Banach Algebras of Matrix Transformations Between Spaces of Strongly Bounded and Summable Sequences

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Abstract

Let $1 \leq p < \infty$. We characterise the classes $(X, Y)$ of all infinite matrices that map $X$ into $Y$ for $X = w_p^\infty(\Lambda)$ or $X = w_0^p(\Lambda)$ and $Y = w_1^\infty(\Lambda')$, for $X = w_1^\infty(\Lambda)$ and $Y = w_p^\infty(\Lambda')$, and for $X = M_p(\Lambda)$ and $Y = M_1(\Lambda')$, the $\beta$-duals of $w_p^\infty(\Lambda)$ and $w_1^\infty(\Lambda')$. As special cases, we obtain the characterisations of the classes of all infinite matrices that map $w_p^\infty$ into $w_1^\infty$, and $w_p$ into $w_1$. Furthermore, we prove that the classes $(w_\infty(\Lambda), w_1(\Lambda))$ and $(w, w)$ are Banach algebras.

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1 Introduction

Maddox [5] introduced the set $w^p$ of all complex sequences $x = (x_k)_{k=0}^\infty$ that are strongly summable with index $p$ by the Cesàro method of order $1$; that is, $w^p$ contains all sequences $x$ for which $\lim_{n \to \infty} \sigma_n^p(x; \xi) = 0$ for some complex number $\xi$, where

$$\sigma_n^p(x; \xi) = \frac{1}{n+1} \sum_{k=0}^{n} |x_k - \xi|^p$$

for all $n = 0, 1, \ldots$.

We will also consider the sets $w_0^p$ and $w_p^\infty$ of all sequences that are strongly summable to zero and strongly bounded, with index $p$; that is, the sets $w_0^p$ and $w_p^\infty$ contain all...
sequences \( x \) for which \( \lim_{n \to \infty} \sigma_n^p(x; 0) = 0 \) and \( \sup_n \sigma_n^p(x; 0) < \infty \), respectively. Maddox also established necessary and sufficient conditions on the entries of an infinite matrix to map \( w^p \) into the space \( c \) of all convergent sequences; his result is similar to the famous classical result by Silverman–Toeplitz which characterises the class \( (c, c) \) of all matrices that map \( c \) into \( c \), the so-called conservative matrices.\footnote{Maddox also established necessary and sufficient conditions on the entries of an infinite matrix to map \( w^p \) into the space \( c \) of all convergent sequences; his result is similar to the famous classical result by Silverman–Toeplitz which characterises the class \( (c, c) \) of all matrices that map \( c \) into \( c \), the so-called conservative matrices.}

Characterisations of classes of matrix transformations between sequence spaces constitute a wide, interesting and important field in both summability and operator theory. These results are needed to determine the corresponding subclasses of compact matrix operators, for instance in [1, 13], and more recently, of general linear operators between the respective sequence spaces, for instance in [2, 7]. They are also applied in studies on the invertibility of operators and the solvability of infinite systems of linear equations, for instance in [6, 8]. To be able to apply methods from the theory of Banach algebras to the solution of those problems, it is essential to determine if a class of linear operators of a sequence space \( X \) into itself is a Banach algebra; this is nontrivial if \( X \) is a BK space that does not have AK. Finally the characterisations of compact operators can be used to establish sufficient conditions for an operator to be a Fredholm operator, as in [3]. The spaces \( w^p_\infty(\Lambda) \) and \( w^p_0(\Lambda) \) for exponentially bounded sequences \( \Lambda \) and \( 1 \leq p < \infty \) were introduced in [10]; they are generalisations of the spaces \( w^p_\infty \) and \( w^p_0 \). Their dual spaces were determined in [11]. In this paper, we establish the new characterisations of the classes \( (X, Y) \) of all infinite matrices that map \( X \) into \( Y \) for \( X = w^p_\infty(\Lambda) \) or \( X = w^p_0(\Lambda) \) and \( Y = w^1_\infty(\Lambda') \), for \( X = w^1_\infty(\Lambda) \) and \( Y = w^p_\infty(\Lambda') \), and when \( X \) is the \( \beta \)-dual of \( w^p_\infty(\Lambda) \) or \( w^p_0(\Lambda) \) and \( Y \) is the \( \beta \)-dual of \( w^1_\infty(\Lambda') \). As a special case, we obtain the characterisations of the classes of all infinite matrices that map \( w^p_\infty \) into \( w^1_\infty \), and \( w^p \) into \( w^1 \), the last result being similar to Maddox’s and the Silverman–Toeplitz theorems. Furthermore, we prove that the classes \( (w_\infty(\Lambda), w_\infty(\Lambda)) \) and \( (w, w) \) are Banach algebras. Our results would be essential for further research in the areas mentioned above.

2 Notations and Known Results

Let \( \omega \) denote the set of all sequences \( x = (x_k)_{k=0}^{\infty} \), and \( \ell_\infty \), \( c_0 \) and \( \phi \) be the sets of all bounded, null and finite complex sequences, respectively; also let \( cs \), \( bs \) and

\[
\ell_p = \left\{ x \in \omega : \sum_{k=0}^{\infty} |x_k|^p < \infty \right\} \quad \text{for } 1 \leq p < \infty
\]

be the sets of all convergent, bounded and absolutely \( p \)-summable series. We write \( e \) and \( e^{(n)} \) \( (n = 0, 1, \ldots) \) for the sequences with \( e_k = 0 \) for all \( k \), and \( e^{(n)}_n = 1 \) and \( e^{(n)}_k = 0 \) for \( k \neq n \). A sequence \( (b_n) \) in a linear metric space \( X \) is called a Schauder basis of \( X \) if for every \( x \in X \) there exists a unique sequence \( \lambda \) of scalars such that \( x = \sum_n \lambda_n b_n \).
A BK space $X$ is a Banach sequence space with continuous coordinates $P_n(x) = x_n$ ($n \in \mathbb{N}_0$) for all $x \in X$; a BK space $X \supset \phi$ is said to have AK if $x = \sum_{k=0}^{\infty} x_k e^{(k)}$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$. Let $X$ be a subset of $\omega$. Then the set

$$X^\beta = \{a \in \omega : ax = (a_k x_k)_{k=0}^{\infty} \in cs \text{ for all } x \in X\}$$

is called the $\beta$-dual of $X$. Let $A = (a_{nk})_{k=0}^{\infty}$ be an infinite matrix of complex numbers and $x = (x_k)_{k=0}^{\infty} \in \omega$. Then we write $A_n = (a_{nk})_{k=0}^{\infty}$ ($n = 0, 1, \ldots$) and $A^k = (a_{nk})_{n=0}^{\infty}$ ($k = 0, 1, \ldots$) for the sequences in the $n$-th row and the $k$-th column of $A$, and $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ provided the series converges. Given any subsets $X$ and $Y$ of $\omega$, then $(X, Y)$ denotes the class of all infinite matrices $A$ that map $X$ into $Y$, that is, $A_n \in X^\beta$ for all $n$, and $Ax = (A_n x)_{n=0}^{\infty} \in Y$.

Let $X$ and $Y$ be Banach spaces and $B_X = \{x \in X : \|x\| \leq 1\}$ denote the unit ball in $X$. Then we write $\mathcal{B}(X, Y)$ for the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup_{x \in B_X} \|L(x)\|$. We write $X^* = \mathcal{B}(X, \mathbb{C})$ for the continuous dual of $X$ with the norm $\|f\| = \sup_{x \in B_X} |f(x)|$ for all $f \in X^*$. The following results and definitions are well known. Since we will frequently apply them, we state them here for the reader’s convenience.

**Proposition 2.1.** Let $X$ and $Y$ be BK spaces.

(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$; this means that if $A \in (X, Y)$, then $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ ($x \in X$) (see [14, Theorem 4.2.8]).

(b) If $X$ has AK then we have $\mathcal{B}(X, Y) \subset (X, Y)$; this means every $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that $L(x) = Ax$ ($x \in X$) (see [4, Theorem 1.9]).

A nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive reals is called exponentially bounded if there is an integer $m \geq 2$ such that for all nonnegative integers $\nu$ there is at least one term $\lambda_n$ in the interval $I^{(\nu)}_m = [m^\nu, m^{\nu+1} - 1]$ ([10]). It was shown ([10, Lemma 1]) that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is exponentially bounded, if and only if the following condition holds

(I) \[ \text{There are reals } s \leq t \text{ such that for some subsequence } (\lambda_{n(\nu)})_{\nu=0}^{\infty} \]

\[ 0 < s \leq \lambda_{n(\nu)}/\lambda_{n(\nu+1)} \leq t < 1 \quad (\nu = 0, 1, \ldots); \]

such a subsequence is called an associated subsequence.

**Example 2.2.** A simple, but important exponentially bounded sequence is the sequence $\Lambda$ with $\lambda_n = n + 1$ for $n = 0, 1, \ldots$; an associated subsequence is given by $\lambda_{n(\nu)} = 2^\nu$, $\nu = 0, 1, \ldots$. 

Throughout, let \( 1 \leq p < \infty \) and \( q \) be the conjugate number of \( p \), that is, \( q = \infty \) for \( p = 1 \) and \( q = p/(p-1) \) for \( 1 < p < \infty \). Also let \((\mu_n)_{n=0}^{\infty}\) be a nondecreasing sequence of positive reals tending to infinity. Furthermore let \( \Lambda = (\lambda_n)_{n=0}^{\infty} \) be an exponentially bounded sequence, and \((\lambda_n)_{n=0}^{\infty}\) an associated subsequence with \( \lambda_n(0) = \lambda_0 \). We write
\[ K^{<\nu>} (\nu = 0, 1, \ldots) \]
for the set of all integers \( k \) with \( n(\nu) \leq k \leq n(\nu+1) - 1 \), and define the sets
\[ \tilde{w}_p^{(0)}(\mu) = \left\{ x \in \omega : \lim_{n \to \infty} \left( \frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \right) = 0 \right\}, \]
\[ \tilde{w}_p^{(\infty)}(\mu) = \left\{ x \in \omega : \sup_n \left( \frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \right) < \infty \right\}, \]
\[ w_0^p(\Lambda) = \left\{ x \in \omega : \lim_{\nu \to \infty} \left( \frac{1}{\lambda_n(\nu+1)} \sum_{k \in K^{<\nu>}} |x_k|^p \right) = 0 \right\}, \]
and
\[ w_\infty^p(\Lambda) = \left\{ x \in \omega : \sup_\nu \left( \frac{1}{\lambda_n(\nu+1)} \sum_{k \in K^{<\nu>}} |x_k|^p \right) < \infty \right\}. \]

If \( p = 1 \), we omit the index \( p \) throughout, that is, we write \( \tilde{w}_0(\mu) = \tilde{w}_1(\mu) \) etc., for short.

**Proposition 2.3** (**See [10, Theorem 1 (a), (b)]**). Let \((\mu_n)_{n=0}^{\infty}\) be a nondecreasing sequence of positive reals tending to infinity, \( \Lambda = (\lambda_n)_{n=0}^{\infty} \) be an exponentially bounded sequence and \((\lambda_n)_{n=0}^{\infty}\) an associated subsequence.

(a) Then \( \tilde{w}_0^p(\mu) \) and \( \tilde{w}_\infty^p(\mu) \) are BK spaces with the sectional norm \( \| \cdot \|_{\tilde{\mu}} \) defined by
\[ \| x \|_{\tilde{\mu}} = \sup_n \left( \frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \right)^{1/p} \]
and \( \tilde{w}_0^p(\mu) \) has AK.

(b) We have \( \tilde{w}_0^p(\Lambda) = w_0^p(\Lambda), \tilde{w}_\infty^p(\Lambda) = w_\infty^p(\Lambda), \) and the sectional norm \( \| \cdot \|_{\Lambda} \) and the block norm \( \| \cdot \|_{\Lambda} \) with
\[ \| x \|_{\Lambda} = \sup_\nu \left( \frac{1}{\lambda_n(\nu+1)} \sum_{k \in K^{<\nu>}} |x_k|^p \right)^{1/p} \]
are equivalent on \( w_0^p(\Lambda) \) and on \( w_\infty^p(\Lambda) \).
Remark 2.4. (a) It can be shown that $w^p_\infty(\Lambda)$ is not separable, and so has no Schauder basis.

(b) It follows from [14, Corollary 4.2.4] and Proposition 2.3, that $w^p_0(\Lambda)$ and $w^p_\infty(\Lambda)$ are $BK$ spaces with the norm $\| \cdot \|_\Lambda$ and that $w^p_0(\Lambda)$ has $AK$.

Example 2.5. We might also define the set
\[ w^p(\Lambda) = w^p_0(\Lambda) \oplus e = \{ x \in \omega : x - \xi \in w^p_0 \text{ for some complex number } \xi \}. \]

It can be shown that the strong $\Lambda$-limit $\xi$ of any $x \in w^p(\Lambda)$ is unique if and only if
\[ \lambda = \limsup_{\nu \to \infty} n(\nu + 1) - n(\nu) < 0, \]
and that $w^p(\Lambda) \subset w^p_\infty(\Lambda)$ if and only if
\[ \lambda = \sup_{\nu} \frac{n(\nu + 1) - n(\nu)}{\lambda_n(\nu+1)} < \infty. \]

In view of Proposition 2.3 (b) and Example 2.2, the sets $w^p_0(\Lambda)$ and $w^p_\infty(\Lambda)$ reduce to the $BK$ spaces $w^p_0$ and $w^p_\infty$ for $\lambda_n = n + 1$ for $n = 0, 1, \ldots$; it is also clear that then $\lambda < \infty$ and $\lambda > 0$, and consequently $w^p$ is a $BK$ space and the strong limit $\xi$ of each sequence $x \in w^p$ is unique.

Throughout, we write $\| \cdot \| = \| \cdot \|_\Lambda$, for short.

The $\beta$-duals play a much more important role than the continuous duals in the theory of sequence spaces and matrix transformations. Let $a$ be a sequence and $X$ be a normed sequence space. Then we write $\| a \|_X^\beta = \sup_{x \in B_X} \left| \sum_{k=0}^\infty a_k x_k \right|$ provided the expression on the right exists and is finite, which is the case whenever $X$ is a $BK$ space and $a \in X^\beta$ by [14, Theorem 7.2.9]. If $\Lambda$ is an exponentially bounded sequence with an associated subsequence $\lambda_n(\nu)$, then we write $\max_\nu$ and $\sum_\nu$ for the maximum and sum taken over all $k \in K^{<\nu}$. We denote by $x^{<\nu} = \sum_\nu x_k e^{(k)} (\nu \in \mathbb{N}_0)$ the $\nu$-block of the sequence $x$.

Parts (a) and (b) of the next result are [11, Theorem 5.5 (a), (b)], Part (c) is [11, Theorem 5.7], and Parts (d) and (e) are [11, Theorem 5.8 (a), (b)].

Proposition 2.6. Suppose $\Lambda = (\lambda_n)_{n=0}^\infty$ is an exponentially bounded sequence and let $(\lambda_n(\nu))_{\nu=0}^\infty$ be an associated subsequence. We write
\[ \mathcal{M}_p(\Lambda) = \left\{ a \in \omega : \| a \|_{\mathcal{M}_p(\Lambda)} = \sum_{\nu=0}^\infty \left( \lambda_n(\nu+1) \right)^{1/p} \cdot \| a^{<\nu} \|_q < \infty \right\}. \]
(a) Then we have \((w_0^p(\Lambda))^\beta = (w_\infty^p(\Lambda))^\beta = \mathcal{M}_p(\Lambda)\) and
\[
\| \cdot \|_{\mathcal{M}_p(\Lambda)} = \| \cdot \|_{w_0^p(\Lambda)} = \| \cdot \|_{w_\infty^p(\Lambda)} \text{ on } \mathcal{M}_p(\Lambda).
\] (2.1)

(b) The continuous dual \(w_0^p(\Lambda)^*\) of \(w_0^p(\Lambda)\) is norm isomorphic to \(\mathcal{M}_p(\Lambda)\) with the norm \(\| \cdot \|_{\mathcal{M}_p(\Lambda)}\).

