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Banach Algebras of Matrix Transformations Between Spaces of Strongly Bounded and Summable Sequences

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Abstract

Let $1 \leq p < \infty$. We characterise the classes (X, Y) of all infinite matrices that map X into Y for $X = w_\infty^p(\Lambda)$ or $X = w_0^p(\Lambda)$ and $Y = w_\infty^1(\Lambda')$, for $X = w_\infty^1(\Lambda)$ and $Y = w_\infty^p(\Lambda')$, and for $X = \mathcal{M}_p(\Lambda)$ and $Y = \mathcal{M}_1(\Lambda')$, the β-duals of $w_\infty^p(\Lambda)$ and $w_\infty^1(\Lambda')$. As special cases, we obtain the characterisations of the classes of all infinite matrices that map w_∞^p into w_∞^1, and w^p into w^1. Furthermore, we prove that the classes $(w_\infty(\Lambda), w_\infty(\Lambda))$ and (w, w) are Banach algebras.

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1 Introduction

Maddox [5] introduced the set w^p of all complex sequences $x = (x_k)_{k=0}^\infty$ that are strongly summable with index p by the Cesàro method of order 1; that is, w^p contains all sequences x for which $\lim_{n\to\infty} \sigma_n^p(x;\xi) = 0$ for some complex number ξ, where

$$\sigma_n^p(x;\xi) = \frac{1}{n+1}\sum_{k=0}^{n} |x_k - \xi|^p \text{ for all } n = 0, 1, \ldots .$$

We will also consider the sets w_0^p and w_∞^p of all sequences that are strongly summable to zero and strongly bounded, with index p; that is, the sets w_0^p and w_∞^p contain all

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sequences x for which $\lim_{n\to\infty} \sigma_n^p(x;0) = 0$ and $\sup_n \sigma_n^p(x;0) < \infty$, respectively. Maddox also established necessary and sufficient conditions on the entries of an infinite matrix to map w^p into the space c of all convergent sequences; his result is similar to the famous classical result by Silverman–Toeplitz which characterises the class (c,c) of all matrices that map c into c, the so-called conservative matrices .

Characterisations of classes of matrix transformations between sequence spaces constitute a wide, interesting and important field in both summability and operator theory. These results are needed to determine the corresponding subclasses of compact matrix operators, for instance in [1, 13], and more recently, of general linear operators between the respective sequence spaces, for instance in [2, 7]. They are also applied in studies on the invertibility of operators and the solvability of infinite systems of linear equations, for instance in [6, 8]. To be able to apply methods from the theory of Banach algebras to the solution of those problems, it is essential to determine if a class of linear operators of a sequence space X into itself is a Banach algebra; this is nontrivial if X is a BK space that does not have AK. Finally the characterisations of compact operators can be used to establish sufficient conditions for an operator to be a Fredholm operator, as in [3].

The spaces $w_\infty^p(\Lambda)$ and $w_0^p(\Lambda)$ for exponentially bounded sequences Λ and $1 \le p < \infty$ were introduced in [10]; they are generalisations of the spaces w_∞^p and w_0^p. Their dual spaces were determined in [11]. In this paper, we establish the new characterisations of the classes (X,Y) of all infinite matrices that map X into Y for $X = w_\infty^p(\Lambda)$ or $X = w_0^p(\Lambda)$ and $Y = w_\infty^1(\Lambda')$, for $X = w_\infty^1(\Lambda)$ and $Y = w_\infty^p(\Lambda')$, and when X is the β-dual of $w_\infty^p(\Lambda)$ or $w_0^p(\Lambda)$ and Y is the β-dual of $w_\infty^1(\Lambda')$. As a special case, we obtain the characterisations of the classes of all infinite matrices that map w_∞^p into w_∞^1, and w^p into w^1, the last result being similar to Maddox's and the Silverman–Toeplitz theorems. Furthermore, we prove that the classes $(w_\infty(\Lambda), w_\infty(\Lambda))$ and (w,w) are Banach algebras. Our results would be essential for further research in the areas mentioned above.

2 Notations and Known Results

Let ω denote the set of all sequences $x = (x_k)_{k=0}^\infty$, and ℓ_∞, c_0 and ϕ be the sets of all bounded, null and finite complex sequences, respectively; also let cs, bs and

$$\ell_p = \left\{ x \in \omega : \sum_{k=0}^\infty |x_k|^p < \infty \right\} \text{ for } 1 \le p < \infty$$

be the sets of all convergent, bounded and absolutely p-summable series. We write e and $e^{(n)}$ $(n = 0, 1, \ldots)$ for the sequences with $e_k = 0$ for all k, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ for $k \ne n$. A sequence (b_n) in a linear metric space X is called a Schauder basis of X if for every $x \in X$ there exists a unique sequence λ_n of scalars such that $x = \sum_n \lambda_n b_n$.

A BK *space* X is a Banach sequence space with continuous coordinates $P_n(x) = x_n$ $(n \in \mathbb{N}_0)$ for all $x \in X$; a BK space $X \supset \phi$ is said to have AK if $x = \sum_{k=0}^{\infty} x_k e^{(k)}$ for every sequence $x = (x_k)_{k=0}^{\infty} \in X$. Let X be a subset of ω. Then the set

$$X^{\beta} = \{a \in \omega : ax = (a_k x_k)_{k=0}^{\infty} \in cs \text{ for all } x \in X\}$$

is called the β-*dual of* X. Let $A = (a_{nk})_{k=0}^{\infty}$ be an infinite matrix of complex numbers and $x = (x_k)_{k=0}^{\infty} \in \omega$. Then we write $A_n = (a_{nk})_{k=0}^{\infty}$ $(n = 0, 1, \ldots)$ and $A^k = (a_{nk})_{n=0}^{\infty}$ $(k = 0, 1, \ldots)$ for the sequences in the n-th row and the k-th column of A, and $A_n x = \sum_{k=0}^{\infty} a_{nk} x_k$ provided the series converges. Given any subsets X and Y of ω, then (X, Y) denotes the class of all infinite matrices A that map X into Y, that is, $A_n \in X^{\beta}$ for all n, and $Ax = (A_n x)_{n=0}^{\infty} \in Y$.

Let X and Y be Banach spaces and $B_X = \{x \in X : \|x\| \leq 1\}$ denote the unit ball in X. Then we write $\mathcal{B}(X, Y)$ for the Banach space of all bounded linear operators $L : X \to Y$ with the operator norm $\|L\| = \sup_{x \in B_X} \|L(x)\|$. We write $X^* = \mathcal{B}(X, \mathbb{C})$ for the *continuous dual* of X with the norm $\|f\| = \sup_{x \in B_X} |f(x)|$ for all $f \in X^*$. The following results and definitions are well known. Since we will frequently apply them, we state them here for the reader's convenience.

Proposition 2.1. *Let X and Y be BK spaces.*

(a) Then we have $(X, Y) \subset \mathcal{B}(X, Y)$; this means that if $A \in (X, Y)$, then $L_A \in \mathcal{B}(X, Y)$, where $L_A(x) = Ax$ $(x \in X)$ (see [14, Theorem 4.2.8]).

(b) If X has AK then we have $\mathcal{B}(X, Y) \subset (X, Y)$; this means every $L \in \mathcal{B}(X, Y)$ is given by a matrix $A \in (X, Y)$ such that $L(x) = Ax$ $(x \in X)$ (see [4, Theorem 1.9]).

A nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ of positive reals is called *exponentially bounded* if there is an integer $m \geq 2$ such that for all nonnegative integers ν there is at least one term λ_n in the interval $I_m^{(\nu)} = [m^{\nu}, m^{\nu+1} - 1]$ ([10]). It was shown ([10, Lemma 1]) that a nondecreasing sequence $\Lambda = (\lambda_n)_{n=0}^{\infty}$ is exponentially bounded, if and only if the following condition holds

$$\text{(I)} \left\{ \begin{array}{c} \text{There are reals } s \leq t \text{ such that for some subsequence } (\lambda_{n(\nu)})_{\nu=0}^{\infty} \\ 0 < s \leq \lambda_{n(\nu)}/\lambda_{n(\nu+1)} \leq t < 1 \ (\nu = 0, 1, \ldots); \end{array} \right.$$

such a subsequence is called an *associated subsequence*.

Example 2.2. A simple, but important exponentially bounded sequence is the sequence Λ with $\lambda_n = n + 1$ for $n = 0, 1, \ldots$; an associated subsequence is given by $\lambda_{n(\nu)} = 2^{\nu}$, $\nu = 0, 1, \ldots$.

Throughout, let $1 \leq p < \infty$ and q be the conjugate number of p, that is, $q = \infty$ for $p = 1$ and $q = p/(p-1)$ for $1 < p < \infty$. Also let $(\mu_n)_{n=0}^{\infty}$ be a nondecreasing sequence of positive reals tending to infinity. Furthermore let $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an exponentially bounded sequence, and $(\lambda_{n(\nu)})_{\nu=0}^{\infty}$ an associated subsequence with $\lambda_{n(0)} = \lambda_0$. We write $K^{<\nu>}$ $(\nu = 0, 1, \ldots)$ for the set of all integers k with $n(\nu) \leq k \leq n(\nu+1) - 1$, and define the sets

$$\tilde{w}_0^p(\mu) = \left\{ x \in \omega : \lim_{n \to \infty} \left(\frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \right) = 0 \right\},$$

$$\tilde{w}_\infty^p(\mu) = \left\{ x \in \omega : \sup_n \left(\frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \right) < \infty \right\},$$

$$w_0^p(\Lambda) = \left\{ x \in \omega : \lim_{\nu \to \infty} \left(\frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K^{<\nu>}} |x_k|^p \right) = 0 \right\},$$

and

$$w_\infty^p(\Lambda) = \left\{ x \in \omega : \sup_\nu \left(\frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K^{<\nu>}} |x_k|^p \right) < \infty \right\}.$$

If $p = 1$, we omit the index p throughout, that is, we write $\tilde{w}_0(\mu) = \tilde{w}_\infty^1(\mu)$ etc., for short.

Proposition 2.3 (See [10, Theorem 1 (a), (b)]). *Let $(\mu_n)_{n=0}^{\infty}$ be a nondecreasing sequence of positive reals tending to infinity, $\Lambda = (\lambda_n)_{n=0}^{\infty}$ be an exponentially bounded sequence and $(\lambda_{n(\nu)})_{n=0}^{\infty}$ be an associated subsequence.*

(a) *Then $\tilde{w}_0^p(\mu)$ and $\tilde{w}_\infty^p(\mu)$ are BK spaces with the sectional norm $\|\cdot\|_\mu^{\tilde{}}$ defined by*

$$\|x\|_\mu^{\tilde{}} = \sup_n \left(\frac{1}{\mu_n} \sum_{k=0}^{n} |x_k|^p \right)^{1/p}$$

and $\tilde{w}_0^p(\mu)$ has AK.

(b) *We have $\tilde{w}_0^p(\Lambda) = w_0^p(\Lambda)$, $\tilde{w}_\infty^p(\Lambda) = w_\infty^p(\Lambda)$, and the sectional norm $\|\cdot\|_\Lambda^{\tilde{}}$ and the block norm $\|\cdot\|_\Lambda$ with*

$$\|x\|_\Lambda = \sup_\nu \left(\frac{1}{\lambda_{n(\nu+1)}} \sum_{k \in K^{<\nu>}} |x_k|^p \right)^{1/p}$$

are equivalent on $w_0^p(\Lambda)$ and on $w_\infty^p(\Lambda)$.

Remark 2.4. (a) It can be shown that $w_\infty^p(\Lambda)$ is not separable, and so has no Schauder basis.

(b) It follows from [14, Corollary 4.2.4] and Proposition 2.3, that $w_0^p(\Lambda)$ and $w_\infty^p(\Lambda)$ are BK spaces with the norm $\|\cdot\|_\Lambda$ and that $w_0^p(\Lambda)$ has AK.

Example 2.5. We might also define the set

$$w^p(\Lambda) = w_0^p(\Lambda) \oplus e = \{x \in \omega : x - \xi \in w_0^p \text{ for some complex number } \xi\}.$$

It can be shown that the *strong* Λ-*limit* ξ of any $x \in w^p(\Lambda)$ is unique if and only if

$$\overline{\lambda} = \limsup_{\nu\to\infty} \frac{n(\nu+1) - n(\nu)}{\lambda_{n(\nu+1)}} > 0,$$

and that $w^p(\Lambda) \subset w_\infty^p(\Lambda)$ if and only if

$$\overline{\Lambda} = \sup_\nu \frac{n(\nu+1) - n(\nu)}{\lambda_{n(\nu+1)}} < \infty.$$

In view of Proposition 2.3 (b) and Example 2.2, the sets $w_0^p(\Lambda)$ and $w_\infty^p(\Lambda)$ reduce to the BK spaces w_0^p and w_∞^p for $\lambda_n = n+1$ for $n = 0, 1, \ldots$; it is also clear that then $\overline{\Lambda} < \infty$ and $\overline{\lambda} > 0$, and consequently w^p is a BK space and the strong limit ξ of each sequence $x \in w^p$ is unique.

Throughout, we write $\|\cdot\| = \|\cdot\|_\Lambda$, for short.

The β-duals play a much more important role than the continuous duals in the theory of sequence spaces and matrix transformations. Let a be a sequence and X be a normed sequence space. Then we write $\|a\|_X^* = \sup_{x\in B_X} |\sum_{k=0}^\infty a_k x_k|$ provided the expression on the right exists and is finite, which is the case whenever X is a BK space and $a \in X^\beta$ by [14, Theorem 7.2.9]. If Λ is an exponentially bounded sequence with an associated subsequence $\lambda_{n(\nu)}$, then we write $\max_\nu$ and $\sum_\nu$ for the maximum and sum taken over all $k \in K^{<\nu>}$. We denote by $x^{<\nu>} = \sum_\nu x_k e^{(k)}$ ($\nu \in \mathbb{N}_0$) the ν-block of the sequence x.

Parts (a) and (b) of the next result are [11, Theorem 5.5 (a), (b)], Part (c) is [11, Theorem 5.7], and Parts (d) and (e) are [11, Theorem 5.8 (a), (b)].

Proposition 2.6. *Suppose* $\Lambda = (\lambda_n)_{n=0}^\infty$ *is an exponentially bounded sequence and let* $(\lambda_{n(\nu)})_{\nu=0}^\infty$ *be an associated subsequence. We write*

$$\mathcal{M}_p(\Lambda) = \left\{ a \in \omega : \|a\|_{\mathcal{M}_p(\Lambda)} = \sum_{\nu=0}^\infty \left(\lambda_{n(\nu+1)}\right)^{1/p} \cdot \left\|a^{<\nu>}\right\|_q < \infty \right\}.$$

(a) Then we have $(w_0^p(\Lambda))^\beta = (w_\infty^p(\Lambda))^\beta = \mathcal{M}_p(\Lambda)$ *and*

$$\|\cdot\|_{\mathcal{M}_p(\Lambda)} = \|\cdot\|^*_{w_\infty^p(\Lambda)} = \|\cdot\|^*_{w_0^p(\Lambda)} \text{ on } \mathcal{M}_p(\Lambda). \tag{2.1}$$

(b) The continuous dual $w_0^p(\Lambda)^*$ *of* $w_0^p(\Lambda)$ *is norm isomorphic to* $\mathcal{M}_p(\Lambda)$ *with the norm* $\|\cdot\|_{\mathcal{M}_p(\Lambda)}$.

