \infty)$, then $[t] \gg [r]$ for any $r \in (-\infty, 0]$ and if $[t] \subset (-\infty, 0]$, then $[r] \gg [t]$ for any $r \in [0, \infty)$. Then, E satisfies property (A1). For $r, t \in (0, \infty)$ with $[r] \neq [t]$ we claim that $[r] \approx [t]$. Note that $[r] \cap (0, 1] \neq \phi$ and $[t] \cap (0, 1] \neq \phi$. Let us put $r_0 := \max\{[r] \cap (0, 1]\}$ and $t_0 := \max\{[t] \cap (0, 1]\}$, then $r_0 \neq t_0$. Assume that $r_0 < t_0$, which implies $t_0 \leq 1 < q^{-1} r_0$. Since the function $f(t) = qt$ is strictly increasing on $\mathbb{R}$, we have

$$\cdots < q^2 r_0 < q^2 t_0 < q r_0 < q t_0 < r_0 < t_0 < q^{-1} r_0 < q^{-1} t_0 < q^{-2} r_0 < q^{-2} t_0 < \cdots .$$

Thus, for $x \in [r]$ and $x' \in [t]$ there exists $k_1, k_2 \in \mathbb{Z}$ such that $x = q^{k_1} r_0$, $x' = q^{k_2} t_0$, and then $\left(q^{k_1+1} t_0, q^{k_1} t_0\right) \cap ([r] \cup [t]) = \{x\}$ and $\left(q^{k_2} r_0, q^{k_2-1} r_0\right) \cap ([r] \cup [t]) = \{x'\}$. Thus, $[r] \approx [t]$. In a similar way, we can show that $[r] \approx [t]$ for any $r, t \in (-\infty, 0)$. While $[r] \gg [0]$ for any $r \in (0, \infty)$ and $[0] \gg [r]$ for any $r \in (-\infty, 0)$. Therefore, E satisfies properties (A2), and so $\langle \mathbb{T}, E, 0 \rangle$ is a time scale. From now on, we call this time scale the *Jackson difference time scale*.

Example 3.9. Fix $\omega \in \mathbb{R}$, $\omega > 0$. Let $\mathbb{T} = \mathbb{R}$, E be defined by

$$xEy \Longleftrightarrow \exists k \in \mathbb{Z} \text{ such that } x = y + k\omega,$$

and $\alpha = \infty$. E is an equivalence relation and for $t \in \mathbb{T}$, we have $[t] = \{t + k\omega : k \in \mathbb{Z}\}$, $\inf[t] = -\infty$, $\sup[t] = \infty$ and $\overline{[t]} = [t]$ which implies $T_t = [t]$, i.e., $\mathbb{S} = \phi$. Here, all classes of $\mathbb{T}$ are nonsingular and $\mathscr{I} = \{\mathbb{T}\}$. We claim that the triple $\langle \mathbb{T}, E, \infty \rangle$ is a time scale. We need to prove that E is arranged. First, E satisfies (A1), since $\inf[t] = -\infty$ and $\sup[t] = \infty$ for any $t \in \mathbb{T}$. Now, for any $r, t \in \mathbb{T}$ we have $[r] \approx [t]$. Indeed, let $t_0 := \min\{[t] \cap [0, \infty)\}$ and $r_0 := \min\{[r] \cap [0, \infty)\}$. We may assume that $r_0 < t_0$. This implies that

$$\cdots < r_0 - 2\omega < t_0 - 2\omega < r_0 - \omega < t_0 - \omega < r_0 < t_0 < r_0 + \omega < t_0 + \omega < r_0 + 2\omega < \cdots . \tag{3.1}$$

For $x \in [r]$ and $x' \in [t]$, there are $k, k' \in \mathbb{Z}$ such that $x = r_0 + k\omega$ and $x' = t_0 + k'\omega$. By (3.1), $\big(t_0+(k-1)\omega, t_0+k\omega\big)\cap([r]\cap[t]) = \{x\}$ and $\big(r_0+k'\omega, r_0+(k'+1)\omega\big)\cap([r]\cap[t]) = \{x'\}$, i.e., $[r] \approx [t]$. Thus, E satisfies (A2), and so E is arranged. Therefore, $\langle \mathbb{T}, E, \infty\rangle$ is a time scale. This time scale will be called *the Nörlund difference time scale.*

Example 3.10. Fix $q \in \mathbb{R}$; $0 < q < 1$ and $\omega > 0$. Assume that $\mathbb{T} = \mathbb{R}$, E is defined by

$$xEy \Longleftrightarrow \exists k \in \mathbb{Z} \text{ such that } x = q^k y + \omega[k]_q, \quad x, y \in \mathbb{T},$$

and $\alpha = \omega_0 := \dfrac{\omega}{1-q}$ where $[k]_q = \dfrac{1-q^k}{1-q}$ for any $k \in \{0, 1, 2, \cdots\}$. Easily, one can check that E is an equivalence relation. Now, we show that E is arranged. Let h denote the function $h(t) = qt + \omega, t \in \mathbb{R}$. One can see that $h(t) < t$ for $t > \omega_0$, $h(t) > t$ for $t < \omega_0$, and $h(\omega_0) = \omega_0$. The function h has the inverse $h^{-1}(t) = q^{-1}(t - \omega), t \in \mathbb{R}$. It is not hard to prove the following results:

$$h^k(t) := \underbrace{h \circ h \circ \cdots \circ h}_{k-\text{times}}(t) = qt^k + \omega[k]_q, \quad t \in \mathbb{R}, \tag{3.2}$$

$$h^{-k}(t) := \underbrace{h^{-1} \circ h^{-1} \circ \cdots \circ h^{-1}}_{k-\text{times}}(t) = \frac{t - \omega[k]_q}{q^k}, \quad t \in \mathbb{R}. \tag{3.3}$$

Furthermore, $\{h^k(t)\}_{k=1}^{\infty}$ is a decreasing (an increasing) sequence in k when $t > \omega_0$ ($t < \omega_0$) with

$$\omega_0 = \inf_{k\in\mathbb{N}} h^k(t) \ \Big(= \sup_{k\in\mathbb{N}} h^k(t)\Big), \tag{3.4}$$

see Figure 3.1 [9]. The sequence $\{h^{-k}(t)\}_{k=1}^{\infty}$ is increasing (decreasing), $t > \omega_0$ ($t < \omega_0$) with

$$\infty = \sup_{k\in\mathbb{N}} h^{-k}(t) \ \Big(-\infty = \inf_{k\in\mathbb{N}} h^{-k}(t)\Big). \tag{3.5}$$

Now, if $t \neq \omega_0$, then $[t] = \{tq^k + \omega[k]_q : k \in \mathbb{Z}\}$, $\overline{[t]} = [t] \cup \{\omega_0\}$, and $t \notin [c]'$ for any $c \in \mathbb{T}$ which implies $T_t = [t]$. Whilst $[\omega_0] = \{\omega_0\}$, $\omega_0 = \inf[t]$ for any $t \in [\omega_0, \infty)$, $\omega_0 = \sup[t]$ for any $t \in (-\infty, \omega_0]$, and $\{\omega_0\} = [t]'$ for any $t \in \mathbb{T} \setminus \{\omega_0\}$, then $T_{\omega_0} = \cup_{t\in\mathbb{T}}\overline{[t]} = \mathbb{R}$. Thus, $\mathbb{S} = \{\omega_0\}$ and all classes are nonsingular except the class of ω_0. For $t \in [\omega_0, \infty)$ $\big(t \in (-\infty, \omega_0]\big)$, we have $[t] \gg [r]$ $\big([r] \gg [t]\big)$ for all $r \in (-\infty, \omega_0]$ $\big(r \in [\omega_0, \infty)\big)$, i.e., E satisfies (A1) property. Now, for $[r] \neq [t]$, with either $r, t \in (\omega_0, \infty)$ or $r, t \in (-\infty, \omega_0)$, we claim that $[r] \approx [t]$. By (3.4), $[r] \cap (\omega_0, 1 + \omega_0] \neq \phi$ and $[t] \cap (\omega_0, 1 + \omega_0] \neq \phi$. Let $r_0 := \max\{[r] \cap (\omega_0, 1 + \omega_0]\}$ and $t_0 := \max\{[t] \cap (\omega_0, 1 + \omega_0]\}$. The values r_0, t_0 exist, since $[r]' = [t]' = \{\omega_0\}$. Here, $r_0 \neq t_0$. We may assume that $r_0 < t_0$, and then $t_0 \leq 1 + \omega_0 < h^{-1}(r_0)$, and since the function h is strictly increasing on $\mathbb{R}$, then $\cdots < h^2(r_0) < h^2(t_0) < h(r_0) < h(t_0) < r_0 < t_0 < h^{-1}(r_0) < h^{-1}(t_0) < h^{-2}(r_0) < h^{-2}(t_0) < \cdots$. This implies for $x \in [r]$ and $x' \in [t]$ there exist $k, k' \in \mathbb{Z}$ such that $x = h^k(r_0), x' = h^{k'}(t_0)$, then