(c) Then \(\mathcal{M}_p(\Lambda)\) is a BK space with AK with respect to \(\| \cdot \|_{\mathcal{M}_p(\Lambda)}\).

(d) We have \((w_\infty^p(\Lambda))^\beta = (w_0^p(\Lambda))^\beta = w_\infty^p(\Lambda)\) and
\[
\| \cdot \|_{\mathcal{M}_p(\Lambda)} = \| \cdot \| \text{ on } (\mathcal{M}_p(\Lambda))^\beta.
\] (2.2)

(e) The continuous dual \((\mathcal{M}_p(\Lambda))^*\) of \(\mathcal{M}_p(\Lambda)\) is norm isomorphic to \(w_\infty^p(\Lambda)\).

Remark 2.7.  
(a) The continuous dual of \(w_\infty(\Lambda)\) is not given by a sequence space.

(b) The set \(w_\infty^p(\Lambda)\) is \(\beta\)-perfect, that is, \((w_\infty^p(\Lambda))^\beta = w_\infty^p(\Lambda)\).

3 Matrix Transformations on \(w_\infty^p(\Lambda)\) and \(w_0^p(\Lambda)\)

Let \(\Lambda = (\lambda_k)_{k=0}^\infty\) and \(\Lambda' = (\lambda'_m)_{m=0}^\infty\) be exponentially bounded sequences and \((\lambda_{k(\nu)})_{\nu=0}^\infty\) and \((\lambda'_{m(\mu)})_{\mu=0}^\infty\) be associated subsequences. Furthermore, let \(K^{<\nu>} = (\nu = 0, 1, \ldots)\) and \(M^{<\mu>} = (\mu = 0, 1, \ldots)\) be the sets of all integers \(k\) and \(m\) with \(k(\nu) \leq k \leq k(\nu + 1) - 1\) and \(m(\mu) \leq m \leq m(\mu + 1) - 1\). If \(A = (a_{m,k})_{m,k=0}^\infty\) is an infinite matrix and \(M = (M_\mu)_{\mu=0}^\infty\) is a sequence of subsets \(M_\mu\) of \(M^{<\mu>}\) for \(\mu = 0, 1, \ldots\), we write \(S^M(A)\) for the matrix with the rows
\[
S^M_{\mu}(A) = \sum_{m \in M_\mu} A_m, \text{ that is, } S^M(\mu)(A) = \sum_{m \in M_\mu} a_{m,k} \text{ for all } \mu, k = 0, 1, \ldots
\]

Here we establish necessary and sufficient conditions for an infinite matrix \(A\) to be in the classes \((w_\infty^p(\Lambda), w_\infty(\Lambda'))\), \((w_0^p(\Lambda), w_\infty(\Lambda'))\) and \((\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))\), and consider the special cases of \((w_\infty^p, w_\infty)\) and \((w_0^p, w_\infty)\). We also estimate the operator norms of \(L_A\) in these cases. Those characterisations and estimates are needed in the proofs of our results on Banach algebras of matrix transformations.

First we characterise the classes \((w_\infty^p(\Lambda), w_\infty(\Lambda'))\) and \((w_0^p(\Lambda), w_\infty(\Lambda'))\), and estimate the operator norm of \(L_A\) when the matrix \(A\) is a member of those classes.

Theorem 3.1. Let \(\Lambda = (\lambda_k)_{k=0}^\infty\) and \(\Lambda' = (\lambda'_m)_{m=0}^\infty\) be exponentially bounded sequences and \((\lambda_{k(\nu)})_{\nu=0}^\infty\) and \((\lambda'_{m(\mu)})_{\mu=0}^\infty\) be associated subsequences. Then we have \(A \in (w_\infty^p(\Lambda), w_\infty(\Lambda'))\) if and only if
\[
\| A \|_{(\Lambda, \Lambda')} = \sup_{\mu} \left( \frac{1}{\lambda_{m(\mu+1)}} \max_{M_\mu \subseteq M^{<\mu}} \| S^M_{\mu}(A) \|_{\mathcal{M}(\Lambda)} \right) < \infty;
\] (3.1)
moreover, we have \((w^p_0(\Lambda), w_\infty(\Lambda')) = (w^p_0(\Lambda), w_\infty(\Lambda')).\) If \(A \in (w^p_\infty(\Lambda), w_\infty(\Lambda'))\), then the operator norm of \(L_A\) satisfies

\[
\|A\|_{(\Lambda,\Lambda')} \leq \|L_A\| \leq 4 \cdot \|A\|_{(\Lambda,\Lambda')},
\]

(3.2)

**Proof.** Throughout the proof, we write \(\|A\| = \|A\|_{(\Lambda,\Lambda')},\) for short.

First we assume that the condition in (3.1) is satisfied. Let \(m \in \mathbb{N}_0\) be given. Then there is a unique \(\mu_m \in \mathbb{N}_0\) such that \(m \in M^{<\mu_m}>.\) We choose \(M_{\mu_m} = \{m\},\) and it follows from (3.1) that \(\|A_m\|_{M_{p}(\Lambda)} < \infty,\) that is, \(A_m \in (w^p_\infty(\Lambda))^\beta\) by Proposition 2.6 (a). Thus we have shown \(A_m \in (w_\infty(\Lambda))^\beta\) for all \(m \in \mathbb{N}_0.\) Now let \(x \in w^p_\infty(\Lambda)\) be given. For each \(\mu \in \mathbb{N}_0,\) we write \(M_\mu(x)\) for a subset of \(M^{<\mu}>\) for which

\[
\left| \sum_{m \in M_\mu(x)} A_m x \right| = \max_{M_\mu \subset M^{<\mu>} \sup_{m \in M_\mu} \left| \sum_{m \in M_\mu} A_m x \right|,
\]

and put \(M(x) = (M_\mu(x))_{\mu=0}^{\infty}.\) Then we have by a well-known inequality (see [12], (2.1) and (3.1)

\[
\frac{1}{\lambda_m'((\mu + 1))} \sum_{m \in M^{<\mu>}} |A_m x| \leq 4 \cdot \frac{1}{\lambda_m'((\mu + 1))} \max_{M_\mu \subset M^{<\mu>}} \sum_{m \in M_\mu} A_m x,
\]

\[
= 4 \cdot \frac{1}{\lambda_m'((\mu + 1))} \left| \sum_{m \in M_\mu(x)} \sum_{k=0}^\infty a_{mk} x_k \right| = 4 \cdot \frac{1}{\lambda_m'((\mu + 1))} \left| \sum_{k=0}^\infty \left( \sum_{m \in M_\mu(x)} a_{mk} \right) x_k \right| \leq 4 \cdot \frac{1}{\lambda_m'((\mu + 1))} \| S_\mu^M(x) (A) \| \cdot \| x \|
\]

\[
\leq 4 \cdot \frac{1}{\lambda_m'((\mu + 1))} \cdot \left( \max_{M_\mu \subset M^{<\mu>}} \| S_\mu^M (x) \| \cdot \| M_{p}(\Lambda) \| \right) \cdot \| x \| \leq 4 \cdot \| A \| \cdot \| x \| < \infty \text{ for all } \mu.
\]

Hence it follows that

\[
\| A x \| = \sup_{\mu} \left( \frac{1}{\lambda_m'((\mu + 1))} \sum_{m \in M^{<\mu>}} |A_m x| \right) \leq 4 \cdot \| A \| \cdot \| x \| < \infty,
\]

(3.3)

and consequently \(A x \in w_\infty(\Lambda')\) for all \(x \in w^p_\infty(\Lambda).\) Thus, we have shown that if the condition in (3.1) is satisfied, then \(A \in (w^p_\infty(\Lambda), w_\infty(\Lambda')) \subset (w^p_\infty(\Lambda), w_\infty(\Lambda')).\)

Conversely, we assume \(A \in (w^p_\infty(\Lambda), w_\infty(\Lambda')).\) Then we have \(A_m \in (w^p_\infty(\Lambda))^\beta\) for all \(m \in \mathbb{N}_0,\) hence \(\|A_m\|_{M_{p}(\Lambda)} < \infty\) for all \(m\) by Proposition 2.6 (a). Since \(w^p_\infty(\Lambda)\) and \(w_\infty(\Lambda')\) are a \(BK\) spaces by Remark 2.4 (b), it follows from Proposition 2.1 (a) that \(L_A \in \mathcal{B}(w^p_\infty(\Lambda), w_\infty(\Lambda'))\), and so \(\|L_A\| < \infty.\) We also have \(L_{M_\mu} \in (w^p_\infty(\Lambda))^\beta\) for all \(M_\mu \subset M^{<\mu>}\) and all \(\mu \in \mathbb{N}_0,\) where \(L_{M_\mu}(x) = (\lambda_m'((\mu + 1))^{-1} \cdot \sum_{m \in M_\mu} A_m x\) for all \(x \in \)
Then it follows from (3.5) that
\[ A \]
Thus we have shown that if \( \mu \) is the unique \( \Lambda = (\lambda_k)_{k=0}^{\infty} \), then (3.3) is satisfied.