(c) Then $\mathcal{M}_p(\Lambda)$ *is a* BK *space with* AK *with respect to* $\|\cdot\|_{\mathcal{M}_p(\Lambda)}$.

(d) We have $(w_\infty^p(\Lambda))^{\beta\beta} = (w_0^p(\Lambda))^{\beta\beta} = w_\infty^p(\Lambda)$ *and*

$$\|\cdot\|^*_{\mathcal{M}_p(\Lambda)} = \|\cdot\| \text{ on } (\mathcal{M}_p(\Lambda))^\beta. \tag{2.2}$$

(e) The continuous dual $(\mathcal{M}_p(\Lambda))^*$ *of* $\mathcal{M}_p(\Lambda)$ *is norm isomorphic to* $w_\infty^p(\Lambda)$.

Remark 2.7. (a) The continuous dual of $w_\infty(\Lambda)$ is not given by a sequence space.

(b) The set $w_\infty^p(\Lambda)$ is β-*perfect*, that is, $(w_\infty^p(\Lambda))^{\beta\beta} = w_\infty^p(\Lambda)$.

3 Matrix Transformations on $w_\infty^p(\Lambda)$ and $w_0^p(\Lambda)$

Let $\Lambda = (\lambda_k)_{k=0}^\infty$ and $\Lambda' = (\lambda'_m)_{m=0}^\infty$ be exponentially bounded sequences and $(\lambda_{k(\nu)})_{\nu=0}^\infty$ and $(\lambda'_{m(\mu)})_{\mu=0}^\infty$ be associated subsequences. Furthermore, let $K^{<\nu>}$ $(\nu = 0, 1, \dots)$ and $M^{<\mu>}$ $(\mu = 0, 1, \dots)$ be the sets of all integers k and m with $k(\nu) \le k \le k(\nu+1) - 1$ and $m(\mu) \le m \le m(\mu+1) - 1$. If $A = (a_{mk})_{m,k=0}^\infty$ is an infinite matrix and $\mathbf{M} = (M_\mu)_{\mu=0}^\infty$ is a sequence of subsets M_μ of $M^{<\mu>}$ for $\mu = 0, 1, \dots$, we write $S^{\mathbf{M}}(A)$ for the matrix with the rows

$$S_\mu^{\mathbf{M}}(A) = \sum_{m \in M_\mu} A_m, \text{ that is, } s_{\mu k}^{\mathbf{M}}(A) = \sum_{m \in M_\mu} a_{mk} \text{ for all } \mu, k = 0, 1, \dots.$$

Here we establish necessary and sufficient conditions for an infinite matrix A to be in the classes $(w_\infty^p(\Lambda), w_\infty(\Lambda'))$, $(w_0^p(\Lambda), w_\infty(\Lambda'))$ and $(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$, and consider the special cases of (w_∞^p, w_∞) and (w^p, w). We also estimate the operator norms of L_A in these cases. Those characterisations and estimates are needed in the proofs of our results on Banach algebras of matrix transformations.

First we characterise the classes $(w_\infty^p(\Lambda), w_\infty(\Lambda'))$ and $(w_0^p(\Lambda), w_\infty(\Lambda'))$, and estimate the operator norm of L_A when the matrix A is a member of those classes.

Theorem 3.1. *Let* $\Lambda = (\lambda_k)_{k=0}^\infty$ *and* $\Lambda' = (\lambda'_m)_{m=0}^\infty$ *be exponentially bounded sequences and* $(\lambda_{k(\nu)})_{\nu=0}^\infty$ *and* $(\lambda'_{m(\mu)})_{\mu=0}^\infty$ *be associated subsequences. Then we have* $A \in (w_\infty^p(\Lambda), w_\infty(\Lambda'))$ *if and only if*

$$\|A\|_{(\Lambda,\Lambda')} = \sup_\mu \left(\frac{1}{\lambda'_{m(\mu+1)}} \max_{M_\mu \subset M^{<\mu>}} \left\| S_\mu^{\mathbf{M}}(A) \right\|_{\mathcal{M}_p(\Lambda)} \right) < \infty; \tag{3.1}$$

moreover, we have $(w_\infty^p(\Lambda), w_\infty(\Lambda')) = (w_0^p(\Lambda), w_\infty(\Lambda'))$. *If* $A \in (w_\infty^p(\Lambda), w_\infty(\Lambda'))$, *then the operator norm of* L_A *satisfies*

$$\|A\|_{(\Lambda,\Lambda')} \leq \|L_A\| \leq 4 \cdot \|A\|_{(\Lambda,\Lambda')}. \tag{3.2}$$

Proof. Throughout the proof, we write $\|A\| = \|A\|_{(\Lambda,\Lambda')}$, for short.

First we assume that the condition in (3.1) is satisfied. Let $m \in \mathbb{N}_0$ be given. Then there is a unique $\mu_m \in \mathbb{N}_0$ such that $m \in M^{<\mu_m>}$. We choose $M_{\mu_m} = \{m\}$, and it follows from (3.1) that $\|A_m\|_{\mathcal{M}_p(\Lambda)} < \infty$, that is, $A_m \in (w_\infty^p(\Lambda))^\beta$ by Proposition 2.6 (a). Thus we have shown $A_m \in (w_\infty(\Lambda))^\beta$ for all $m \in \mathbb{N}_0$. Now let $x \in w_\infty^p(\Lambda)$ be given. For each $\mu \in \mathbb{N}_0$, we write $M_{\mu(x)}$ for a subset of $M^{<\mu>}$ for which

$$\left|\sum_{m \in M_{\mu(x)}} A_m x\right| = \max_{M_\mu \subset M^{<\mu>}} \left|\sum_{m \in M_\mu} A_m x\right|,$$

and put $\mathbf{M(x)} = (M_{\mu(x)})_{\mu=0}^\infty$. Then we have by a well-known inequality (see [12]), (2.1) and (3.1)

$$\begin{aligned}
&\frac{1}{\lambda'_m(\mu+1)} \sum_{m \in M^{<\mu>}} |A_m x| \leq 4 \cdot \frac{1}{\lambda'_m(\mu+1)} \max_{M_\mu \subset M^{<\mu>}} \left|\sum_{m \in M_\mu} A_m x\right| \\
&= 4 \cdot \frac{1}{\lambda'_m(\mu+1)} \left|\sum_{m \in M_{\mu(x)}} \sum_{k=0}^\infty a_{mk} x_k\right| = 4 \cdot \frac{1}{\lambda'_m(\mu+1)} \left|\sum_{k=0}^\infty \left(\sum_{m \in M_{\mu(x)}} a_{mk}\right) x_k\right| \\
&\leq 4 \cdot \frac{1}{\lambda'_m(\mu+1)} \left|S_\mu^{\mathbf{M(x)}}(A) x\right| \leq 4 \cdot \frac{1}{\lambda'_m(\mu+1)} \left\|S_\mu^{\mathbf{M(x)}}(A)\right\|_{\mathcal{M}_p(\Lambda)} \cdot \|x\| \\
&\leq 4 \cdot \frac{1}{\lambda'_m(\mu+1)} \cdot \left(\max_{M_\mu \subset M^{<\mu>}} \left\|S_\mu^{\mathbf{M}}(A)\right\|_{\mathcal{M}_p(\Lambda)}\right) \cdot \|x\| \leq 4 \cdot \|A\| \cdot \|x\| < \infty \text{ for all } \mu.
\end{aligned}$$

Hence it follows that

$$\|Ax\| = \sup_\mu \left(\frac{1}{\lambda'_m(\mu+1)} \sum_{m \in M^{<\mu>}} |A_m x|\right) \leq 4 \cdot \|A\| \cdot \|x\| < \infty, \tag{3.3}$$

and consequently $Ax \in w_\infty(\Lambda')$ for all $x \in w_\infty^p(\Lambda)$. Thus, we have shown that if the condition in (3.1) is satisfied, then $A \in (w_\infty^p(\Lambda), w_\infty(\Lambda')) \subset (w_0^p(\Lambda), w_\infty(\Lambda'))$.