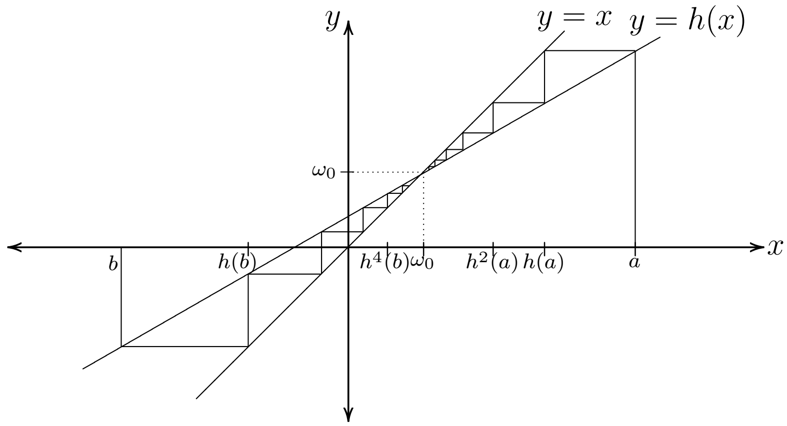

Figure 3.1: The iteration of h for $0 < q < 1, \omega > 0$

$\big(h^{k+1}(t_0), h^k(t_0)\big) \cap ([r] \cup [t]) = \{x\}$ and $\big(h^{k'}(r_0), h^{k'-1}r_0\big) \cap ([r] \cup [t]) = \{x'\}$. That is $[r] \approx [t]$. Similarly, we can see that $[r] \approx [t]$ for any $r, t \in (-\infty, \omega_0)$. Also, we have $[r] > [t]$ for any $r \in (\omega_0, \infty)$ and $t \in (-\infty, \omega_0]$. So, (A2) property is true which implies that E is arranged. Therefore the triple $\langle \mathbb{T}, E, \omega_0 \rangle$ is a time scale. We call this time scale the *Hahn difference time scale*.

Example 3.11. Consider $\langle \mathbb{T}, E, \alpha \rangle$, where $\mathbb{T} = \mathbb{Z} + \dfrac{1}{\mathbb{N}} := \left\{ z + \dfrac{1}{n} : z \in \mathbb{Z} \text{ and } n \in \mathbb{N} \right\}$, E is defined by $xEy \iff \exists z \in \mathbb{Z}$ such that $x, y \in (z, z+1]$ and $\alpha \in \mathbb{Z} \cup \{-\infty, \infty\}$. One can see that $\mathbb{T}$ is closed and E is an equivalence relation. For $z \in \mathbb{Z}$, we have

$$[z] = \left\{ z - \frac{n-1}{n} : n \in \mathbb{N} \right\} = (z-1, z],\ T_z = [z] \cup [z+1] = (z-1, z+1],$$

and $T_t = [t]$ for all $t \in \mathbb{T} \setminus \mathbb{Z}$ which imply $\mathbb{S} = \mathbb{Z}$ and $z = \sup[z] = \inf[z+1]$. Consequently, for $z \in \mathbb{Z}$ we have $\mathbb{T} = \bigcup\limits_{t \in \mathbb{Z}} [t]$, and $[z+1] \gg [z]$. For $z_1, z_2 \in \mathbb{Z}, z_1 > z_2$ we note that $[z_1] > [z_2]$. Hence E is arranged. Therefore $\langle \mathbb{T}, E, \alpha \rangle$ is a time scale for any $\alpha \in \mathbb{Z} \cup \{-\infty, \infty\}$.

Example 3.12. Let $\mathbb{T}$ be defined as in Example 3.11 and E be the equivalence relation on $\mathbb{T}$ defined by $xEy \iff \exists z \in \mathbb{Z}$ such that $x, y \in [z, z+1)$ and $\alpha \in \{-\infty, \infty\}$. For $z \in \mathbb{Z}$ we have $[z] = \left\{ z, z + \dfrac{1}{2}, z + \dfrac{1}{3}, \dots \right\} = \left[z, z + \dfrac{1}{2} \right]$. Here $T_t = [t]$ for all $t \in \mathbb{T}$, that is $\mathbb{S} = \phi$. For $z_1, z_2 \in \mathbb{Z}, z_1 > z_2$, we have $[z_1] > [z_2]$, that is E satisfies property (A2). Since $z + \dfrac{1}{2} = \sup[z] < \inf[z+1] = z+1$, $z \in \mathbb{Z}$, then there is no $z' \in \mathbb{Z}$ such that $[z'] \gg [z]$. Thus, (A1) is not true and hence $\langle \mathbb{T}, E, \alpha \rangle$ is not a time scale.

One of the main properties of a time scale is the following.

Theorem 3.13. *Assume that $\langle \mathbb{T}, E, \alpha\rangle$ is a time scale and $x, y \in \mathbb{T},\ x > y$. Then, there is a sequence of nonsingular intervals $\{I_i\}_{i=1}^{n}$ such that*

$$I_1 \gg I_2 \gg \cdots \gg I_n,\ x \in \overline{I_1} \text{ and } y \in \overline{I_n}. \tag{3.6}$$

In addition, either $[x, y] \cap \mathbb{S} = \phi$, or $[x, y] \cap \mathbb{S} = \{s_i\}_{i=1}^{m},\ m \in \mathbb{N}$ such that

$$[x, s_1] \gg [s_1, s_2] \gg \cdots \gg [s_{m-1}, s_m] \gg [s_m, y]. \tag{3.7}$$

We postpone the proof of this theorem to Section 4. The following lemma will be needed in order to define jump operators. Also, it plays an essential role in this work.

Lemma 3.14. *In a time scale $\langle \mathbb{T}, E, \alpha\rangle$, either*

$$\overline{[x]} \subseteq [\alpha, \sup \mathbb{T}] \qquad \text{or} \qquad \overline{[x]} \subseteq [\inf \mathbb{T}, \alpha], \qquad x \in \mathbb{T}. \tag{3.8}$$

Proof. First, we can see that $\alpha \in \{\inf[\alpha], \sup[\alpha]\}$. Indeed, assume the contrary: i.e., $\alpha \in (\inf[\alpha], \sup[\alpha])$. Then $\alpha \notin \{\inf \mathbb{T}, \sup \mathbb{T}\}$, that is $\alpha \in \mathbb{S} \setminus \{\inf \mathbb{T}, \sup \mathbb{T}\}$. Thus, there is $c \in T_\alpha \setminus [\alpha]$, i.e., $\alpha \in \overline{[c]} \setminus [c]$. Since $\alpha \in [c]'$, then $(\inf[\alpha], \sup[\alpha]) \cap [c] \neq \phi$. Hence, neither $[\alpha] > [c]$ nor $[c] > [\alpha]$. By property (A2), $[\alpha] \approx [c]$. Then there are $y, z \in [c]$ such that $(y, z) \cap ([\alpha] \cup [c]) = \{\alpha\}$, which contradicts $\alpha \in [c]'$. Therefore $\alpha \in \{\inf[\alpha], \sup[\alpha]\}$. Let $x \in \mathbb{T}$. We distinguish between the following three cases: First case: $x = \alpha$. Statement (3.8) is true, since $\alpha \in \{\inf[\alpha], \sup[\alpha]\}$. Second case: $x > \alpha$. If $x \in [\alpha]$, then $[x] = [\alpha]$. Hence (3.8) is true, by the first case. If $\inf[x] \geq \alpha$, thus $\overline{[x]} \subseteq [\alpha, \sup \mathbb{T}]$. When $\inf[x] < \alpha$, neither $[\alpha] > [x]$ nor $[x] > [\alpha]$. So, $[x] \approx [\alpha]$, by property (A2). Recall that $\alpha \in \{\inf[\alpha], \sup[\alpha]\}$. For $\alpha = \inf[\alpha]$ ($\alpha = \sup[\alpha]$), we have $t \in [\inf[x], \alpha)_x$ ($t \in (\alpha, \sup[x]\,]_x$). Then there are no $y, z \in [\alpha]$ such that $(y, z) \cap ([\alpha] \cup [x]) = \{t\}$. This leads to a contradiction with $[\alpha] \approx [x]$. Therefore $\overline{[x]} \subseteq [\alpha, \sup \mathbb{T}]$. Third case: $x < \alpha$. In a similar way, we can show that $\overline{[x]} \subseteq [\inf \mathbb{T}, \alpha]$. $\square$

Definition 3.15. Let $\langle \mathbb{T}, E, \alpha\rangle$ be a time scale. We define the jump operators $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$ as follows:
The *forward jump operator*

$$\sigma(t) = \begin{cases} \sup\{z \in [t] : z < t\}, & \text{if } \overline{[t]} \subseteq [\alpha, \sup \mathbb{T}],\ t \neq \inf[t], \\ \inf\{z \in [t] : z > t\}, & \text{if } \overline{[t]} \subseteq [\inf \mathbb{T}, \alpha],\ t \neq \sup[t], \\ t, & \text{otherwise.} \end{cases}$$

The *backward jump operator*

$$\rho(t) = \begin{cases} \inf\{z \in [t] : z > t\}, & \text{if } \overline{[t]} \subseteq [\alpha, \sup \mathbb{T}],\ t \neq \sup[t], \\ \sup\{z \in [t] : z < t\}, & \text{if } \overline{[t]} \subseteq [\inf \mathbb{T}, \alpha],\ t \neq \inf[t], \\ t, & \text{otherwise.} \end{cases}$$

We also define the following functions: the *forward stepsize function* $\mu : \mathbb{T} \longrightarrow \mathbb{R}$; $\mu(t) = \sigma(t) - t$, and the *backward stepsize function* $\nu : \mathbb{T} \longrightarrow \mathbb{R}$; $\nu(t) = t - \rho(t)$.