It remains to show that if \( A \in (w_0^p(\Lambda), w_{\infty}(\Lambda')) \), then (3.2) holds. But the first and second inequalities in (3.2) follow from (3.4) and (3.3), respectively.

Now we characterise the class \( (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda')) \), and estimate the operator norm of \( L_A \) when \( A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda')) \). We write \( T \) for the set of all sequences \( t = (t_\mu)_{\mu=0}^{\infty} \) such that for each \( \mu = 0, 1, \ldots \) there is one and only one \( t_\mu \in M^{<\mu>} \).

**Theorem 3.2.** Let \( \Lambda = (\lambda_k)_{k=0}^{\infty} \) and \( \Lambda' = (\lambda'_m)_{m=0}^{\infty} \) be exponentially bounded sequences and \( (\lambda_k(\nu))_{\nu=0}^{\infty} \) and \( (\lambda'_m(\mu))_{\mu=0}^{\infty} \) be associated subsequences. Then we have \( A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda')) \) if and only if

\[
\| A \|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))} = \sup_{N \in \mathbb{N}_0} \left( \sup_{\mu \in \mathcal{N}} \left( \sum_{\mu \in \mathcal{N}} \lambda'_m(\mu+1) A_{t_\mu} \right) \right) < \infty, \tag{3.5}
\]

where, of course,

\[
\left\| \sum_{\mu \in \mathcal{N}} \lambda'_m(\mu+1) A_{t_\mu} \right\|_{\Lambda} = \sup_{\nu} \left( \frac{1}{\lambda_k(\nu+1)} \sum_{k \in K^{<\nu>}} \left| \sum_{\mu \in \mathcal{N}} \lambda'_m(\mu+1) A_{t_{\mu,k}} \right| \right)^{1/p}.
\]

If \( A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda')) \), then the operator norm of \( L_A \) satisfies

\[
\| A \|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))} \leq \| L_A \| \leq 4 \cdot \| A \|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))}. \tag{3.6}
\]

**Proof.** Throughout the proof, we write \( \| A \| = \| A \|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))} \), for short.

First we assume that the condition in (3.5) is satisfied. Let \( m \in \mathbb{N}_0 \) be given. Then there is a unique \( \mu_m \in \mathbb{N}_0 \) such that \( m \in M^{<\mu_m>} \). We choose \( N = \{ m \} \) and \( t_{\mu_m} = m \). Then it follows from (3.5) that

\[
\| A_m \|_{\Lambda} = \sup_{\nu} \left( \frac{1}{\lambda_k(\nu+1)} \sum_{k \in K^{<\nu>}} |a_{\nu,k}|^p \right)^{1/p}
\]

\[
= \frac{1}{\lambda'_m(\mu_m+1)} \cdot \sup_{\nu} \left\| \sum_{\mu \in \mathcal{N}} \lambda'_m(\mu_m+1) A_{t_{\mu_m}} \right\|_{\Lambda} < \infty,
\]

since trivially \( |L_{M_\mu}(x)| \leq \| L_A \| \cdot \| x \| \) for all \( x \in w_0^p(\Lambda) \), all \( M_\mu \in M^{<\mu>} \) and all \( \mu \in \mathbb{N}_0 \), it follows by (2.1) in Proposition 2.6 (a) that

\[
\| L_{M_\mu} \|_{\mathcal{M}_p(\Lambda)} = \| L_{M_\mu} \|_{w_0^p(\Lambda)} \leq \| L_A \| \text{ for all } M_\mu \in M^{<\mu>} \text{ and } \mu \in \mathbb{N}_0,
\]

and so

\[
\sup_{\mu \in M^{<\mu>}} \left( \max_{M_\mu \in M^{<\mu>}} \| L_{M_\mu} \|_{\mathcal{M}_p(\Lambda)} \right) = \sup_{\mu \in M^{<\mu>}} \left( \frac{1}{\lambda'_m(\mu+1)} \max_{M_\mu \in M^{<\mu>}} \left\| \sum_{\mu \in M_\mu} A_\mu \right\|_{\mathcal{M}_p(\Lambda)} \right)
\]

\[
= \| A \| \leq \| L_A \| < \infty. \tag{3.4}
\]
and so $A_m \in w_p^\infty(\Lambda) = (\mathcal{M}_p(\Lambda))^\beta$ by Proposition 2.6 (a) and (d). Now let $\mu_0 \in \mathbb{N}_0$ and $x \in \mathcal{M}_p(\Lambda)$ be given. For each $\mu \in \mathbb{N}_0$ with $0 \leq \mu \leq \mu_0$, let $m(\mu; x)$ be the smallest integer in $M^{<\mu}$ such that $\max_{m \in M^{<\mu}} |A_m x| = |A_{m(\mu; x)} x|$. Then we have by a well-known inequality (see [12]) and (2.2) in Proposition 2.6 (d)

$$\sum_{\mu=0}^{\mu_0} \lambda'_{m(\mu+1)} \max_{m \in M^{<\mu}} |A_m x| = \sum_{\mu=0}^{\mu_0} \lambda'_{m(\mu+1)} |A_{m(\mu; x)} x|$$

$$\leq 4 \cdot \max_{N \subseteq \{0, \ldots, \mu_0\}} \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{m(\mu; x)} x$$

$$= 4 \cdot \max_{N \subseteq \{0, \ldots, \mu_0\}} \left| \sum_{k=0}^{\infty} \left( \sum_{\mu \in N} \lambda'_{m(\mu+1)} a_{m(\mu; x), k} \right) x_k \right|$$

$$\leq 4 \cdot \max_{N \subseteq \{0, \ldots, \mu_0\}} \left( \sup_{p} \left( \frac{1}{\lambda_k(\mu+1)} \sum_{k \in K^{<\mu}} \left| \sum_{\mu \in N} \lambda'_{m(\mu+1)} a_{m(\mu; x), k} \right|^p \right)^{1/p} \right) \|x\|_{\mathcal{M}_p(\Lambda)}$$

$$\leq 4 \cdot \max_{N \subseteq \{0, \ldots, \mu_0\}} \left( \sup_{t \in T} \left( \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t, \Lambda} x \right) \right) \cdot \|x\|_{\mathcal{M}_p(\Lambda)}$$

$$\leq 4 \cdot \sup_{N \subseteq \mathbb{N}_0, \mu_0} \left( \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t, \Lambda} x \right) \cdot \|x\|_{\mathcal{M}_p(\Lambda)} = 4 \cdot \|A\| \cdot \|x\|_{\mathcal{M}_p(\Lambda)} < \infty.$$

Since $\mu_0 \in \mathbb{N}_0$ was arbitrary, we obtain

$$\|Ax\|_{\mathcal{M}(\Lambda')} \leq 4 \cdot \|A\| \cdot \|x\|_{\mathcal{M}_p(\Lambda)} < \infty \text{ for all } x \in \mathcal{M}_p(\Lambda), \quad (3.7)$$

and consequently $Ax \in \mathcal{M}(\Lambda')$ for all $x \in \mathcal{M}_p(\Lambda)$. Thus we have shown that if the condition in (3.5) is satisfied, then $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$.