Conversely, we assume $A \in (w_0^p(\Lambda), w_\infty(\Lambda'))$. Then we have $A_m \in (w_0^p(\Lambda))^\beta$ for all $m \in \mathbb{N}_0$, hence $\|A_m\|_{\mathcal{M}_p(\Lambda)} < \infty$ for all m by Proposition 2.6 (a). Since $w_0^p(\Lambda)$ and $w_\infty(\Lambda')$ are a BK spaces by Remark 2.4 (b), it follows from Proposition 2.1 (a) that $L_A \in \mathcal{B}(w_0^p(\Lambda), w_\infty(\Lambda'))$, and so $\|L_A\| < \infty$. We also have $L_{M_\mu} \in (w_0^p(\Lambda))^*$ for all $M_\mu \subset M^{<\mu>}$ and all $\mu \in \mathbb{N}_0$, where $L_{M_\mu}(x) = (\lambda'_{m(\mu+1)})^{-1} \cdot \sum_{m \in M_\mu} A_m x$ for all $x \in$

$w_0^p(\Lambda)$. Since trivially $|L_{M_\mu}(x)| \leq \|L_A(x)\| \leq \|L_A\| \cdot \|x\|$ for all $x \in w_0^p(\Lambda)$, all $M_\mu \subset M^{<\mu>}$ and all $\mu \in \mathbb{N}_0$, it follows by (2.1) in Proposition 2.6 (a) that $\|L_{M_\mu}\|_{\mathcal{M}_p(\Lambda)} = \|L_{M_\mu}\|^*_{w_0^p(\Lambda)} \leq \|L_A\|$ for all $M_\mu \subset M^{<\mu>}$ and $\mu \in \mathbb{N}_0$, and so

$$\sup_\mu \left(\max_{M_\mu \subset M^{<\mu>}} \left\| L_{M_\mu} \right\|_{\mathcal{M}_p(\Lambda)} \right) = \sup_\mu \left(\frac{1}{\lambda'_{m(\mu+1)}} \max_{M_\mu \subset M^{<\mu>}} \left\| \sum_{m \in M_\mu} A_m \right\|_{\mathcal{M}_p(\Lambda)} \right)$$
$$= \|A\| \leq \|L_A\| < \infty. \tag{3.4}$$

Thus we have shown that if $A \in (w_0^p(\Lambda), w_\infty(\Lambda'))$, then (3.3) is satisfied.

It remains to show that if $A \in (w_\infty^p(\Lambda), w_\infty(\Lambda'))$, then (3.2) holds. But the first and second inequalities in (3.2) follow from (3.4) and (3.3), respectively. □

Now we characterise the class $(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$, and estimate the operator norm of L_A when $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$. We write $\mathcal{T}$ for the set of all sequences $t = (t_\mu)_{\mu=0}^\infty$ such that for each $\mu = 0, 1, \ldots$ there is one and only one $t_\mu \in M^{<\mu>}$.

Theorem 3.2. *Let $\Lambda = (\lambda_k)_{k=0}^\infty$ and $\Lambda' = (\lambda'_m)_{m=0}^\infty$ be exponentially bounded sequences and $(\lambda_{k(\nu)})_{\nu=0}^\infty$ and $(\lambda'_{m(\mu)})_{\mu=0}^\infty$ be associated subsequences. Then we have $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$ if and only if*

$$\|A\|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))} = \sup_{\substack{N \subset \mathbb{N}_0 \\ N \text{ finite}}} \left(\sup_{t \in \mathcal{T}} \left\| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t_\mu} \right\|_\Lambda \right) < \infty, \tag{3.5}$$

where, of course,

$$\left\| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t_\mu} \right\|_\Lambda = \sup_\nu \left(\frac{1}{\lambda_{k(\nu+1)}} \sum_{k \in K^{<\nu>}} \left| \sum_{\mu \in N} \lambda'_{m(\mu+1)} a_{t_\mu, k} \right|^p \right)^{1/p}.$$

If $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$, then the operator norm of L_A satisfies

$$\|A\|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))} \leq \|L_A\| \leq 4 \cdot \|A\|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))}. \tag{3.6}$$

Proof. Throughout the proof, we write $\|A\| = \|A\|_{(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))}$, for short.

First we assume that the condition in (3.5) is satisfied. Let $m \in \mathbb{N}_0$ be given. Then there is a unique $\mu_m \in \mathbb{N}_0$ such that $m \in M^{<\mu_m>}$. We choose $N = \{m\}$ and $t_{\mu_m} = m$. Then it follows from (3.5) that

$$\|A_m\|_\Lambda = \sup_\nu \left(\frac{1}{\lambda_{k(\nu+1)}} \sum_{k \in K^{<\nu>}} |a_{mk}|^p \right)^{1/p}$$
$$= \frac{1}{\lambda'_{m(\mu_m+1)}} \cdot \sup_\nu \left\| \sum_{m \in N} \lambda'_{m(\mu_m+1)} A_{t_{\mu_m}} \right\|_\Lambda < \infty,$$

and so $A_m \in w_\infty^p(\Lambda) = (\mathcal{M}_p(\Lambda))^\beta$ by Proposition 2.6 (a) and (d). Now let $\mu_0 \in \mathbb{N}_0$ and $x \in \mathcal{M}_p(\Lambda)$ be given. For each $\mu \in \mathbb{N}_0$ with $0 \leq \mu \leq \mu_0$, let $m(\mu; x)$ be the smallest integer in $M^{<\mu>}$ such that $\max_{m \in M^{<\mu>}} |A_m x| = |A_{m(\mu;x)}x|$. Then we have by a well-known inequality (see [12]) and (2.2) in Proposition 2.6 (d)

$$\begin{aligned}
&\sum_{\mu=0}^{\mu_0} \lambda'_{m(\mu+1)} \max_{m \in M^{<\mu>}} |A_m x| = \sum_{\mu=0}^{\mu_0} \lambda'_{m(\mu+1)} |A_{m(\mu;x)}x| \\
&\leq 4 \cdot \max_{N \subset \{0,\dots,\mu_0\}} \left| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{m(\mu;x)} x \right| \\
&= 4 \cdot \max_{N \subset \{0,\dots,\mu_0\}} \left| \sum_{k=0}^{\infty} \left(\sum_{\mu \in N} \lambda'_{m(\mu+1)} a_{m(\mu;x),k} \right) x_k \right| \\
&\leq 4 \cdot \max_{N \subset \{0,\dots,\mu_0\}} \left[\sup_\nu \left(\frac{1}{\lambda_{k(\nu+1)}} \sum_{k \in K^{<\nu>}} \left| \sum_{\mu \in N} \lambda'_{m(\mu+1)} a_{m(\mu;x),k} \right|^p \right)^{1/p} \right] \|x\|_{\mathcal{M}_p(\Lambda)} \\
&\leq 4 \cdot \max_{N \subset \{0,\dots,\mu_0\}} \left(\left\| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{m(\mu;x)} \right\|_\Lambda \right) \cdot \|x\|_{\mathcal{M}_p(\Lambda)} \\
&\leq 4 \cdot \sup_{\substack{N \subset \mathbb{N}_0 \\ N \text{ finite}}} \left(\sup_{t \in \mathcal{T}} \left\| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t_\mu} \right\|_\Lambda \right) \cdot \|x\|_{\mathcal{M}_p(\Lambda)} = 4 \cdot \|A\| \cdot \|x\|_{\mathcal{M}_p(\Lambda)} < \infty.
\end{aligned}$$

Since $\mu_0 \in \mathbb{N}_0$ was arbitrary, we obtain

$$\|Ax\|_{\mathcal{M}(\Lambda')} \leq 4 \cdot \|A\| \cdot \|x\|_{\mathcal{M}_p(\Lambda)} < \infty \text{ for all } x \in \mathcal{M}_p(\Lambda), \tag{3.7}$$

and consequently $Ax \in \mathcal{M}(\Lambda')$ for all $x \in \mathcal{M}_p(\Lambda)$. Thus we have shown that if the condition in (3.5) is satisfied, then $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$.