We can see that for $t \in \mathbb{T} \setminus \{\alpha\}$:

$$\sigma(t) = r \text{ iff } r \in [t,\alpha]_t \text{ and } |t-r| = \inf\{|t-z| : z \in (t,\alpha)_t\},$$
$$\rho(t) = r \text{ iff } r \in [t] \setminus (t,\alpha) \text{ and } |t-r| = \inf\{|t-z| : z \in [t] \setminus [t,\alpha]\}.$$

For a function $f : \mathbb{T} \longrightarrow \mathbb{R}$, we denote by $f^{\infty}(t) := \lim_{n \longrightarrow \infty} f^n(t)$ whenever the limit exists. If f^{-1} exists, then $f^{-\infty}(t) := (f^{-1})^{\infty}(t)$. Here, f^n denotes the n-th iteration of f for $n \in \mathbb{N}$ and f^0 is the identity function.

Definition 3.16. For $t \in \mathbb{T}$, we say that t is *forward scattered*, *forward dense*, *backward scattered*, *backward dense*, *isolated* and *dense* if $\sigma(t) \neq t$, $\sigma(t) = t$, $\rho(t) \neq t$, $\rho(t) = t$, $\rho(t) \neq t \neq \sigma(t)$, and $\rho(t) = t = \sigma(t)$, respectively.

Note that $\alpha = \inf[\alpha]$ $\big(\alpha = \sup[\alpha]\big)$ when $\alpha \in \mathbb{T}$ and $\overline{[\alpha]} \subseteq [\alpha, \sup \mathbb{T}]$ $\big(\overline{[\alpha]} \subseteq [\inf \mathbb{T}, \alpha]\big)$. Thus, we have always $\sigma(\alpha) = \alpha$ while it is not true in general for $\rho(\alpha)$.

Example 3.17. On the following time scales, we find σ and ρ:

(i) $\mathbb{T}$ is a closed subset of $\mathbb{R}$ and E is the universal equivalence relation. In this case, we have only one equivalence class $\mathbb{T}$, and hence $\mathbb{S} = \phi$. Also, $\mathbb{T}$ is the only nonsingular interval when $\mathbb{T}$ has more than one point. For $\alpha = \sup \mathbb{T}$, we have $\sigma(t) = \inf\{z \in \mathbb{T} : z > t\}$ and $\rho(t) = \sup\{z \in \mathbb{T} : z < t\}$ which are the forward and backward jump operators in the classical time scale calculus, respectively. For $\alpha = \inf \mathbb{T}$, we have $\sigma(t) = \sup\{z \in \mathbb{T} : z < t\}$ and $\rho(t) = \inf\{z \in \mathbb{T} : z > t\}$. That is to say that σ and ρ interchange their roles when α changes between $\inf \mathbb{T}$ and $\sup \mathbb{T}$.

(ii) $\langle \mathbb{T}, E, 0 \rangle$ is the q-difference time scale. We note that for $t \in (0, \infty)$,

$$0 < \cdots < q^2 t < qt < t < q^{-1}t < q^{-2}t < \cdots$$

which implies that $\sigma(t) = \sup\left\{q^k t : k \in \mathbb{N}\right\} = qt$, $\rho(t) = \inf\{q^{-k}t : k \in \mathbb{N}\} = q^{-1}t$, $\mu(t) = t(q-1)$, $\nu(t) = t(1-q^{-1})$, $\sigma^{\infty}(t) = 0$, and $\rho^{\infty}(t) = \infty$. If $t \in (-\infty, 0)$, then

$$\cdots < q^{-2}t < q^{-1}t < t < qt < q^2 t \cdots < 0,$$

$\sigma(t) = qt$, $\rho(t) = q^{-1}t$, $\mu(t) = t(q-1)$, $\nu(t) = t(1-q^{-1})$, $\sigma^{\infty}(t) = 0$, and $\rho^{\infty}(t) = -\infty$.

(iii) $\langle \mathbb{T}, E, \infty \rangle$ is the Nörlund difference time scale. For $t \in \mathbb{T}$, we have

$$\cdots < t - 2\omega < t - \omega < t < t + \omega < t + 2\omega < \cdots,$$

and $\lim_{k \longrightarrow \infty} t + k\omega = \infty$ and $\lim_{k \longrightarrow \infty} t - k\omega = -\infty$. Thus, $\sigma(t) = t + \omega$, $\rho(t) = t - \omega$, $\mu(t) = \nu(t) = \omega$, $\sigma^{\infty}(t) = \infty$, and $\rho^{\infty}(t) = -\infty$. This implies that all points of $\mathbb{T}$ are isolated.

(iv) $\langle \mathbb{T}, E, \omega_0 \rangle$ is the Hahn difference time scale. First, for $t \in (\omega_0, \infty)$, we have $\omega_0 < \cdots < h^2(t) < h(t) < t < h^{-1}(t) < h^{-2}(t) < \cdots$ and for $t \in (-\infty, \omega_0)$, $\cdots < h^{-2}(t) < h^{-1}(t) < t < h(t) < h^2(t) \cdots < \omega_0$. Then $\sigma(t) = h(t) = qt + \omega$, $\rho(t) = h^{-1}(t) = q^{-1}(t - \omega)$, $\mu(t) = t(q-1) + \omega$, $\nu(t) = t(1 - q^{-1}) + q^{-1}\omega$,

$$\sigma^\infty(t) = \omega_0 \quad \text{and} \quad \rho^\infty(t) = \begin{cases} \infty, & \text{if} \quad t > \omega_0, \\ \omega_0, & \text{if} \quad t = \omega_0, \\ -\infty, & \text{if} \quad t < \omega_0. \end{cases}$$

(v) Fix a positive integer $n > 1$. Assume that the triple $\langle \mathbb{T}, E, \alpha \rangle$ with $\mathbb{T} = \mathbb{R}^+$, xEy if there exists $k \in \mathbb{Z}$ such that $x = y^{n^k}$ for $x, y \in \mathbb{R}^+$, and $\alpha \in \{0, 1, \infty\}$. Easily, one check that E is an equivalence relation. Note that $[t] = \left\{ t^{n^k} : k \in \mathbb{Z} \right\}$ and $T_t = [t]$ for all $t \in \mathbb{T} \setminus \{0, 1\}$, and $[0] = \{0\}$, $[1] = \{1\}$, $T_0 = [0, 1)$ and $T_1 = (0, \infty)$. Furthermore, 0 and 1 are the special points. Also, the intervals $(0, 1)$ and $(1, \infty)$ are the nonsingular intervals of $\mathbb{T}$. We can show easily that E is arranged. If $\alpha = \infty$, then we have

$$\sigma(t) = \begin{cases} \sqrt[n]{t}, & \text{if } 0 \le t \le 1, \\ t^n, & \text{if } t \ge 1, \end{cases} \qquad \rho(t) = \begin{cases} t^n, & \text{if } 0 \le t \le 1, \\ \sqrt[n]{t}, & \text{if } t \ge 1. \end{cases}$$

So, that $\sigma^\infty(t) = \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } 0 < t \le 1, \\ \infty, & \text{if } t > 1, \end{cases}$ $\rho^\infty(t) = \begin{cases} 0, & \text{if } 0 \le t < 1, \\ 1, & \text{if } t \ge 1. \end{cases}$

If $\alpha = 1$, then $\sigma(t) = \sqrt[n]{t}$, $\rho(t) = t^n$, for all $t \in \mathbb{T}$. This implies that

$$\sigma^\infty(t) = \begin{cases} t, & \text{if } t = 0, \\ 1, & \text{otherwise}, \end{cases} \qquad \rho^\infty(t) = \begin{cases} 0, & \text{if } 0 \le t < 1, \\ 1, & \text{if } t = 1, \\ \infty, & \text{if } t > 1. \end{cases}$$

Finally, when $\alpha = 0$, we get

$$\sigma(t) = \begin{cases} t^n, & \text{if } 0 \le t \le 1, \\ \sqrt[n]{t}, & \text{if } t \ge 1, \end{cases} \qquad \rho(t) = \begin{cases} \sqrt[n]{t}, & \text{if } 0 \le t \le 1, \\ t^n, & \text{if } t \ge 1, \end{cases}$$

and hence $\sigma^\infty(t) = \begin{cases} 0, & \text{if } 0 \le t < 1, \\ 1, & \text{if } t \ge 1, \end{cases}$ $\rho^\infty(t) = \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } 0 < t \le 1, \\ \infty, & \text{if } t > 1. \end{cases}$

4 Preliminary Results

In this section, we study some properties of time scales that will be needed in later sections. In the beginning, we state some preliminary lemmas, for the proofs see [5].