Conversely, we assume $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$. Then $A_m \in (\mathcal{M}_p(\Lambda))^{\beta} = w_p^\infty(\Lambda)$ for all $m \in \mathbb{N}_0$ by Proposition 2.6 (a) and (d). Furthermore, since $\mathcal{M}_p(\Lambda)$ and $\mathcal{M}(\Lambda')$ are BK spaces by Proposition 2.6 (c), it follows from Proposition 2.1 (a) that $L_A \in B(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$. We also have $L_{N,t} \in (\mathcal{M}_p(\Lambda))^\ast$ for all finite subsets $N$ of $\mathbb{N}_0$ and all sequences $t \in T$, where $L_{N,t}(x) = \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t, \Lambda} x$ for all $x \in \mathcal{M}_p(\Lambda)$. Since

$$|L_{N,t}(x)| \leq \sum_{\mu=0}^{\infty} \lambda'_{m(\mu+1)} \max_{m \in M^{<\mu}} |A_m x| = \|Ax\|_{\mathcal{M}(\Lambda')} \leq \|L_A\| \cdot \|x\|_{\Lambda}$$

for all finite subsets $N$ of $\mathbb{N}_0$ and all $t \in T$, it follows by (2.2) in Proposition 2.6 (c) that

$$\|L_{N,t}\|_{\Lambda}^* = \|L_{N,t}\|_{\Lambda} = \left\| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t, \Lambda} \right\|_{\Lambda} \leq \|L_A\| < \infty.$$
Since this holds for all finite subsets $N$ of $\mathbb{N}_0$ and all $t \in T$, we conclude

$$
\|A\| = \sup_{N \subset \mathbb{N}_0} \left( \sup_{t \in T} \left\| \sum_{\mu \in N} \lambda'_\mu A_{\mu t} \right\|_A \right) \leq \|L\| < \infty. \tag{3.8}
$$

Thus we have shown that if $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$, then (3.5) is satisfied.

Finally, if $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$, then (3.6) follows from (3.8) and (3.7).

Using the transpose $A^T$ of a matrix $A$, we obtain an alternative characterisation of the class $(w_\infty(\Lambda), w^p_\infty(\Lambda'))$.

**Theorem 3.3.** We have $A \in (w_\infty(\Lambda), w^p_\infty(\Lambda'))$ if and only if

$$
\|A^T\|_{(\mathcal{M}_p(\Lambda'), \mathcal{M}(\Lambda))} < \infty. \tag{3.9}
$$

**Proof.** Since $X = w_0(\Lambda)$ and $Z = \mathcal{M}_p(\Lambda')$ are $BK$ spaces with $AK$ by Remark 2.4 (b) and Proposition 2.6 (c), and $Y = Z^\beta = w^p_\infty(\Lambda')$ by Proposition 2.6 (d), it follows from [14, Theorem 8.3.9] that $A \in (w_0(\Lambda), w^p_\infty(\Lambda')) = (X, Y) = (X^\beta, Y) = (w_\infty(\Lambda), w^p_\infty(\Lambda'))$ and $A \in ((w_0(\Lambda), w^p_\infty(\Lambda'))$ if and only if $A^T \in (Z, X^\beta) = (\mathcal{M}_p(\Lambda'), \mathcal{M}(\Lambda))$, and, by (3.5) in Theorem 3.2, this is the case if and only if (3.9) holds.

We consider an application to the characterisations of the classes $(w^p_\infty, w_\infty)$, $(w^p, w)$ and $(w_\infty, w^p_\infty)$. Let $\Lambda = \Lambda'$ and $\lambda_n = n + 1$ for $n = 0, 1, \ldots$ as in Examples 2.2 and 2.5. Then we may choose the subsequences given by $\lambda_k(\nu) = 2^\nu$ and $\lambda_m(\mu) = 2^\mu$ for all $\nu, \mu = 0, 1, \ldots$, and consequently the sets $K^{<\nu>}$ and $M^{<\mu>}$ are the sets of all integers $k$ and $m$ with $2^\nu \leq k \leq 2^{\nu+1} - 1$ and $2^\mu \leq m \leq 2^{\mu+1} - 1$. We also write $\mathcal{M}_p = \mathcal{M}_p(\Lambda)$.

**Remark 3.4.** (a) We obviously have $w_0^p \subset w^p \subset w^p_\infty$.

(b) For each $x \in w^p$, the strong limit $\xi$, that is, the complex number $\xi$ with

$$
\lim_{\nu \to \infty} \frac{1}{2^\nu} \sum_{\nu} |x_k - \xi|^p = 0
$$

is unique (see [5]).

(c) Every sequence $x = (x_k)_{k=0}^\infty \in w^p$ has a unique representation

$$
x = \xi \cdot e + \sum_{k=0}^{\infty} (x_k - \xi)e^{(k)} \tag{[5]).
$$

**Example 3.5.** (a) It follows from Theorem 3.1 that $A \in (w^p_\infty, w_\infty) = (w_0^p, w_\infty)$ if and only if

$$
\|A\|_{(w^p_\infty, w_\infty)} = \sup_{\mu} \left( \frac{1}{2^\mu} \max_{M_\mu \subset M^{<\mu>}} \left\| \sum_{m \in M_\mu} A_m \right\|_{\mathcal{M}_p} \right) < \infty, \tag{3.10}
$$
where

\[
\left\| \sum_{m \in M_\mu} A_m \right\|_{\mathcal{M}_p} = \begin{cases}
\sum_{\nu=0}^{\infty} 2^{\nu} \max_{k \in K^{<\nu>}} \left| \sum_{m \in M_\mu} a_{mk} \right| & (p = 1) \\
\sum_{\nu=0}^{\infty} 2^{\nu/p} \left( \sum_{k \in K^{<\nu>}} \sum_{m \in M_\mu} a_{mk} \right)^{q/2} & (1 < p < \infty).
\end{cases}
\]

(b) It follows from Part (a) and \([14, 8.3.6, 8.3.7]\) that \(A \in (w^p, w)\) if and only if

\[
\begin{cases}
\text{for each } k \text{ there exists a complex number } \alpha_k \text{ with } \\
\lim_{\mu \to \infty} 2^{\nu} \sum_{m \in M^{<\nu>}} |a_{mk} - \alpha_k| = 0
\end{cases}
\]

and

\[
\lim_{\mu \to \infty} 2^{\nu} \sum_{m \in M^{<\nu>}} \left| \sum_{k=0}^{\infty} a_{mk} - \hat{\alpha} \right| = 0 \text{ for some complex number } \hat{\alpha}
\]

hold.

(c) We obtain from Theorems 3.2 and 3.3, interchanging the roles of \(N\) and \(K\), and \(\mu\) and \(\nu\), that \(A \in (w^\infty, w^p_\infty)\) if and only if

\[
\sup_{K \subset \text{finite}} \left( \sup_{t \in T} \left\| \sum_{\nu \in K} 2^\nu A^{t\nu} \right\|_\Lambda \right) < \infty,
\]

where

\[
\left\| \sum_{\nu \in K} 2^\nu A^{t\nu} \right\|_\Lambda = \sup_{\mu} \left( \frac{1}{2^\mu} \sum_{m \in M^{<\mu>}} \left| \sum_{\nu \in K} 2^\nu a_{m,tk} \right| \right).
\]

We also give a formula for the strong limit of \(Ax\) when \(A \in (w^p, w)\) and \(x \in w^p\).

**Theorem 3.6.** If \(A \in (w^p, w)\), then the strong limit \(\eta\) of \(Ax\) for each sequence \(x \in w^p\) is given by

\[
\eta = \hat{\alpha} \cdot \xi + \sum_{k=0}^{\infty} \alpha_k (x_k - \xi),
\]

where \(\xi\) is the strong limit of the sequence \(x\), and the complex numbers \(\hat{\alpha}\) and \(\alpha_k\) for \(k = 0, 1, \ldots\) are given by (3.12) and (3.11) in Example 3.5 (b).
Proof. We assume \( A \in (w^p, w) \) and write \( \| \cdot \| = \| \cdot \|_{(w^p, w)} \), for short. The complex numbers \( \tilde{o} \) and \( \alpha_k \) for \( k = 0, 1, \ldots \) exist by Example 3.5 (b).