Conversely, we assume $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$. Then $A_m \in (\mathcal{M}_p(\Lambda))^\beta = w_\infty^p(\Lambda)$ for all $m \in \mathbb{N}_0$ by Proposition 2.6 (a) and (d). Furthermore, since $\mathcal{M}_p(\Lambda)$ and $\mathcal{M}(\Lambda')$ are BK spaces by Proposition 2.6 (c), it follows from Proposition 2.1 (a) that $L_A \in \mathcal{B}(\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$. We also have $L_{N,t} \in (\mathcal{M}_p(\Lambda))^*$ for all finite subsets N of $\mathbb{N}_0$ and all sequences $t \in \mathcal{T}$, where $L_{N,t}(x) = \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t_\mu} x$ for all $x \in \mathcal{M}_p(\Lambda)$. Since trivially $|L_{N,t}(x)| \leq \sum_{\mu=0}^{\infty} \lambda'_{m(\mu+1)} \max_{m \in M^{<\mu>}} |A_m x| = \|Ax\|_{\mathcal{M}(\Lambda')} \leq \|L_A\| \cdot \|x\|_\Lambda$ for all finite subsets N of $\mathbb{N}_0$ and all $t \in \mathcal{T}$, it follows by (2.2) in Proposition 2.6 (c) that

$$\|L_{N,t}\|^*_{\mathcal{M}_p(\Lambda)} = \|L_{N,t}\|_\Lambda = \left\| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t_\mu} \right\|_\Lambda \leq \|L_A\| < \infty.$$

Since this holds for all finite subsets N of $\mathbb{N}_0$ and all $t \in \mathcal{T}$, we conclude

$$\|A\| = \sup_{\substack{N \subset \mathbb{N}_0 \\ N \text{ finite}}} \left(\sup_{t \in \mathcal{T}} \left\| \sum_{\mu \in N} \lambda'_{m(\mu+1)} A_{t_\mu} \right\|_{\Lambda} \right) \leq \|L_A\| < \infty. \tag{3.8}$$

Thus we have shown that if $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$, then (3.5) is satisfied.

Finally, if $A \in (\mathcal{M}_p(\Lambda), \mathcal{M}(\Lambda'))$, then (3.6) follows from (3.8) and (3.7). □

Using the transpose A^T of a matrix A, we obtain an alternative characterisation of the class $(w_\infty(\Lambda), w_\infty^p(\Lambda'))$.

Theorem 3.3. *We have* $A \in (w_\infty(\Lambda), w_\infty^p(\Lambda'))$ *if and only if*

$$\|A^T\|_{(\mathcal{M}_p(\Lambda'), \mathcal{M}(\Lambda))} < \infty. \tag{3.9}$$

Proof. Since $X = w_0(\Lambda)$ and $Z = \mathcal{M}_p(\Lambda')$ are BK spaces with AK by Remark 2.4 (b) and Proposition 2.6 (c), and $Y = Z^\beta = w_\infty^p(\Lambda')$ by Proposition 2.6 (d), it follows from [14, Theorem 8.3.9] that $A \in (w_0(\Lambda), w_\infty^p(\Lambda')) = (X, Y) = (X^{\beta\beta}, Y) = (w_\infty(\Lambda), w_\infty^p(\Lambda'))$ and $A \in ((w_0(\Lambda), w_\infty^p(\Lambda'))$ if and only if $A^T \in (Z, X^\beta) = (\mathcal{M}_p(\Lambda'), \mathcal{M}(\Lambda))$, and, by (3.5) in Theorem 3.2, this is the case if and only if (3.9) holds. □

We consider an application to the characterisations of the classes (w_∞^p, w_∞), (w^p, w) and (w_∞, w_∞^p). Let $\Lambda = \Lambda'$ and $\lambda_n = n + 1$ for $n = 0, 1, \ldots$ as in Examples 2.2 and 2.5. Then we may choose the subsequences given by $\lambda_{k(\nu)} = 2^\nu$ and $\lambda_{m(\mu)} = 2^\mu$ for all $\nu, \mu = 0, 1, \ldots$, and consequently the sets $K^{<\nu>}$ and $M^{<\mu>}$ are the sets of all integers k and m with $2^\nu \leq k \leq 2^{\nu+1} - 1$ and $2^\mu \leq m \leq 2^{\mu+1} - 1$. We also write $\mathcal{M}_p = \mathcal{M}_p(\Lambda)$.

Remark 3.4. (a) We obviously have $w_0^p \subset w^p \subset w_\infty^p$.

(b) For each $x \in w^p$, the *strong limit* ξ, that is, the complex number ξ with

$$\lim_{\nu \to \infty} \frac{1}{2^\nu} \sum_\nu |x_k - \xi|^p = 0$$

is unique (see [5]).

(c) Every sequence $x = (x_k)_{k=0}^\infty \in w^p$ has a unique representation

$$x = \xi \cdot e + \sum_{k=0}^\infty (x_k - \xi) e^{(k)} \text{ ([5]).}$$

Example 3.5. (a) It follows from Theorem 3.1 that $A \in (w_\infty^p, w_\infty) = (w_0^p, w_\infty)$ if and only if

$$\|A\|_{(w_\infty^p, w_\infty)} = \sup_\mu \left(\frac{1}{2^\mu} \max_{M_\mu \subset M^{<mu>}} \left\| \sum_{m \in M_\mu} A_m \right\|_{\mathcal{M}_p} \right) < \infty, \tag{3.10}$$

where

$$\left\| \sum_{m \in M_\mu} A_m \right\|_{\mathcal{M}_p} = \begin{cases} \displaystyle\sum_{\nu=0}^{\infty} 2^\nu \max_{k \in K^{<\nu>}} \left| \sum_{m \in M_\mu} a_{mk} \right| & (p = 1) \\ \displaystyle\sum_{\nu=0}^{\infty} 2^{\nu/p} \left(\sum_{k \in K^{<\nu>}} \left| \sum_{m \in M_\mu} a_{mk} \right|^q \right)^{1/q} & (1 < p < \infty). \end{cases}$$

(b) It follows from Part (a) and [14, 8.3.6, 8.3.7] that $A \in (w^p, w)$ if and only if (3.10),

$$\left\{ \begin{array}{c} \text{for each } k \text{ there exists a complex number } \alpha_k \text{ with} \\ \displaystyle\lim_{\mu \to \infty} \frac{1}{2^\mu} \sum_{m \in M^{<\mu>}} |a_{mk} - \alpha_k| = 0 \end{array} \right\} \tag{3.11}$$

and

$$\lim_{\mu \to \infty} \frac{1}{2^\mu} \sum_{m \in M^{<\mu>}} \left| \sum_{k=0}^{\infty} a_{mk} - \tilde{\alpha} \right| = 0 \text{ for some comlex number } \tilde{\alpha} \tag{3.12}$$

hold.