The following lemma gives us equivalent statements for a class $[t]$ to be alternating with itself.

Lemma 4.1. *Let $\langle \mathbb{T}, E, \alpha \rangle$ be a time scale and $t \in \mathbb{T}$. Then the following statements are equivalent:*

(i) $[t] \approx [t]$.

(ii) $[t]' = \{\inf[t], \sup[t]\} \cap \mathbb{T}$ *and both* $\inf[t], \sup[t] \notin [t]$.

(iii) *All points of* $[t]$ *are isolated.*

(iv) $[t] = \{\cdots, \rho^2(t), \rho(t), t, \sigma(t), \sigma^2(t), \cdots\}$ *whose elements are distinct.*

(v) $[t] \approx [d]$ *for some* $d \in \mathbb{T}$.

Lemmas 4.2, 4.3, 4.4, 4.5 describe some properties of alternating classes, general classes, special points and nonsingular intervals of a time scale, respectively.

Lemma 4.2. *Let $\langle \mathbb{T}, E, \alpha \rangle$ be a time scale and $t \in \mathbb{T}$ be such that $[t] \approx [t]$. The following statements are true.*

(i) *If* $[t] \approx [d]$ *for* $d \in \mathbb{T}$, *then* $[t]' = [d]'$.

(ii) $t \notin [c]'$ *for every* $c \in \mathbb{T}$.

(iii) $T_t = [t]$.

(iv) $\sigma^\infty(t) \in \mathbb{S} \cup \{\alpha\}$ *and* $\rho^\infty(t) \in \mathbb{S} \cup \{\inf \mathbb{T}, \sup \mathbb{T}\}$.

(v) *If* $c \in (\inf[t], \sup[t])$, *then* $[c] \approx [t]$.

The following lemma indicates possible forms and some properties of the classes of E.

Lemma 4.3. *Let $\langle \mathbb{T}, E, \alpha \rangle$ be a time scale and $t \in \mathbb{T}$. Then we have:*

(i) $[t]$ *is of the following types:* $[t]$ *is an interval of* $\mathbb{T}$, $[t] = \{t\}$, *and*

$$[t] = \{\cdots, \rho^2(t), \rho(t), t, \sigma(t), \sigma^2(t), \cdots\}.$$

(ii) $[t] = [\rho(t)] = [\sigma(t)]$.

(iii) *If* $[t]$ *is nonsingular, then there exists a unique nonsingular interval* I *such that* $[t] \subseteq I$.

(iv) *If* $\sup[t] \neq \sup \mathbb{T}$ $(\inf[t] \neq \inf \mathbb{T})$, *then* $\sup[t] \in \mathbb{S}$ $(\inf[t] \in \mathbb{S})$.

(v) *If* $[t]$ *is nonsingular, then* $\overline{[t]} = \big([t] \cup \{\inf[t], \sup[t]\}\big) \cap \mathbb{T}$ *and* $[t]' = I' \subseteq I \cup \{\inf I, \sup I\}$ *where* I *is the nonsingular interval which contains* t.

Lemma 4.4. *Let $\langle \mathbb{T}, E, \alpha \rangle$ be a time scale. Assume that $\mathbb{S} \neq \phi$. Then:*

(i) *E is not the universal equivalence relation.*

(ii) *For $s \in \mathbb{S}$, $T_s = \cup\{I \in \mathscr{I} : s \in \overline{I} \setminus I\} \cup [s]$.*

(iii) *For $s \in \mathbb{S}$, either $s = \inf I = \sup[s]$ or $s = \sup J = \inf[s]$ for some $I, J \in \mathscr{I}$.*

(iv) *If $\inf \mathbb{S} \neq \inf \mathbb{T}$, then either $[\inf \mathbb{T}, \inf \mathbb{S})$ or $[\inf \mathbb{T}, \inf \mathbb{S}]$ is a nonsingular interval.*

(v) *If $\sup \mathbb{S} \neq \sup \mathbb{T}$, then either $(\sup \mathbb{S}, \sup \mathbb{T}]$ or $[\sup \mathbb{S}, \sup \mathbb{T}]$ is a nonsingular interval.*

(vi) *If $[t] = \{t\}$, then $t \in \mathbb{S} \setminus \cup\mathscr{I}$.*

In the following lemma, a detailed description of the nonsingular intervals is given.

Lemma 4.5. *Assume that $\langle \mathbb{T}, E, \alpha \rangle$ is a time scale and $I \in \mathscr{I}$. Then,*

(i) *If all points of I are isolated, then I is the union of some alternating classes, i.e., classes which are alternating with each other.*

(ii) *If $s \in \mathbb{S} \cap I$, then $I = [s]$, $s \in \{\inf I, \sup I\}$ and s is not isolated.*

(iii) *If $\inf I \in I$ $(\sup I \in I)$, then $I = [\inf I]$ $(I = [\sup I])$.*

(iv) *If $\inf I \neq \inf \mathbb{T}$ $(\sup I \neq \sup \mathbb{T})$, then $\inf I \in \mathbb{S}$ and $I \subseteq T_{\inf I}$ $(\sup I \in \mathbb{S}$ and $I \subseteq T_{\sup I})$.*

(v) $\mathbb{S} \cap (\inf I, \sup I) = \phi$.

(vi) *If $\mathbb{T}$ is not singleton, then $\mathscr{I}$ is an arranged countable family of pairwise disjoint intervals and $\mathbb{T} = \overline{\cup\mathscr{I}} = \cup\mathscr{I} \cup \mathbb{S}$.*

The domain sets $T_t,\ t \in \mathbb{T}$ play an important role in the calculus of time scales suggested in this paper. Recall that $T_t = [t]$ iff $t \notin \mathbb{S}$. The following lemma combines the most important properties for domain sets.

Lemma 4.6. *For a time scale $\langle \mathbb{T}, E, \alpha \rangle$, $t \in \mathbb{T}$ we have the following:*

(i) $(\rho(t), t) \cap T_t = (\sigma(t), t) \cap T_t = \phi$.

(ii) *If $t \neq \alpha$ and $\overline{[t]} \subseteq [\alpha, \sup \mathbb{T}]$ $(\overline{[t]} \subseteq [\inf \mathbb{T}, \alpha])$, then $T_t \subseteq [\alpha, \sup \mathbb{T}]$ $(T_t \subseteq [\inf \mathbb{T}, \alpha])$.*

(iii) *If $c \in T_t$ and V is a neighborhood of t in T_t, then $V \cap \overline{[c]}$ is a neighborhood of t in $\overline{[c]}$.*

We are now in a position to prove Theorem 3.13.

Proof of Theorem 3.13. Assume that $x, y \in \mathbb{T},\ x > y$. By Lemma 4.5 (vi), there are $I, J \in \mathscr{I}$ such that $x \in \overline{I},\ y \in \overline{J}$. There are two cases: $I = J$ or $I > J$. If $I = J$, then (3.6) is true and $n = 1$. If not, then $I > J$. We put $I_1 := I$. By Lemma 4.5 (vi), $\mathscr{I}$ is arranged, then by property (A1) there is $I_2 \in \mathscr{I}$ with $I_1 \gg I_2$. Now, given $\{I_1, \cdots, I_n\}, n \in \mathbb{N}$, there exists I_{n+1} such that $I_n \gg I_{n+1}$. So, we construct a sequence $\{I_n\} \subseteq \mathscr{I}$ such that $I_1 \gg I_2 \gg \cdots$. We have the following two possibilities: 1) $I_n > J$ for all $n \in \mathbb{N}$. 2) $I_n = J$ for some n. Assume, towards a contradiction, that $I_n > J$ for all n. The sequence $\{\inf I_n\}_{n\in\mathbb{N}}$ is decreasing and bounded below by $\sup J$, then it converges to a point of $\mathbb{T}$ say z. Now, there are two cases. Case 1: $\sup[z] \neq z$. Then $[z] \neq \{z\}$. By using Lemma 4.3 (iii), there is $I_z \in \mathscr{I}$ containing $[z]$ and hence $\sup I_z \neq z$. Now, we have $z \in (\inf I_z, \sup I_z)$ and $(\inf I_z, \sup I_z) \cap (\{\inf I_n\}_{n\in\mathbb{N}} \setminus \{z\}) = \phi$ which contradicts our definition of z. Case 2: $\sup[z] = z$. Now, by property (A1), there is $[t] \neq \{t\}$, $[t] \gg [z]$. Let I_t be the nonsingular interval which contains t, in view of Lemma 4.3 (iii). Then $\inf I_t = z$, this contradicts the definition of z. Therefore (3.6) is true. Now, if $[x, y] \cap \mathbb{S} \neq \phi$, then by using (3.6) and Lemma 4.5 (iv), (v), we have (3.7) is true. □

5 Limits and Continuity

In this section, we generalize the concepts of limits and continuity in the classical real analysis by introducing dense limit (d-limit) and dense continuity (d-continuity). Furthermore, we give analogous concepts of right dense and left dense continuity in the classical time scale calculus. Throughout the remainder of this work, we suppose that $\langle \mathbb{T}, E, \alpha \rangle$ is a time scale. Often, if there is no confusion concerning E and α, we denote $\langle \mathbb{T}, E, \alpha \rangle$ simply by $\mathbb{T}$. When we write $\lim\limits_{r \longrightarrow t}$ we mean $t, r \in \mathbb{T}$.