First, we show \((\alpha_k)_{k=0}^{\infty} \in \mathcal{M}_p\). Let \( x \in w^p \) and \( k_0 \in \mathbb{N}_0 \) be given. Then there exists an integer \( \nu(k_0) \) with \( k_0 \in K^{<\nu(k_0)>} \) and we have by the inequality in [9, Lemma 1]

\[
\sum_{k=0}^{k_0} |\alpha_k x_k| = \sum_{k=0}^{k_0} \left( \frac{1}{2^\mu} \sum_\mu |\alpha_k| \cdot |x_k| \right)
\leq \frac{1}{2^\mu} \sum_{k=0}^{k_0} \left( \sum_\mu |a_{nk} - \alpha_k| \cdot |x_k| \right) + \nu(k_0) \left( \frac{1}{2^\mu} \sum_\mu |a_{nk}| \cdot |x_k| \right)
\leq \sum_{k=0}^{k_0} \left( \frac{1}{2^\mu} \sum_\mu |a_{nk} - \alpha_k| \right) \cdot |x_k| + 4 \cdot \max_{\mu \in M_{<\nu>}} \sum_{\nu=0}^{\infty} \sum_\nu \left( \frac{1}{2^\mu} \sum_\mu |a_{nk}| \cdot |x_k| \right)
\leq \sum_{k=0}^{k_0} \left( \frac{1}{2^\mu} \sum_\mu |a_{nk} - \alpha_k| \right) \cdot |x_k| + 4 \cdot \sup_\mu \left( \frac{1}{2^\mu} \max_{\mu \in M_{<\nu>}} \left\| \sum_{n \in M_{\mu}} A_n \right\|_{\mathcal{M}_p} \right) \cdot \|x\|.
\]

Letting \( \mu \) tend to \( \infty \), we obtain \( \sum_{k=0}^{k_0} |\alpha_k x_k| \leq 0 + 4 \cdot \|A\| < \infty \) from (3.11) and (3.10).

Since \( k_0 \in \mathbb{N}_0 \) was arbitrary, it follows that \( \sum_{k=0}^{\infty} |\alpha_k x_k| < \infty \) for all \( x \in w^p \), that is, \((\alpha_k)_{k=0}^{\infty} \in (w^p)^\beta = \mathcal{M}_p\).

Now we write \( \hat{o}(x) = \sum_{k=0}^{\infty} \alpha_k x_k \) and \( B = (b_{nk})_{n,k=0}^{\infty} \) for the matrix with \( b_{nk} = a_{nk} - \alpha_k \) for all \( n \) and \( k \), and show

\[
\lim_{\mu \to \infty} \frac{1}{2^\mu} \sum_\mu |B_n x| = 0 \quad \text{for all} \quad x \in w^p_0.
\] (3.14)

Let \( x \in w^p_0 \) and \( \varepsilon > 0 \) be given. Since \( w^p_0 \) has \( AK \), there is \( k_0 \in \mathbb{N}_0 \) such that

\[
\|x - x^{[k_0]}\| < \frac{\varepsilon}{\|A\| + \|(\alpha_k)_{k=0}^{\infty}\|_{\mathcal{M}_p} + 1} \quad \text{for} \quad x^{[k_0]} = \sum_{k=0}^{k_0} x_k e^{(k)}.
\]

It also follows from (3.11) that there is \( \mu_0 \in \mathbb{N}_0 \) such that

\[
\frac{1}{2^\mu} \sum_\mu |B_n x^{[k_0]}| = \frac{1}{2^\mu} \sum_\mu \left| \sum_{k=0}^{k_0} b_{nk} x_k \right| < \varepsilon \quad \text{for all} \quad \mu \geq \mu_0.
\]

Let \( \mu \geq \mu_0 \) be given. Then we have

\[
\frac{1}{2^\mu} \sum_\mu |B_n x| \leq \frac{1}{2^\mu} \sum_\mu |B_n x^{[k_0]}| + \frac{1}{2^\mu} \sum_\mu |B_n (x - x^{[k_0]})| \]

\[
\langle \varepsilon + 4 \cdot \max_{M, \mu \in M^{<\mu}} \left( \frac{1}{2\mu} \sum_{n \in M, n \mu} B_n \left( x - x^{[k_0]} \right) \right) \rangle \\
\leq \varepsilon + 4 \cdot \max_{M, \mu \in M^{<\mu}} \left( \frac{1}{2\mu} \sum_{n \in M, n \mu} B_n \right) \| x - x^{[k_0]} \| < 5 \cdot \varepsilon.
\]

Thus we have shown (3.14).

Finally, let \( x \in w^p \) be given. Then there is a unique complex number \( \xi \) such that \( x^{(0)} = x - \xi \cdot e \in w_0^p \), by Remark 3.4 (b), and we obtain by (3.14) and (3.12)

\[
0 \leq \frac{1}{2\mu} \sum_{\mu} |A_n x - \eta| = \frac{1}{2\mu} \sum_{\mu} \left( A_n x^{(0)} + \xi \cdot A_n(e) - \left( \tilde{\alpha} \cdot \xi + \sum_{k=0}^{\infty} \alpha_k x^{(0)} \right) \right) \\
\leq \frac{1}{2\mu} \sum_{\mu} |A_n x^{(0)} - \tilde{\alpha}(x^{(0)})| + |\xi| \cdot \frac{1}{2\mu} \sum_{\mu} |A_n e - \tilde{\alpha}| \\
= \frac{1}{2\mu} \sum_{\mu} |B_n x^{(0)}| + |\xi| \cdot \frac{1}{2\mu} \sum_{\mu} \left| \sum_{k=0}^{\infty} a_{nk} - \tilde{\alpha} \right| \rightarrow 0 + 0 = 0 \ (\mu \rightarrow \infty).
\]

This completes the proof. \( \square \)

4 The Banach Algebra \( (w_\infty(L), w_\infty(L)) \)

In this section, we show that \( (w_\infty(L), w_\infty(L)) \) is a Banach algebra with respect to the norm \( \| \cdot \| \) defined by \( \| A \| = \| L_A \| \) for all \( A \in (w_\infty(L), w_\infty(L)) \). We also consider the nontrivial special case of \( (w, w^p) \).

We need the following results.

**Lemma 4.1.**

(a) The matrix product \( B \cdot A \) is defined for all \( A, B \in (w_\infty(L), w_\infty(L)) \); in fact

\[
\sum_{m=0}^{\infty} |b_{nm} a_{mk}| \leq \| B_n \|_{M(L)} \| A_k \| \text{ for all } n \text{ and } k.
\]  

(b) Matrix multiplication is associative in \( (w_\infty(L), w_\infty(L)) \).

(c) The space \( (w_\infty(L), w_\infty(L)) \) is a Banach space with respect to

\[
\| A \|_{(\Lambda, \Lambda)} = \sup_{\mu} \left( \frac{1}{\lambda_{m(\mu+1)}} \max_{M, \mu \in M^{<\mu}} \sum_{\nu=0}^{\infty} \lambda_{k(\nu+1)} \max_{k \in K^{<\nu}} \left| \sum_{m \in M, m \mu} a_{mk} \right| \right). \]  

(4.2)
Proof. (a) Let $A, B \in (w_{\infty}(\Lambda), w_{\infty}(\Lambda))$. First we observe that $e^{(k)} \in w_{\infty}(\Lambda)$ implies $Ae^{(k)} = (A_{m}e^{(k)})_{m=0} = (a_{mk})_{m=0} = A^{k} \in w_{\infty}(\Lambda)$ for all $k$. Therefore we have

$$\|A^{k}\| = \sup_{\mu} \left( \frac{1}{\lambda_{m}(\mu+1)} \sum_{m \in M^{\nu}} |a_{mk}| \right) < \infty \text{ for all } k. \quad (4.3)$$

Furthermore $B \in (w_{\infty}(\Lambda), w_{\infty}(\Lambda))$ implies $B_{n} \in (w_{\infty}(\Lambda))^{\nu} = M(\Lambda)$ for all $n$, that is, by Proposition 2.6 (a)

$$\|B_{n}\|_{M(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{m}(\mu+1) \max_{m \in M^{\nu}} |b_{nm}| < \infty \text{ for all } n. \quad (4.4)$$