(c) We obtain from Theorems 3.2 and 3.3, interchanging the roles of N and K, and μ and ν, that $A \in (w_\infty, w_\infty^p)$ if and only if

$$\sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \left(\sup_{t \in \mathcal{T}} \left\| \sum_{\nu \in K} 2^\nu A^{t_\nu} \right\|_\Lambda \right) < \infty,$$

where

$$\left\| \sum_{\nu \in K} 2^\nu A^{t_\nu} \right\|_\Lambda = \sup_\mu \left(\frac{1}{2^\mu} \sum_{m \in M^{<\mu>}} \left| \sum_{\nu \in K} 2^\nu a_{m,t_\nu} \right| \right).$$

We also give a formula for the strong limit of Ax when $A \in (w^p, w)$ and $x \in w^p$.

Theorem 3.6. *If $A \in (w^p, w)$, then the strong limit η of Ax for each sequence $x \in w^p$ is given by*

$$\eta = \tilde{\alpha} \cdot \xi + \sum_{k=0}^{\infty} \alpha_k (x_k - \xi), \tag{3.13}$$

where ξ is the strong limit of the sequence x, and the complex numbers $\tilde{\alpha}$ and α_k for $k = 0, 1, \ldots$ are given by (3.12) *and* (3.11) *in Example 3.5 (b).*

Proof. We assume $A \in (w^p, w)$ and write $\|\cdot\| = \|\cdot\|_{(w^p_\infty, w_\infty)}$, for short. The complex numbers $\tilde{\alpha}$ and α_k for $k = 0, 1, \dots$ exist by Example 3.5 (b).

First, we show $(\alpha_k)_{k=0}^\infty \in \mathcal{M}_p$. Let $x \in w^p$ and $k_0 \in \mathbb{N}_0$ be given. Then there exists an integer $\nu(k_0)$ with $k_0 \in K^{<\nu(k_0)>}$ and we have by the inequality in [9, Lemma 1]

$$\begin{aligned}
\sum_{k=0}^{k_0} |\alpha_k x_k| &= \sum_{k=0}^{k_0} \left(\frac{1}{2^\mu} \sum_\mu |\alpha_k| \cdot |x_k| \right) \\
&\leq \frac{1}{2^\mu} \sum_{k=0}^{k_0} \left(\sum_\mu |a_{nk} - \alpha_k| \cdot |x_k| \right) + \sum_{\nu=0}^{\nu(k_0)} \sum_\nu \left(\frac{1}{2^\mu} \sum_\mu |a_{nk}| \cdot |x_k| \right) \\
&\leq \sum_{k=0}^{k_0} \left(\frac{1}{2^\mu} \sum_\mu |a_{nk} - \alpha_k| \right) \cdot |x_k| + 4 \cdot \max_{M_\mu \in M^{<\mu>}} \sum_{\nu=0}^{\infty} \sum_\nu \left(\frac{1}{2^\mu} \left| \sum_{n \in M_\mu} a_{nk} \right| \cdot |x_k| \right) \\
&\leq \sum_{k=0}^{k_0} \left(\frac{1}{2^\mu} \sum_\mu |a_{nk} - \alpha_k| \right) \cdot |x_k| + 4 \cdot \sup_\mu \left(\frac{1}{2^\mu} \max_{M_\mu \subset M^{<\mu>}} \left\| \sum_{n \in M_\mu} A_n \right\|_{\mathcal{M}_p} \right) \cdot \|x\|.
\end{aligned}$$

Letting μ tend to ∞, we obtain $\displaystyle\sum_{k=0}^{k_0} |\alpha_k x_k| \leq 0 + 4 \cdot \|A\| < \infty$ from (3.11) and (3.10).

Since $k_0 \in \mathbb{N}_0$ was arbitrary, it follows that $\displaystyle\sum_{k=0}^{\infty} |\alpha_k x_k| < \infty$ for all $x \in w^p$, that is, $(\alpha_k)_{k=1}^\infty \in (w^p)^\beta = \mathcal{M}_p$.

Now we write $\hat{\alpha}(x) = \displaystyle\sum_{k=0}^{\infty} \alpha_k x_k$ and $B = (b_{nk})_{n,k=0}^\infty$ for the matrix with $b_{nk} = a_{nk} - \alpha_k$ for all n and k, and show

$$\lim_{\mu \to \infty} \frac{1}{2^\mu} \sum_\mu |B_n x| = 0 \text{ for all } x \in w_0^p. \tag{3.14}$$

Let $x \in w_0^p$ and $\varepsilon > 0$ be given. Since w_0^p has AK, there is $k_0 \in \mathbb{N}_0$ such that

$$\left\| x - x^{[k_0]} \right\| < \frac{\varepsilon}{\|A\| + \|(\alpha_k)_{k=0}^\infty\|_{\mathcal{M}_p} + 1} \text{ for } x^{[k_0]} = \sum_{k=0}^{k_0} x_k e^{(k)}.$$

It also follows from (3.11) that there is $\mu_0 \in \mathbb{N}_0$ such that

$$\frac{1}{2^\mu} \sum_\mu \left| B_n x^{[k_0]} \right| = \frac{1}{2^\mu} \sum_\mu \left| \sum_{k=0}^{k_0} b_{nk} x_k \right| < \varepsilon \text{ for all } \mu \geq \mu_0.$$

Let $\mu \geq \mu_0$ be given. Then we have

$$\frac{1}{2^\mu} \sum_\mu |B_n x| \leq \frac{1}{2^\mu} \sum_\mu \left| B_n x^{[k_0]} \right| + \frac{1}{2^\mu} \sum_\mu \left| B_n \left(x - x^{[k_0]} \right) \right|$$

$$
\begin{aligned}
&< \varepsilon + 4 \cdot \max_{M_\mu \subset M^{<\mu>}} \left(\frac{1}{2^\mu} \left| \sum_{n \in M_\mu} B_n \left(x - x^{[k_0]} \right) \right| \right) \\
&\leq \varepsilon + 4 \cdot \max_{M_\mu \subset M^{<\mu>}} \left(\frac{1}{2^\mu} \left\| \sum_{n \in M_\mu} B_n \right\|_{\mathcal{M}_p} \right) \left\| x - x^{[k_0]} \right\| < 5 \cdot \varepsilon.
\end{aligned}
$$

Thus we have shown (3.14).

Finally, let $x \in w^p$ be given. Then there is a unique complex number ξ such that $x^{(0)} = x - \xi \cdot e \in w_0^p$, by Remark 3.4 (b), and we obtain by (3.14) and (3.12)

$$
\begin{aligned}
0 &\leq \frac{1}{2^\mu} \sum_\mu |A_n x - \eta| = \frac{1}{2^\mu} \sum_\mu \left| A_n x^{(0)} + \xi \cdot A_n(e) - \left(\tilde{\alpha} \cdot \xi + \sum_{k=0}^\infty \alpha_k x^{(0)} \right) \right| \\
&\leq \frac{1}{2^\mu} \sum_\mu \left| A_n x^{(0)} - \hat{\alpha}(x^{(0)}) \right| + |\xi| \cdot \frac{1}{2^\mu} \sum_\mu |A_n e - \tilde{\alpha}| \\
&= \frac{1}{2^\mu} \sum_\mu |B_n x^{(0)}| + |\xi| \cdot \frac{1}{2^\mu} \sum_\mu \left| \sum_{k=0}^\infty a_{nk} - \tilde{\alpha} \right| \to 0 + 0 = 0 \ (\mu \to \infty).
\end{aligned}
$$

This completes the proof. □

4 The Banach Algebra $(w_\infty(\Lambda), w_\infty(\Lambda))$

In this section, we show that $(w_\infty(\Lambda), w_\infty(\Lambda))$ is a Banach algebra with respect to the norm $\|\cdot\|$ defined by $\|A\| = \|L_A\|$ for all $A \in (w_\infty(\Lambda), w_\infty(\Lambda))$. We also consider the nontrivial special case of (w, w).