Definition 5.1. Let A be a subset of $\mathbb{T}$. A point $t \in \mathbb{T}$ is called:

(i) a *forward dense limit point*, or shortly a *fd-limit point*, of A if $t \in \mathbb{T} \setminus \{\alpha\}$ and it is a limit point of $A \cap T_t \cap (\alpha, t)$,

(ii) a *backward dense limit point*, or shortly a *bd-limit point*, of A if $t \in \mathbb{T} \setminus \{\alpha, \inf \mathbb{T}, \sup \mathbb{T}\}$ and it is a limit point of $A \cap T_t \setminus [\alpha, t]$, and

(iii) a *dense limit point*, or shortly *d-limit point*, of A if it is a limit point of $A \cap T_t$.

The sets fd-L(A), bd-L(A) and d-L(A) denote all fd-limit, bd-limit, and d-limit points of A, respectively.

Note that d-L(A) = fd-L$(A) \cup$ bd-L(A), $A \subseteq \mathbb{T}$. In the Jackson, Nörlund, Hahn difference time scales, d-L$(\mathbb{T}) = \{0\}$, d-L$(\mathbb{T}) = \phi$, d-L$(\mathbb{T}) = \{\omega_0\}$, respectively.

The following lemma investigates some properties of the dense limit points.

Lemma 5.2. *Let* $\mathbb{T} \supsetneq \{t\}$. *Then the following statements hold:*

(i) *If* $t \in$ d-L$(\mathbb{T})$*, then* T_t *is a neighborhood of* t.

(ii) $t \in$ d-L$(\mathbb{T})$ *iff one of the following cases holds:* t *is dense,* $t \notin \{\inf \mathbb{T}, \sup \mathbb{T}\}$ *is forward dense,* $t \notin \{\inf \mathbb{T}, \sup \mathbb{T}\}$ *is backward dense.*

(iii) *If* $t \neq \alpha$*, then* t *is a forward dense point iff* $t \in$ fd-L$(\mathbb{T})$.

(iv) *If* $t \notin \{\alpha, \inf \mathbb{T}, \sup \mathbb{T}\}$*, then* t *is a backward dense point iff* $t \in$ bd-L$(\mathbb{T})$.

Throughout the rest of the paper, X denotes a Banach space.

Definition 5.3. Let $f : \mathbb{T} \longrightarrow X$. Assume that $t \in$ fd-L$(\mathbb{T})$ ($t \in$ bd-L$(\mathbb{T})$). We say that the *forward* (*backward*) *dense limit*; or just *fd-limit* (*bd-limit*); of f exists at t, if there is an element $l \in X$ such that for every $\varepsilon > 0$ there is a neighborhood U of t in T_t such that:

$$\|f(r) - l\| < \varepsilon \quad \text{for all } r \in U \cap (\alpha, t) \qquad \big(r \in U \setminus [\alpha, t]\big).$$

In this case, we write fd-$\lim\limits_{r \longrightarrow t} f(r) = l$ $\big($bd-$\lim\limits_{r \longrightarrow t} f(r) = l\big)$. For $t \in$ d-L$(\mathbb{T})$, we say that the *dense limit*; or just *d-limit*; of f exists at t, if there is $l \in X$ such that for every $\varepsilon > 0$ there is a neighborhood U of t in T_t such that:

$$\|f(r) - l\| < \varepsilon \quad \text{for all } r \in U \setminus \{t\}.$$

Here, we write d-$\lim\limits_{r \longrightarrow t} f(r) = l$. When $t \notin$ d-L$(\mathbb{T})$, then we put d-$\lim\limits_{r \longrightarrow t} f(r) := f(t)$, and when $t \notin$ fd-L$(\mathbb{T})$ ($t \notin$ bd-L$(\mathbb{T})$) and $t \neq \alpha$, then we put fd-$\lim\limits_{r \longrightarrow t} f(r) := f(t)$ (bd-$\lim\limits_{r \longrightarrow t} f(r) := f(t)$).

For more details about d-limits, see [5, 6].

Example 5.4. Consider the time scale $\langle \mathbb{T}, E, \alpha \rangle$ which is defined in Example 3.17(v). Recall that $T_1 = (0, \infty)$, $T_0 = [0, 1)$, and $T_t = [t]$ for any $t \in \mathbb{T} \setminus \{0, 1\}$. Note that $t \notin T_t'$ for $t \in \mathbb{T} \setminus \{0, 1\}$, $0 \in T_0'$ and $1 \in T_1'$, i.e., d-L$(\mathbb{T}) = \mathbb{S} = \{0, 1\}$. Assume that $f : \mathbb{T} \longrightarrow \mathbb{R}$, then

$$\text{d-}\lim_{r \longrightarrow 1} f(r) = \lim_{r \longrightarrow 1} f(r), \quad \text{d-}\lim_{r \longrightarrow 0} f(r) = \lim_{r \longrightarrow 0^+} f(r),$$

when these limits exist. Now, if we define f by

$$f(t) = \begin{cases} t, & \text{if } t \in \mathbb{T} \cap \mathbb{Q}, \\ -t, & \text{otherwise}, \end{cases}$$

then $\lim\limits_{r \longrightarrow t} f(r)$ does not exist for any $t \in \mathbb{T}$, but d-$\lim\limits_{r \longrightarrow t} f(r) = f(t)$ for $t \in \mathbb{T} \setminus \{0, 1\}$.

Definition 5.5. Let $f : \mathbb{T} \longrightarrow X$. Then f is called:

(i) *Regulated* on $A \subseteq \mathbb{T}$ if the limits fd-$\lim\limits_{r \longrightarrow t} f(r)$ for $t \in$ fd-L(A) and bd-$\lim\limits_{r \longrightarrow t} f(r)$ for $t \in$ bd-L(A) exist, and if $\alpha \in$ d-L(A) then d-$\lim\limits_{r \longrightarrow \alpha} f(r)$ exist.

(ii) *Dense continuous*; or just *d-continuous*; at a point $t \in \mathbb{T}$ if for any $\varepsilon > 0$, there is a neighborhood U of t in T_t such that $\|f(r) - f(t)\| < \varepsilon$ whenever $r \in U$ and f is said to be *d-continuous* on $A \subseteq \mathbb{T}$ if it is *d-continuous* at all points of A.

(iii) *Forward dense continuous*; or just *fd-continuous*; on $A \subseteq \mathbb{T}$ provided it is regulated on A and d-continuous at all forward dense points in A.

(iv) *Backward dense continuous*; or just *bd-continuous*; on $A \subseteq \mathbb{T}$ provided it is regulated on A and d-continuous at all backward dense points in A.

Remark 5.6. For any function $f : \mathbb{T} \longrightarrow X$, we note the following:

(i) f is d-continuous on $A \subseteq \mathbb{T}$ iff it is continuous at all points of d-L(A), in view of Lemma 5.2 (i).

(ii) f is d-continuous on $A \subseteq \mathbb{T}$ iff it is d-continuous at all forward dense points in A and at all backward dense points in A, i.e., f is d-continuous on A iff f is fd-continuous and bd-continuous on A.

(iii) In general, fd-continuity (bd-continuity) does not imply d-continuity.

(iv) Continuity implies d-continuity, the converse is not necessarily true. In Example 5.4, f is d-continuous at points of $\mathbb{T} \setminus \{0, 1\}$ and discontinuous at all points of $\mathbb{T}$.

(v) f is fd-continuous on $A \subseteq \mathbb{T}$ iff f is d-continuous at all forward dense points in A and all its bd-limits exist at all backward dense points in $A \setminus \{\inf \mathbb{T}, \sup \mathbb{T}\}$.

We can show the following lemma, see [5].

Lemma 5.7. *The following statements are true:*

(i) *σ and ρ are nondecreasing.*

(ii) *σ and ρ are fd-continuous and bd-continuous, respectively.*

(iii) *If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a regulated (fd-continuous) function, then so is $f \circ \sigma$.*

(iv) *Let $\{f_n\}_{n \in \mathbb{N}}$ be a sequence of functions which is locally uniformly convergent to f. If f_n is a regulated, fd-continuous or d-continuous for all $n \in \mathbb{N}$, then so is f.*

6 Differentiation

In this section, we define the Δ-derivative on a time scale $\langle \mathbb{T}, E, \alpha \rangle$ and we study their main properties. In this setting, the Δ-derivative generalizes both the Δ and ∇-derivatives of the classical time scales calculus. Throughout this section, we suppose that $\langle \mathbb{T}, E, \alpha \rangle$ is a time scale and X is a Banach space.

We denote by

$$\mathbb{T}^k := \begin{cases} \mathbb{T} \setminus \{\alpha\}, & \text{if } \alpha \in \{\inf \mathbb{T}, \sup \mathbb{T}\} \text{ is backward scattered,} \\ \mathbb{T}, & \text{otherwise.} \end{cases}$$

For a subset $A \subseteq \mathbb{T}$, we denote the set $A \cap \mathbb{T}^k$ by A^k.