Now it follows from (4.3) and (4.4) that

$$|B_{n}A^{k}| \leq \sum_{m=0}^{\infty} \sum_{m \in M^{\nu}} |b_{nm}a_{mk}| = \sum_{\mu=0}^{\infty} \sum_{m \in M^{\nu}} \lambda_{m}(\mu+1) |b_{nm}| \cdot \frac{1}{\lambda_{m}(\mu+1)} |a_{mk}|$$

$$\leq \sum_{\mu=0}^{\infty} \left( \lambda_{m}(\mu+1) \max_{m \in M^{\nu}} |b_{nm}| \right) \cdot \frac{1}{\lambda_{m}(\mu+1)} \sum_{m \in M^{\nu}} |a_{mk}|$$

$$\leq \left( \sum_{\mu=0}^{\infty} \lambda_{m}(\mu+1) \max_{m \in M^{\nu}} |b_{nm}| \right) \cdot \sup_{\mu} \left( \frac{1}{\lambda_{m}(\mu+1)} \sum_{m \in M^{\nu}} |a_{mk}| \right)$$

$$= \|B_{n}\|_{M(\Lambda)} \cdot \|A^{k}\| < \infty \text{ for all } n \text{ and } k.$$

(b) Let $A, B, C \in (w_{\infty}(\Lambda), w_{\infty}(\Lambda))$. We write for $D \in (w_{\infty}(\Lambda), w_{\infty}(\Lambda))$

$$M^{T}(D) = \|D^{T}\|_{(M(\Lambda), M(\Lambda))} = \sup_{K \subset \mathbb{N}_{0}} \left( \sup_{t \in T} \left( \sup_{\nu \in K} \left( \frac{1}{\lambda_{m}(\mu+1)} \sum_{m \in M^{\nu}} \left| \sum_{\nu \in K} \lambda_{m}(\nu+1) d_{m,t_{\nu}} \right| \right) \right) \right)$$

and note that $M^{T}(D) < \infty$ by Theorem 3.3. We are going to show that the series $\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} a_{mk} b_{mk} c_{kj}$ are absolutely convergent for all $n$ and $j$. We fix $n$ and $j$ and write $s = A_{n}$ and $t = C^{j}$ for the sequences in the $n$-th row of $A$ and the $j$-th column of $C$. Then we have $s \in M(\Lambda)$ and $t \in w_{\infty}(\Lambda)$. We define the matrix $D = (d_{\mu k})_{\mu,k=0}^{\infty}$ by

$$d_{\mu k} = \frac{1}{\lambda_{m}(\mu+1)} \sum_{m \in M^{\nu}} |b_{mk}| \text{ for } \mu, k = 0, 1, \ldots.$$ 

Furthermore, given $\mu \in \mathbb{N}_{0}$, for every $\nu = 0, 1, \ldots$, let $k_{\nu} = k_{\nu}(\mu) \in K^{\nu}$ be the smallest integer with $\max_{k \in K^{\nu}} d_{\mu k} = d_{\mu k_{\nu}}$. Then by the inequality in [9, Lemma 1],

$$\lambda_{m}(\mu+1) \|D_{\mu}\|_{M(\Lambda)} = \sum_{\nu=0}^{\infty} \lambda_{k_{\nu}(\nu+1)} \sum_{m \in M^{\nu}} |b_{mk_{\nu}}|.$$
\[ \leq 4 \cdot \sup_{K \subset \mathbb{N}_0 \text{ finite}} \left( \max_{M_{\mu} \subset \mathcal{M}^{\lambda}} \left| \sum_{\nu \in K} \lambda_{k(\nu+1)} \sum_{m \in M_{\mu}} b_{mk} \right| \right) \]

\[ \leq 4 \cdot \sup_{K \subset \mathbb{N}_0 \text{ finite}} \sum_{m \in M^{\lambda}} \left| \sum_{\nu \in K} \lambda_{k(\nu+1)} b_{mk} \right| . \]

hence

\[ \|D_{\mu}\| \leq 4 \cdot M^T(B) < \infty \text{ for } \mu = 0, 1, \ldots \] (4.5)

It also follows that for \( \mu = 0, 1, \ldots \)

\[ \frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{\lambda}} \sum_{k=0}^{\infty} |b_{mk} t_k| = \sum_{k=0}^{\infty} |t_k| \cdot |d_{\mu k}| \leq \|D_{\mu}\| \cdot \|t\|. \] (4.6)

Therefore, we obtain from (4.6) and (4.5)

\[ \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} |s_m b_{mk} t_k| \leq \left( \sup_{\mu} \left( \frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{\lambda}} \sum_{k=0}^{\infty} |b_{mk} t_k| \right) \right) \cdot \|s\| \leq \sup_{\mu} \left( \|D_{\mu}\| \cdot \|t\| \cdot \|s\| \right) \leq 4 \cdot M^T(B) \cdot \|t\| \cdot \|s\| < \infty. \]

Thus we have shown that \( \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} s_m b_{mk} t_k \) is absolutely convergent, and consequently matrix multiplication is associative in \((w_\infty(\Lambda), w_\infty(\Lambda))\).

(c) We assume that \((A^{(j)})_{j=0}^{\infty}\) is a Cauchy sequence in \((w_\infty(\Lambda), w_\infty(\Lambda))\). Since \((w_\infty(\Lambda), w_\infty(\Lambda)) = (w_0(\Lambda), w_\infty(\Lambda))\) by Theorem 3.1 and \(w_0(\Lambda)\) has AK by Remark 2.4 (b), it is a Cauchy sequence in \((w_0(\Lambda), w_\infty(\Lambda)) = B(w_0(\Lambda), w_\infty(\Lambda))\), by Proposition 2.1. Consequently there is \(L_A \in B(w_0(\Lambda), w_\infty(\Lambda))\) with \(L_A^{(j)} \to L_A\). Since \(w_0(\Lambda)\) has AK there is a matrix \(A \in (w_0(\Lambda), w_\infty(\Lambda))\) by Proposition 2.1 (b) such that \(Ax = L_A(x)\) for all \(x \in w_0(\Lambda)\). Finally \(w_0(\Lambda), w_\infty(\Lambda)) = (w_\infty(\Lambda), w_\infty(\Lambda))\) implies \(A \in (w_\infty(\Lambda), w_\infty(\Lambda))\).

The following result is obtained as an immediate consequence of Lemma 4.1.

**Theorem 4.2.** The class \((w_\infty(\Lambda), w_\infty(\Lambda))\) is a Banach algebra with respect to the norm \(\|A\| = \|L_A\|\) for all \(A \in (w_\infty(\Lambda), w_\infty(\Lambda))\).

The following example is obtained from Theorem 4.2.

**Example 4.3.** Let \(\lambda_n = n + 1\) for \(n = 0, 1, \ldots\) as in Examples 2.2, 2.5 and 3.5. Then \((w_\infty, w_\infty)\) is a Banach algebra with \(\|A\| = \|L_A\|\).

Finally, we show that \((w, w)\) is a Banach algebra.
**Theorem 4.4.** The class \((w, w)\) is a Banach algebra with \(\|A\| = \|L_A\|\).

**Proof.** We have to show in view of Theorem 4.2 that

(i) \((w, w)\) is complete;

(ii) if \(A, B \in (w, w)\), then \(B \cdot A \in (w, w)\).