We need the following results.

Lemma 4.1. *(a) The matrix product $B \cdot A$ is defined for all $A, B \in (w_\infty(\Lambda), w_\infty(\Lambda))$; in fact*

$$
\sum_{m=0}^\infty |b_{nm} a_{mk}| \leq \|B_n\|_{\mathcal{M}(\Lambda)} \|A^k\| \text{ for all } n \text{ and } k. \tag{4.1}
$$

(b) Matrix multiplication is associative in $(w_\infty(\Lambda), w_\infty(\Lambda))$.

(c) The space $(w_\infty(\Lambda), w_\infty(\Lambda))$ is a Banach space with respect to

$$
\|A\|_{(\Lambda,\Lambda)} = \sup_\mu \left(\frac{1}{\lambda_{m(\mu+1)}} \max_{M_\mu \subset M^{<\mu>}} \sum_{\nu=0}^\infty \lambda_{k(\nu+1)} \max_{k \in K^{<\nu>}} \left| \sum_{m \in M_\mu} a_{mk} \right| \right). \tag{4.2}
$$

Proof. (a) Let $A, B \in (w_\infty(\Lambda), w_\infty(\Lambda))$. First we observe that $e^{(k)} \in w_\infty(\Lambda)$ implies $Ae^{(k)} = (A_m e^{(k)})_{m=0}^\infty = (a_{mk})_{m=0}^\infty = A^k \in w_\infty(\Lambda)$ for all k. Therefore we have

$$\|A^k\| = \sup_\mu \left(\frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{<\mu>}} |a_{mk}| \right) < \infty \text{ for all } k. \tag{4.3}$$

Furthermore $B \in (w_\infty(\Lambda), w_\infty(\Lambda))$ implies $B_n \in (w_\infty(\Lambda))^\beta = \mathcal{M}(\Lambda)$ for all n, that is, by Proposition 2.6 (a)

$$\|B_n\|_{\mathcal{M}(\Lambda)} = \sum_{\mu=0}^\infty \lambda_{m(\mu+1)} \max_{m \in M^{<\mu>}} |b_{nm}| < \infty \text{ for all } n. \tag{4.4}$$

Now it follows from (4.3) and (4.4) that

$$\begin{aligned}
\left|B_n A^k\right| &\leq \sum_{m=0}^\infty |b_{nm} a_{mk}| = \sum_{\mu=0}^\infty \sum_{m \in M^{<\mu>}} \lambda_{m(\mu+1)} |b_{nm}| \cdot \frac{1}{\lambda_{m(\mu+1)}} |a_{mk}| \\
&\leq \sum_{\mu=0}^\infty \left[\left(\lambda_{m(\mu+1)} \max_{m \in M^{<\mu>}} |b_{nm}| \right) \cdot \left(\frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{<\mu>}} |a_{mk}| \right) \right] \\
&\leq \left(\sum_{\mu=0}^\infty \left(\lambda_{m(\mu+1)} \max_{m \in M^{<\mu>}} |b_{nm}| \right) \right) \cdot \sup_\mu \left(\frac{1}{\lambda_{m(\mu)+1}} \sum_{m \in M^{<\mu>}} |a_{mk}| \right) \\
&= \|B_n\|_{\mathcal{M}(\Lambda)} \cdot \|A^k\| < \infty \text{ for all } n \text{ and } k.
\end{aligned}$$

(b) Let $A, B, C \in (w_\infty(\Lambda), w_\infty(\Lambda))$. We write for $D \in (w_\infty(\Lambda), w_\infty(\Lambda))$

$$M^T(D) = \|D^T\|_{(\mathcal{M}(\Lambda), \mathcal{M}(\Lambda))} = \sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \left(\sup_{t \in \mathcal{T}} \left(\sup_\mu \frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{<\mu>}} \left| \sum_{\nu \in K} \lambda_{n(\nu+1)} d_{m,t_\nu} \right| \right) \right)$$

and note that $M^T(D) < \infty$ by Theorem 3.3. We are going to show that the series $\sum_{k=0}^\infty \sum_{m=0}^\infty a_{nm} b_{mk} c_{kj}$ are absolutely convergent for all n and j. We fix n and j and write $s = A_n$ and $t = C^j$ for the sequences in the n-th row of A and the j-th column of C. Then we have $s \in \mathcal{M}(\Lambda)$ and $t \in w_\infty(\Lambda)$. We define the matrix $D = (d_{\mu k})_{\mu,k=0}^\infty$ by

$$d_{\mu k} = \frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{<\mu>}} |b_{mk}| \text{ for } \mu, k = 0, 1, \ldots.$$

Furthermore, given $\mu \in \mathbb{N}_0$, for every $\nu = 0, 1, \ldots$, let $k_\nu = k_\nu(\mu) \in K^{<\nu>}$ be the smallest integer with $\max_{k \in K^{<\nu>}} d_{\mu k} = d_{\mu k_\nu}$. Then by the inequality in [9, Lemma 1],

$$\lambda_{m(\mu+1)} \|D_\mu\|_{\mathcal{M}(\Lambda)} = \sum_{\nu=0}^\infty \lambda_{k(\nu+1)} \sum_{m \in M^{<\mu>}} |b_{mk_\nu}|$$

$$\leq 4 \cdot \sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \left(\max_{M_\mu \subset M^{<\mu>}} \left| \sum_{\nu \in K} \lambda_{k(\nu+1)} \sum_{m \in M_\mu} b_{mk_\nu} \right| \right)$$
$$\leq 4 \cdot \sup_{\substack{K \subset \mathbb{N}_0 \\ K \text{ finite}}} \sum_{m \in M^{<\mu>}} \left| \sum_{\nu \in K} \lambda_{k(\nu+1)} b_{mk_\nu} \right|,$$

hence

$$\|D_\mu\|_{\mathcal{M}(\Lambda)} \leq 4 \cdot M^T(B) < \infty \text{ for } \mu = 0, 1, \ldots. \tag{4.5}$$

It also follows that for $\mu = 0, 1, \ldots$

$$\frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{<\mu>}} \sum_{k=0}^{\infty} |b_{mk} t_k| = \sum_{k=0}^{\infty} |t_k| \cdot |d_{\mu k}| \leq \|D_\mu\|_{\mathcal{M}(\Lambda)} \cdot \|t\|. \tag{4.6}$$

Therefore, we obtain from (4.6) and (4.5)

$$\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} |s_m b_{mk} t_k| \leq \left(\sup_\mu \left(\frac{1}{\lambda_{m(\mu+1)}} \sum_{m \in M^{<\mu>}} \sum_{k=0}^{\infty} |b_{mk} t_k| \right) \right) \cdot \|s\|_{\mathcal{M}(\Lambda)}$$
$$\leq \sup_\mu \left(\|D_\mu\|_{\mathcal{M}(\Lambda)} \right) \cdot \|t\| \cdot \|s\|_{\mathcal{M}(\Lambda)} \leq 4 \cdot M^T(B) \cdot \|t\| \cdot \|s\|_{\mathcal{M}(\Lambda)} < \infty.$$

Thus we have shown that $\sum_{m=0}^{\infty} \sum_{k=0}^{\infty} s_m b_{mk} t_k$ is absolutely convergent, and consequently matrix multiplication is associative in $(w_\infty(\Lambda), w_\infty(\Lambda))$.