Definition 6.1 (The delta derivative)**.** A function $f : \mathbb{T} \longrightarrow X$ is called *delta differentiable* (Δ-differentiable) at $t \in \mathbb{T}^k$ if f is d-continuous at t and

$$f^{\Delta}(t) := \text{d-}\lim_{r \longrightarrow t} \frac{f(\sigma(t)) - f(r)}{\sigma(t) - r}$$

exists. We say that $f^{\Delta}(t)$ is the *delta derivative* (Δ-derivative) of f at t. For $A \subseteq \mathbb{T}$, if f is Δ-differentiable at every $t \in A^k$, then f is said to be Δ-differentiable on A^k.

Now, we state without proofs some relationships concerning the Δ-derivative, see [5].

Theorem 6.2. *Assume that $f : \mathbb{T} \longrightarrow X$ and $t \in \mathbb{T}^k$. The following statements are true:*

(i) *If f is Δ-differentiable at t, then f is d-continuous at t.*

(ii) *If f is d-continuous at a forward scattered point t, then it is Δ-differentiable at t and*
$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}.$$

(iii) *If f is Δ-differentiable at a forward dense point t, then*

$$f^{\Delta}(t) = \text{d-}\lim_{r \longrightarrow t} \frac{f(t) - f(r)}{t - r}.$$

(iv) *If f is Δ-differentiable at t, then*

$$f(\sigma(t)) = f(t) + \mu(t) f^{\Delta}(t). \tag{6.1}$$

Remark 6.3. In Example 5.4, the function

$$f(t) = \begin{cases} t, & \text{if } t \in \mathbb{T} \cap \mathbb{Q}, \\ -t, & \text{otherwise,} \end{cases}$$

is Δ-differentiable on $\mathbb{T} \setminus \{0, 1\}$, but it is not continuous, in the ordinary sense, at any point of $\mathbb{T}$ in stark contrast to the classical time scales calculus, where f is Δ-differentiable at $t \in \mathbb{T}$ implies that it is continuous at that point.

Example 6.4. (i) In the Jackson difference time scale, we have $T'_0 = \mathbb{R}$ and $T'_t = [t]' = \{0\}$ which implies that $\text{d-}\lim\limits_{r \longrightarrow 0} f(r) = \lim\limits_{r \longrightarrow 0} f(r)$ and $\text{d-}\lim\limits_{r \longrightarrow t} f(r) = f(t)$ for any $t \in \mathbb{R} \setminus \{0\}$ when these limits exist. Note also that $\mathbb{T}^k = \mathbb{T}$, since $\alpha = 0$, and then

$$f^{\Delta}(t) = \begin{cases} \dfrac{f(qt) - f(t)}{t(q-1)}, & \text{if} \quad t \neq 0, \\ f'(0), & \text{if} \quad t = 0. \end{cases}$$

Therefore for $t \in \mathbb{R}$, we have $f^{\Delta}(t) = D_q f(t)$, where D_q is the Jackson q-difference operator which is defined in (2.1).

(ii) Consider the Nörlund difference time scale $\langle \mathbb{T}, E, \infty \rangle$. Here, for any $t \in \mathbb{T}$, we have $T'_t = [t]' = \phi$ which implies fd-L(A) = bd-L(A) = d-L$(A) = \phi$ for $A \subseteq \mathbb{R}$. For a function $f : \mathbb{T} \longrightarrow X$ and $t \in \mathbb{T}$, we have $\text{d-}\lim\limits_{r \longrightarrow t} f(r) = f(t)$. Easily, we can see that $\mathbb{T}^k = \mathbb{T}$. Recall that $\sigma(t) = t + \omega$ for any $t \in \mathbb{T}$, then by Definition 6.1 we have

$$f^{\Delta}(t) = \frac{f(t+\omega) - f(t)}{\omega} = \Delta_{\omega} f(t),$$

where Δ_{ω} is the usual forward difference operator with step size ω, see [12, 30, 31, 36].

(iii) In the Hahn difference time scale $\langle \mathbb{T}, E, \omega_0 \rangle$, $T_{\omega_0} = \{t \in \mathbb{T} : \omega_0 \in \overline{[t]}\} = \mathbb{R}$, $T_t = [t]$ for any $t \neq \omega_0$, and $\mathbb{T}^k = \mathbb{T}$. Here, fd-L$(\mathbb{T})$ = bd-L$(\mathbb{T})$ = d-L$(\mathbb{T}) = \{\omega_0\}$. Then by Definition 6.1,

$$f^{\Delta}(t) = \begin{cases} \dfrac{f(qt+\omega) - f(t)}{t(q-1)+\omega}, & \text{if} \quad t \neq \omega_0, \\ f'(\omega_0), & \text{if} \quad t = \omega_0. \end{cases}$$

Therefore for $t \in \mathbb{R}$, we have $D_{q,\omega} f(t) = f^{\Delta}(t)$, where $D_{q,\omega}$ is the Hahn difference operator which is defined in (2.10).

Lemma 6.5. *Let $f : \mathbb{T} \longrightarrow X$ and $t \in \mathbb{T}^k$. Then f is Δ-differentiable at t iff there is $l \in X$ with the property that given $\varepsilon > 0$, there is a neighborhood U of t in T_t such that:*

$$\|f(\sigma(t)) - f(r) - l(\sigma(t) - r)\| \leq \varepsilon \mid \sigma(t) - r \mid \text{ for all } r \in U. \tag{6.2}$$

In this case $l = f^{\Delta}(t)$.

In the next theorem, we calculate the Δ-derivative of sums, products and quotients of Δ-differentiable functions.

Theorem 6.6. *Assume that $f, g : \mathbb{T} \longrightarrow X$ are Δ-differentiable at $t \in \mathbb{T}^k$ then:*

(i) *The sum $f + g$ is Δ-differentiable at t and*

$$(f+g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t).$$

(ii) *For any constant c, $cf : \mathbb{T} \longrightarrow X$ is Δ-differentiable at t and*

$$(cf)^{\Delta}(t) = cf^{\Delta}(t).$$

(iii) *If $X = \mathbb{R}$, the product $fg : \mathbb{T} \longrightarrow \mathbb{R}$ is Δ-differentiable at t and*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(iv) *If $X = \mathbb{R}$ and $g(t)g(\sigma(t)) \neq 0$, then f/g is Δ-differentiable at t and*

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

Example 6.7. Let $\langle \mathbb{T}, E, \alpha \rangle$ be a time scale. For $n \in \mathbb{N}_0$, by induction one can show easily that

$$\left((at+b)^n\right)^{\Delta} = a \sum_{k=0}^{n-1} (a\sigma(t)+b)^k \, (at+b)^{n-k-1}. \tag{6.3}$$

By using Theorem 6.6 (iv) and (6.3), we obtain that

$$\left((at+b)^{-n}\right)^{\Delta} = -a \sum_{k=0}^{n-1} (a\sigma(t)+b)^{-n+k} \, (at+b)^{-k-1}, \tag{6.4}$$

where $a, b \in \mathbb{R}$, provided that $(a\sigma(t)+b)\,(at+b) \neq 0$.

Definition 6.8. Let A be a closed subset of $\mathbb{T}$. A function $F : \mathbb{T} \longrightarrow X$ is called *pre Δ-differentiable* on A with region D if:

(D1) F is d-continuous on A,

(D2) F is Δ-differentiable on D, and

(D3) $D \subseteq A^k$ and $A^k \setminus D$ is a countable subset, that contains no forward scattered points from A.

A pre Δ-differentiable function F on A with region D is called *pre Δ-antiderivative* of $f : \mathbb{T} \longrightarrow X$ on A if:

$$F^{\Delta}(t) = f(t) \text{ for all } t \in D.$$

Finally, we say that a function $F : A \longrightarrow X$ is a Δ-*antiderivative* of f on A if $F^{\Delta}(t) = f(t)$ for all $t \in A^k$.

Remark 6.9. In [5], the author defined ∇-derivative and established analogues of the previous theorems and relationships of this section. Also, he gave analogues of Leibniz' formula, mean value theorem and chain rules of the classical time scale.

7 Integration

This section introduces the theory of Δ-integration on a time scale $\langle \mathbb{T}, E, \alpha \rangle$. This theory extends the theory of integration derived by Hilger [24, 25] to include some new examples. To simplify the discussion, we provide the following notation:

$$\alpha_c := \begin{cases} \inf \overline{[c]}, & \text{if } \overline{[c]} \subseteq [\alpha, \sup \mathbb{T}], \\ \sup \overline{[c]}, & \text{if } \overline{[c]} \subseteq [\inf \mathbb{T}, \alpha], \end{cases} \quad \text{and} \quad \gamma_c := \begin{cases} \sup \overline{[c]}, & \text{if } \overline{[c]} \subseteq [\alpha, \sup \mathbb{T}], \\ \inf \overline{[c]}, & \text{if } \overline{[c]} \subseteq [\inf \mathbb{T}, \alpha]. \end{cases} \tag{7.1}$$

That is, α_c and γ_c are the nearest and the farthest points respectively in $\overline{[c]}$ from α. Assume that $\langle \mathbb{T}, E, \alpha \rangle$ is a time scale throughout this section.