First we show (i). Let \((A^j)_{j=1}^\infty\) be a Cauchy sequence in \((w, w)\). Since \((w, w) \subset (w_\infty, w_\infty)\) and the operator norm on \(B(w_\infty, w_\infty)\) is the same as that on \(B(w, w)\), it follows that \((A^j)_{j=1}^\infty\) is a Cauchy sequence in \((w_\infty, w_\infty)\), and so \(A = \lim_{j \to \infty} A^j \in (w_\infty, w_\infty)\) by Lemma 4.1 (c). We have to show \(A \in (w, w)\). Let \(\varepsilon > 0\) be given. Since \((A^j)_{j=1}^\infty\) is a Cauchy sequence in \((w, w)\) there exists a \(j_0 \in \mathbb{N}_0\) such that

\[
\|A^j - A^\ell\|_{(w_\infty, w_\infty)} = \sup_{\mu} \left( \frac{1}{2^\mu} \max_{M_\mu \subseteq M < \mu} \left\| \sum_{n \in M_\mu} (A^j_n - A^\ell_n) \right\|_M \right) < \frac{\varepsilon}{4} \text{ for all } j, \ell \geq j_0; \quad (4.7)
\]

Also, by (3.11) and (3.12), for each fixed \(j\) there exist complex numbers \(\alpha_k^j\) \((k = 0, 1, \ldots)\) and \(\tilde{\alpha}_k^j\) such that

\[
\lim_{\mu \to \infty} \left( \frac{1}{2^\mu} \sum_{\mu} \left| a_{nk}^j - \alpha_k^j \right| \right) = 0 \text{ for each } k \quad (4.8)
\]

and

\[
\lim_{\mu \to \infty} \left( \frac{1}{2^\mu} \sum_{\mu} \sum_{k=0}^{\infty} \left| a_{nk}^j - \tilde{\alpha}_k^j \right| \right) = 0. \quad (4.9)
\]

Let \(j, \ell \geq j_0\) be given. Then we have for each fixed \(k \in \mathbb{N}_0\) by (4.7)

\[
\left| \alpha_k^j - \alpha_k^\ell \right| = \frac{1}{2^\mu} \sum_{\mu} \left| a_{nk}^j - \alpha_k^j \right| \leq \frac{1}{2^\mu} \sum_{\mu} \left| a_{nk}^j - \alpha_k^j \right| + \frac{1}{2^\mu} \sum_{\mu} \left| a_{nk}^\ell - \alpha_k^\ell \right| + \frac{1}{2^\mu} \sum_{\mu} \left| a_{nk}^j - a_{nk}^\ell \right| \leq \frac{1}{2^\mu} \sum_{\mu} \left| a_{nk}^j - \alpha_k^j \right| + \frac{1}{2^\mu} \sum_{\mu} \left| a_{nk}^\ell - \alpha_k^\ell \right| + 4 \cdot \max_{M_\mu \subseteq M < \mu} \left| \sum_{n \in M_\mu} (A_n^j - A_n^\ell)(e^{(k)}) \right|
\]
\[ \leq \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(j)} - \alpha_k^{(j)}| + \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(\ell)} - \alpha_k^{(\ell)}| \\
+ 4 \cdot \sup_{\mu} \left( \frac{1}{2\mu} \max_{M_{\mu} \subset M^{<\mu}} \| A^{(j)} - A^{(\ell)} \|_M \right) \| e^{(k)} \| \\
\leq \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(j)} - \alpha_k^{(j)}| + \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(\ell)} - \alpha_k^{(\ell)}| + \varepsilon \text{ for all } \mu \in \mathbb{N}_0. \]

Letting \( \mu \to \infty \), we obtain from (4.8)

\[ |\alpha_k^{(j)} - \alpha_k^{(\ell)}| \leq \varepsilon \text{ for all } j, \ell \geq j_0. \]

Thus \( (\alpha_k^{(j)})_{j=1}^{\infty} \) is a Cauchy sequence of complex numbers for each fixed \( k \in \mathbb{N}_0 \) and so

\[ \alpha_k = \lim_{j \to \infty} \alpha_k^{(j)} \text{ exists for each } k \in \mathbb{N}_0. \quad (4.10) \]

Now let \( k \in \mathbb{N}_0 \) be fixed. Then we obtain for all sufficiently large \( j \) and for all \( \mu \) by (4.10) and since \( A = \lim_{j \to \infty} A^{(j)} \)

\[ \frac{1}{2\mu} \sum_{\mu} |a_{nk} - \alpha_k| \leq \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(j)} - a_{nk}| + \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(\ell)} - \alpha_k^{(j)}| + \frac{1}{2\mu} \sum_{\mu} |\alpha_k - \alpha_k^{(j)}| \\
\leq \| A^{(j)} - A \|_{(w, w)} + \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(j)} - \alpha_k^{(j)}| + \varepsilon \\
< 2 \cdot \varepsilon + \frac{1}{2\mu} \sum_{\mu} |a_{nk}^{(j)} - \alpha_k^{(j)}|. \]

Letting \( \mu \to \infty \), we obtain from (4.8)

\[ \lim_{\mu \to \infty} \left( \frac{1}{2\mu} \sum_{\mu} |a_{nk} - \alpha_k| \right) \leq \varepsilon. \]

Since \( \varepsilon > 0 \) was arbitrary, it follows that \( \alpha_k \) satisfies the condition in (3.11) of Example 3.5 (b). Using exactly the same argument as before with \( a_{nk}^{(j)} \) and \( \alpha_k^{(j)} \) replaced by \( \sum_{k=0}^{\infty} a_{nk}^{(j)} \) and \( \tilde{\alpha}^{(j)} \), and applying (4.9) instead of (4.8), we conclude that \( \tilde{\alpha} = \lim_{j \to \infty} \tilde{\alpha}^{(j)} \)
exists and satisfies the condition in (3.12) of Example 3.5 (b). Finally \( A \in (w, w) \) and (3.11) and (3.12) imply \( A \in (w, w) \) by Example 3.5 (b). Thus we have shown that \( (w, w) \) is complete. This completes the proof of (i).

Now we show that \( A, B \in (w, w) \) implies \( B \cdot A \in (w, w) \). Since \( A, B \in (w, w) \), by Example 3.5 (b), there are complex numbers \( \alpha_k, \tilde{\alpha} \) that satisfy (3.11) and (3.12), and
complex numbers $\beta_k, \tilde{\beta}$ that satisfy (3.11) and (3.12) with $b_{nk}, \tilde{\beta}_k$ and $\tilde{\beta}$ instead of $a_{nk}, \tilde{\alpha}_k$ and $\tilde{\alpha}$. Let $x \in w$ be given and $\xi$ be the strong limit of $x$. We put
\[
\zeta = \left( \tilde{\beta} - \sum_{n=0}^{\infty} \beta_n \right) \cdot \left( \left( \tilde{\alpha} - \sum_{k=0}^{\infty} \alpha_k \right) \cdot \xi + \sum_{k=0}^{\infty} \alpha_k x_k \right) + \sum_{n=0}^{\infty} \beta_n A_n x.
\]
We observe that $(\alpha_k)_{k=0}^{\infty}, (\beta_n)_{n=0}^{\infty} \in \mathcal{M}$ by the proof of Theorem 3.6, and also trivially $\mathcal{M} \subset \ell_1 \subset cs$. Therefore all the series in the definition of $\zeta$ converge. We write $C = B \cdot A$, $y = Ax$, $\eta$ for the strong limit of the sequence $y$, and $\zeta'$ for the strong limit of the sequence $z = By$. Since $Cx = B(Ax)$ by Lemma 4.1 (b), we obtain by (3.13) in Theorem 3.6
\[
\left| C_m x - \zeta \right| = \left| B_m y - \zeta \right|
\]
\[
= \left| B_m y - \sum_{n=0}^{\infty} \beta_n y_n - \left( \tilde{\beta} - \sum_{n=0}^{\infty} \beta_n \right) \cdot \left( \left( \tilde{\alpha} - \sum_{k=0}^{\infty} \alpha_k \right) \cdot \xi + \sum_{k=0}^{\infty} \alpha_k x_k \right) \right|
\]
\[
= \left| B_m y - \sum_{n=0}^{\infty} \beta_n y_n - \left( \tilde{\beta} - \sum_{n=0}^{\infty} \beta_n \right) \cdot \eta \right|
\]
\[
= \left| z_m - \sum_{n=0}^{\infty} \beta_n (y_n - \eta) + \eta \tilde{\beta} \right| = \left| z_m - \zeta' \right| \text{ for all } m,
\]
hence
\[
\lim_{\mu \to \infty} \left( \frac{1}{2^\mu} \sum_{m \in M<\mu>} \left| C_m x - \zeta \right| \right) = \lim_{\mu \to \infty} \left( \frac{1}{2^\mu} \sum_{m \in M<\mu>} \left| z_m - \zeta' \right| \right) = 0.
\]
This shows that $Cx \in w$, and completes the proof of (ii).

### References