Remark 7.1. For any class $[c]$ of $\mathbb{T}$, we denote by E_c, the universal equivalence relation on $\overline{[c]}$. The triple $\left\langle \overline{[c]}, E_c, \alpha_c \right\rangle$ is a time scale. For any function f defined on $\mathbb{T}$ we have $f^{\Delta}(t) = f^{\Delta_c}(t)$ for any $t \in \overline{[c]} \setminus \mathbb{S}$ where $^{\Delta}$ and $^{\Delta_c}$ are the delta derivatives in the time scales $\langle \mathbb{T}, E, \alpha \rangle$ and $\left\langle \overline{[c]}, E_c, \alpha_c \right\rangle$, respectively. Here $\mathbb{S}$ is the set of special points in the time scale $\langle \mathbb{T}, E, \alpha \rangle$.

Now, we give the main existence theorems for Δ-antiderivatives, which is the precursor to defining Δ-integration, see [5]. Throughout this section, X is a Banach space.

Theorem 7.2. *Let $f : \mathbb{T} \longrightarrow X$ be a regulated function and $c \in \mathbb{T}$. For $\tau \in \overline{[c]}$ and $x \in X$ there exists a unique pre Δ_c-differentiable function $F_c : \overline{[c]} \longrightarrow X$ with region $D_c \subseteq \overline{[c]}$ such that*

$$F_c^{\Delta_c}(t) = f(t) \quad \textit{for all} \quad t \in D_c \;\; \textit{and} \;\; F_c(\tau) = x.$$

Theorem 7.3 (Existence of Δ-antiderivative)**.** *Assume that $f : \mathbb{T} \longrightarrow X$ is fd-continuous and $c \in \mathbb{T}$. Then f has a Δ_c-antiderivative function F_c on $\overline{[c]}$.*

The previous theorems are analogues for the results in classical time scales calculus, see [13, 25].

Assume that $\mathrm{RG}(\mathbb{T}, X)$ is the set of all regulated functions from $\mathbb{T}$ to X. Let $f \in \mathrm{RG}(\mathbb{T}, X)$ and F_c be a pre Δ_c-antiderivative of f on $\overline{[c]}, c \in \mathbb{T}$ which exists by Theorem 7.2. In view of Remark 7.1, F_c is a pre Δ-antiderivative of f on $\overline{[c]} \setminus \mathbb{S}$.

Our approach for integration depends on the choice of some points $c_I,\ I \in \mathscr{I}$. This is similar to (2.6) and (2.7). From now on, for any $I \in \mathscr{I}$, we fix a point $c_I \in I$. For $a, b \in \mathbb{T}$, by Theorem 3.13 and Lemma 4.5 (iv), there are m special points in the interval $[a, b]$, $m \in \mathbb{N}_0$. If $m \geq 1$, then we denote those special points by $\{s_i\}_{i=1}^m$ such that either

$$[a, s_1] \gg [s_1, s_2] \gg \cdots \gg [s_{m-1}, s_m] \gg [s_m, b] \tag{7.2}$$

or

$$[s_m, b] \gg [s_{m-1}, s_m] \gg \cdots \gg [s_1, s_2] \gg [a, s_1]. \tag{7.3}$$

For $m \geq 1$, put

$$\left.\begin{array}{c} s_0 := c_0 := c_0' := a, \\ s_{m+1} := c_m := c_m' := b, \\ c_i = c_i' = c_{I_i} \text{ for } i = 1, \cdots, m-1,\ m \geq 2, \end{array}\right\} \tag{7.4}$$

where $I_i \in \mathscr{I},\ \overline{I_i} = [s_i, s_{i+1}]$ for $i = 1, \cdots, m$. For $m = 0$, we use the convention

$$s_0 = c_0 = a \text{ and } s_1 = c_0' = b. \tag{7.5}$$

Define the *Cauchy Δ-integral* of f from a to b by

$$\int_a^b f(t)\Delta t := \sum_{i=0}^{m} \left\{ F_{c_i'}(s_{i+1}) - F_{c_i}(s_i) \right\}. \tag{7.6}$$

Remark 7.4. Notice that this definition yields the Cauchy Δ-integral (resp. ∇-integral) in the classical time scales calculus when E is the universal equivalence relation and $\alpha = \sup \mathbb{T}$ (resp. $\alpha = \inf \mathbb{T}$). Indeed, we have no special points, i.e., $m = 0$. Also, there is only one class $[c] = \mathbb{T},\ c \in \mathbb{T}$. Consequently, if F is a pre Δ-antiderivative of f on $\mathbb{T}$, then F is also a pre Δ-antiderivative of f on $\overline{[t]}$ for any $t \in \mathbb{T}$.

By applying (7.6), one can obtain directly the following lemma.

Lemma 7.5. *For $a, b \in \mathbb{T}$, the Cauchy Δ-integral satisfies the following properties:*

(i) *If $[a, b] \cap \mathbb{S} = \phi$, then* $\int_a^b f(t)\Delta t = F_b(b) - F_a(a)$.

(ii) *If $[a, b] \cap \mathbb{S} = \{a, b\}$, then* $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t = F_{c_I}(b) - F_{c_I}(a)$, *where $I \in \mathscr{I}$ with $\overline{I} = [a, b]$.*

(iii) *If $[a, b] \cap \mathbb{S} = \{b\}$, then* $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t = F_a(b) - F_a(a)$.

(iv) *If* $[a,b]\cap\mathbb{S}=\{s_i\}_{i=1}^m,\ m\geq 1$, *then*

$$\int_a^b f(t)\Delta t = F_b(b)-F_b(s_m)+\sum_{i=1}^{m-1}\Big\{F_{c_i}(s_{i+1})-F_{c_i}(s_i)\Big\}+F_a(s_1)-F_a(a).$$

Using these relations, (7.6) yields

$$\int_a^b f(t)\Delta t := \int_a^{s_1} f(t)\Delta t+\sum_{i=1}^{m-1}\left\{\int_{s_i}^{s_{i+1}} f(t)\Delta t\right\}+\int_{s_m}^b f(t)\Delta t, \tag{7.7}$$

when $m\geq 1$.

In the following, we define the improper Δ-integrals.

Definition 7.6. Let $a\in\mathbb{T}$. If $\sup\mathbb{T}=\infty$, we define

$$\int_a^\infty f(t)\Delta t := \begin{cases} \lim\limits_{r\longrightarrow\infty, r\in[a]} \displaystyle\int_a^r f(t)\Delta t, & \text{if } (a,\infty)\cap\mathbb{S}=\phi,\\ \lim\limits_{r\longrightarrow\infty} \displaystyle\int_a^r f(t)\Delta t, & \text{otherwise,}\end{cases} \tag{7.8}$$

whenever the limits exist. Similarly, when $\inf\mathbb{T}=-\infty$, we define

$$\int_{-\infty}^a f(t)\Delta t := \begin{cases} \lim\limits_{r\longrightarrow-\infty, r\in[a]} \displaystyle\int_r^a f(t)\Delta t, & \text{if } (-\infty,a)\cap\mathbb{S}=\phi,\\ \lim\limits_{r\longrightarrow-\infty} \displaystyle\int_r^a f(t)\Delta t, & \text{otherwise,}\end{cases} \tag{7.9}$$

whenever the limits exist.

In the next theorems, we give some properties of the Cauchy Δ-integral, for the proofs see [5]. The first theorem is an analogue of [13, Theorem 1.77],

Theorem 7.7. *Let $f,g:\mathbb{T}\longrightarrow X$ be fd-continuous functions. For $k\in\mathbb{R}$ and $a,b\in\mathbb{T}$, we have the following:*

(i) $\displaystyle\int_a^a f(t)\Delta t = 0.$

(ii) $\displaystyle\int_a^b kf(t)\Delta t = k\int_a^b f(t)\Delta t.$

(iii) $\displaystyle\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t.$

(iv) $\displaystyle\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t+\int_c^b f(t)\Delta t$ *when* $[a,c]\cap\mathbb{S}=\phi$, $[b,c]\cap\mathbb{S}=\phi$ *or* $c\in\mathbb{S}\cup\{\inf\mathbb{T},\sup\mathbb{T}\}$.

(v) $\displaystyle\int_a^b (f(t)+g(t))\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t.$

(vi) *If* $X=\mathbb{R}$, *then* $\displaystyle\int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t = f(t)g(t)\big|_a^b - \int_a^b f^{\Delta}(t)g(t)\Delta t$ (*integration by parts formula*).

(vii) *If* $X=\mathbb{R}$, $a,b\in\overline{[c]}$ *and* $g(t)\geq|f(t)|$ *on* $[a,b]_c$, *then* $\displaystyle\int_a^b g(t)\Delta t \geq \left|\int_a^b f(t)\Delta t\right|.$

Theorem 7.8. *Let* f *and* g *be fd-continuous functions defined on a time scale* $\mathbb{T}$. *Then, the following statements are true:*

(i) *If* $t\in\mathbb{T}$ *is isolated, then*

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t) \quad \textit{and} \quad \int_{\rho(t)}^t f(s)\Delta s = \mu(\rho(t))f(\rho(t)).$$

(ii) *If* $a,b\in[c]$ *such that* $b\in(a,\alpha)$ *and all points of* $[a,b]_c$ *are isolated, then*

$$\int_a^b f(t)\Delta t = \sum_{t\in[a,b)_c} \mu(t)f(t).$$

(iii) *If* $[c]\approx[c]$, *then for any* $a\in[c]$ *we have*

$$\int_a^{\alpha_c} f(t)\Delta t = \sum_{k=0}^{\infty}\mu(\sigma^k(a))f(\sigma^k(a)),\ \int_a^{\gamma_c} f(t)\Delta t = -\sum_{k=1}^{\infty}\mu(\rho^k(a))f(\rho^k(a)), \tag{7.10}$$

provided that these sums are convergent.

(iv) *If* I *is an isolated nonsingular interval,* $\overline{I}=[r,s]$ *and* $r\in(s,\alpha]$, *then*

$$\int_r^s f(t)\Delta t = -\sum_{k=-\infty}^{\infty}\mu(\sigma^k(c_I))f(\sigma^k(c_I)), \tag{7.11}$$

provided that this sum is convergent (here $\sigma^{-1}:=\rho$*).*

Example 7.9. In q-difference time scale, if we choose $c_{(0,\infty)}:=c$ and $c_{(-\infty,0)}:=-c$

for fixed $c \in (0, \infty)$, then by Theorem 7.8, we obtain

$$\int_0^a f(t)\Delta t = a(1-q)\sum_{k=0}^{\infty} q^k f(q^k a), \quad a \in \mathbb{R}. \tag{7.12}$$

$$\int_a^b f(t)\Delta t = \int_0^b f(t)\Delta t - \int_0^a f(t)\Delta t, \quad a, b \in \mathbb{R}. \tag{7.13}$$

$$\int_0^{\infty} f(t)\Delta t = c(1-q)\sum_{k=-\infty}^{\infty} q^k f(cq^k). \tag{7.14}$$

$$\int_{-\infty}^0 f(t)\Delta t = c(1-q)\sum_{k=-\infty}^{\infty} q^{-k} f(-cq^{-k}). \tag{7.15}$$

$$\int_{-\infty}^{\infty} f(t)\Delta t = c(q-1)\sum_{k=-\infty}^{\infty} q^k \Big[f(cq^k) + f(-cq^k)\Big]. \tag{7.16}$$

Formulas (7.12), (7.13), (7.14) and (7.16) are the Jackson q-integrals, setting $c = 1$, provided that the sums are convergent [1,4,29,34]. Therefore, if we fix $c_{(0,\infty)} = -c_{(-\infty,0)} = 1$, then we have

$$\int_a^b f(t)d_q t = \int_a^b f(t)\Delta t \quad \text{for } a, b \in \mathbb{R} \cup \{-\infty, \infty\}.$$

Here, one can check that for any fd-continuous f, the function $F(t) := \sum_{k=0}^{\infty} tq^k(1-q)f(tq^k)$ is a Δ-antiderivative of f on $\mathbb{T}$.

Example 7.10. Assume that $\mathbb{T}$ is the Nörlund difference time scale. For any $t \in \mathbb{T}$, $\sigma^{\infty}(t) = \infty$, then by Theorem 7.8 (iii), the Δ-antiderivative of any function $f : \mathbb{T} \longrightarrow X$ is

$$F(t) = -\omega \sum_{k=0}^{\infty} f(t + k\omega),$$

provided this sum is convergent. This function is the well known Nörlund sum, see [20, 30, 38]. If $\omega = 1$, then obtain the so called indefinite sum in [31]. Therefore, for $a, b \in \mathbb{T}$, we have

$$\int_a^b f(t)\Delta t = \sum_{j=0}^{\infty} \omega\big[f(a + j\omega) - f(b + j\omega)\big],$$

and if we choose $c_{(-\infty,\infty)} = 0$, then

$$\int_{-\infty}^{\infty} f(t)\Delta t = \omega \sum_{k=-\infty}^{\infty} f(k\omega).$$

Example 7.11. Let $\mathbb{T}$ be the Hahn difference time scale. For any $x \in \mathbb{T}$, $\alpha_x = \lim_{n \longrightarrow \infty} h^n(x) = \omega_0$, and $[x] \approx [x]$, then by Theorem 7.8 (iii),

$$\int_{\omega_0}^{x} f(t)\Delta t = (x(1-q)-\omega)\sum_{k=0}^{\infty} q^k f(xq^k + \omega[k]_q),$$

and by Theorem 7.7 (iv), we have

$$\int_a^b f(t)\Delta t = \int_{\omega_0}^{b} f(t)\Delta t - \int_{\omega_0}^{a} f(t)\Delta t,$$

provided that the series converges at $x = a$ and $x = b$. Therefore,

$$\int_a^b f(t)\Delta t = \int_a^b f(t) d_{q,\omega} t, \quad a, b \in \mathbb{R} \tag{7.17}$$

where q, ω-integral is defined in (2.11) and (2.12).

Example 7.12. In Example 3.17 (v), recall that there are two nonsingular intervals $(0,1)$, $(1,\infty)$ and two special points 0, 1. For simplicity, we put $c_{(0,1)} := c_1$ and $c_{(1,\infty)} := c_2$. We have two cases. First case: $\alpha = \infty$. By (7.11) we get

$$\begin{aligned} \int_0^1 f(t)\Delta t &= \sum_{k=-\infty}^{\infty} \left(\sqrt[n^{k+1}]{c_1} - \sqrt[n^k]{c_1} \right) f\left(\sqrt[n^k]{c_1} \right), \\ \int_1^{\infty} f(t)\Delta t &= \sum_{k=-\infty}^{\infty} \left(c_2^{n^{k+1}} - c_2^{n^k} \right) f\left(c_2^{n^k} \right). \end{aligned} \tag{7.18}$$

If $x \in (0,1)$ ($x \in (1,\infty)$), then $\alpha_x = 0$ ($\alpha_x = \infty$) and $\gamma_x = 0$ ($\gamma_x = 1$). By (7.10), we obtain that

$$\int_x^1 f(t)\Delta t = \begin{cases} \sum_{k=0}^{\infty} \left(\sqrt[n^{k+1}]{x} - \sqrt[n^k]{x} \right) f\left(\sqrt[n^k]{x} \right), & \text{if } 0 < x \le 1, \\ \sum_{k=1}^{\infty} \left(\sqrt[n^k]{x} - \sqrt[n^{k-1}]{x} \right) f\left(\sqrt[n^k]{x} \right), & \text{if } 1 < x, \end{cases} \tag{7.19}$$

$$\int_x^{\infty} f(t)\Delta t = \sum_{k=0}^{\infty} \left(x^{n^{k+1}} - x^{n^k} \right) f\left(x^{n^k} \right), \qquad 1 < x, \tag{7.20}$$

$$\int_0^{x} f(t)\Delta t = \sum_{k=1}^{\infty} \left(x^{n^{k-1}} - x^{n^k} \right) f\left(x^{n^k} \right), \qquad 0 < x < 1. \tag{7.21}$$

Second case $\alpha = 1$. In this case $\alpha_x = \alpha$ for all $x \in (0, \infty)$. By Theorem 7.8 (iii), we have the following integrals:

$$\int_x^1 f(t)\Delta t = \sum_{k=0}^{\infty} \left(\sqrt[n^{k+1}]{x} - \sqrt[n^k]{x} \right) f\left(\sqrt[n^k]{x} \right), \qquad x \in (0, \infty), \tag{7.22}$$

$$\int_x^{\infty} f(t)\Delta t = \sum_{k=1}^{\infty} \left(x^{n^k} - x^{n^{k-1}} \right) f\left(x^{n^k} \right), \qquad 1 < x, \tag{7.23}$$

$$\int_0^x f(t)\Delta t = \sum_{k=1}^{\infty} \left(x^{n^{k-1}} - x^{n^k} \right) f\left(x^{n^k} \right), \qquad 0 < x < 1. \tag{7.24}$$

Finally, by (7.11), we obtain that

$$\int_0^1 f(t)\Delta t = \sum_{k=-\infty}^{\infty} \left(\sqrt[n^{k+1}]{c_1} - \sqrt[n^k]{c_1} \right) f\left(\sqrt[n^k]{c_1} \right), \tag{7.25}$$

$$\int_1^{\infty} f(t)\Delta t = \sum_{k=-\infty}^{\infty} \left(\sqrt[n^{k+1}]{c_2} - \sqrt[n^k]{c_2} \right) f\left(\sqrt[n^k]{c_2} \right). \tag{7.26}$$

In the two cases, $\alpha = \infty$ and $\alpha = 1$, using (7.7) and Theorem 7.7 (iii), we have

$$\int_a^b f(t)\Delta t = \int_a^1 f(t)\Delta t - \int_b^1 f(t)\Delta t, \qquad a, b \in (0, \infty). \tag{7.27}$$

In the above, we assumed that all sums are convergent.

For the definition and theorems of ∇-integral of this setting, see [5].

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