(c) We assume that $(A^{(j)})_{j=0}^{\infty}$ is a Cauchy sequence in $(w_\infty(\Lambda), w_\infty(\Lambda))$. Since $(w_\infty(\Lambda), w_\infty(\Lambda)) = (w_0(\Lambda), w_\infty(\Lambda))$ by Theorem 3.1 and $w_0(\Lambda)$ has AK by Remark 2.4 (b), it is a Cauchy sequence in $(w_0(\Lambda), w_\infty(\Lambda)) = \mathcal{B}(w_0(\Lambda), w_\infty(\Lambda))$, by Proposition 2.1. Consequently there is $L_A \in \mathcal{B}(w_0(\Lambda), w_\infty(\Lambda))$ with $L_{A^{(j)}} \to L_A$. Since $w_0(\Lambda)$ has AK there is a matrix $A \in (w_0(\Lambda), w_\infty(\Lambda))$ by Proposition 2.1 (b) such that $Ax = L_A(x)$ for all $x \in w_0(\Lambda)$. Finally $(w_0(\Lambda), w_\infty(\Lambda)) = (w_\infty(\Lambda), w_\infty(\Lambda))$ implies $A \in (w_\infty(\Lambda), w_\infty(\Lambda))$. □

The following result is obtained as an immediate consequence of Lemma 4.1.

Theorem 4.2. *The class $(w_\infty(\Lambda), w_\infty(\Lambda))$ is a Banach algebra with respect to the norm $\|A\| = \|L_A\|$ for all $A \in (w_\infty(\Lambda), w_\infty(\Lambda))$.*

The following example is obtained from Theorem 4.2.

Example 4.3. Let $\lambda_n = n + 1$ for $n = 0, 1, \ldots$ as in Examples 2.2, 2.5 and 3.5. Then (w_∞, w_∞) is a Banach algebra with $\|A\| = \|L_A\|$.

Finally, we show that (w, w) is a Banach algebra.

Theorem 4.4. *The class (w,w) is a Banach algebra with $\|A\| = \|L_A\|$.*

Proof. We have to show in view of Theorem 4.2 that

(i) (w,w) is complete;

(ii) if $A, B \in (w,w)$, then $B \cdot A \in (w,w)$.

First we show (i). Let $(A^{(j)})_{j=1}^{\infty}$ be a Cauchy sequence in (w,w). Since $(w,w) \subset (w_\infty, w_\infty)$ and the operator norm on $\mathcal{B}(w_\infty, w_\infty)$ is the same as that on $\mathcal{B}(w,w)$, it follows that $(A^{(j)})_{j=1}^{\infty}$ is a Cauchy sequence in (w_∞, w_∞), and so $A = \lim_{j\to\infty} A^{(j)} \in (w_\infty, w_\infty)$ by Lemma 4.1 (c). We have to show $A \in (w,w)$. Let $\varepsilon > 0$ be given. Since $(A^{(j)})_{j=1}^{\infty}$ is a Cauchy sequence in (w,w) there exists a $j_0 \in \mathbb{N}_0$ such that

$$\|A^{(j)} - A^{(\ell)}\|_{(w_\infty,w_\infty)} = \sup_\mu \left(\frac{1}{2^\mu} \max_{M_\mu \subset M^{<\mu>}} \left\| \sum_{n\in M_\mu} (A_n^{(j)} - A_n^{(\ell)}) \right\|_{\mathcal{M}} \right) < \frac{\varepsilon}{4} \text{ for all } j, \ell \geq j_0; \tag{4.7}$$

Also, by (3.11) and (3.12), for each fixed j there exist complex numbers $\alpha_k^{(j)}$ $(k = 0, 1, \ldots)$ and $\tilde{\alpha}^{(j)}$ such that

$$\lim_{\mu\to\infty} \left(\frac{1}{2^\mu} \sum_\mu \left| a_{nk}^{(j)} - \alpha_k^{(j)} \right| \right) = 0 \text{ for each } k \tag{4.8}$$

and

$$\lim_{\mu\to\infty} \left(\frac{1}{2^\mu} \sum_\mu \left| \sum_{k=0}^{\infty} a_{nk}^{(j)} - \tilde{\alpha}^{(j)} \right| \right) = 0. \tag{4.9}$$

Let $j, \ell \geq j_0$ be given. Then we have for each fixed $k \in \mathbb{N}_0$ by (4.7)

$$\begin{aligned}
\left| \alpha_k^{(j)} - \alpha_k^{(\ell)} \right| &= \frac{1}{2^\mu} \sum_\mu \left| \alpha_k^{(j)} - \alpha_k^{(\ell)} \right| \\
&\leq \frac{1}{2^\mu} \sum_\mu \left| a_{nk}^{(j)} - \alpha_k^{(j)} \right| + \frac{1}{2^\mu} \sum_\mu \left| a_{nk}^{(\ell)} - \alpha_k^{(\ell)} \right| + \frac{1}{2^\mu} \sum_\mu \left| a_{nk}^{(j)} - a_{nk}^{(\ell)} \right| \\
&\leq \frac{1}{2^\mu} \sum_\mu \left| a_{nk}^{(j)} - \alpha_k^{(j)} \right| + \frac{1}{2^\mu} \sum_\mu \left| a_{nk}^{(\ell)} - \alpha_k^{(\ell)} \right| \\
&\qquad + 4 \cdot \max_{M_\mu \subset M^{<\mu>}} \left| \frac{1}{2^\mu} \sum_{n\in M_\mu} \left(A_n^{(j)} - A_n^{(\ell)} \right) (e^{(k)}) \right|
\end{aligned}$$

$$\leq \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(j)} - \alpha_k^{(j)}\right| + \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(\ell)} - \alpha_k^{(\ell)}\right|$$

$$+ 4 \cdot \sup_\mu \left(\frac{1}{2^\mu} \max_{M_\mu \subset M^{<\mu>}} \left\|A^{(j)} - A^{(\ell)}\right\|_{\mathcal{M}}\right) \|e^{(k)}\|$$

$$\leq \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(j)} - \alpha_k^{(j)}\right| + \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(\ell)} - \alpha_k^{(\ell)}\right| + \varepsilon \text{ for all } \mu \in \mathbb{N}_0.$$

Letting $\mu \to \infty$, we obtain from (4.8)

$$\left|\alpha_k^{(j)} - \alpha_k^{(\ell)}\right| \leq \varepsilon \text{ for all } j, \ell \geq j_0.$$

Thus $(\alpha_k^{(j)})_{j=1}^\infty$ is a Cauchy sequence of complex numbers for each fixed $k \in \mathbb{N}_0$ and so

$$\alpha_k = \lim_{j\to\infty} \alpha_k^{(j)} \text{ exists for each } k \in \mathbb{N}_0. \tag{4.10}$$

Now let $k \in \mathbb{N}_0$ be fixed. Then we obtain for all sufficiently large j and for all μ by (4.10) and since $A = \lim_{j\to\infty} A^{(j)}$

$$\frac{1}{2^\mu}\sum_\mu |a_{nk} - \alpha_k| \leq \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(j)} - a_{nk}\right| + \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(j)} - \alpha_k^{(j)}\right| + \frac{1}{2^\mu}\sum_\mu \left|\alpha_k - \alpha_k^{(j)}\right|$$

$$\leq \left\|A^{(j)} - A\right\|_{(w_\infty, w_\infty)} + \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(j)} - \alpha_k^{(j)}\right| + \varepsilon$$

$$< 2 \cdot \varepsilon + \frac{1}{2^\mu}\sum_\mu \left|a_{nk}^{(j)} - \alpha_k^{(j)}\right|.$$

Letting $\mu \to \infty$, we obtain from (4.8)

$$\overline{\lim_{\mu\to\infty}} \left(\frac{1}{2^\mu}\sum_\mu |a_{nk} - \alpha_k|\right) \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that α_k satisfies the condition in (3.11) of Example 3.5 (b). Using exactly the same argument as before with $a_{nk}^{(j)}$ and $\alpha_k^{(j)}$ replaced by $\sum_{k=0}^\infty a_{nk}^{(j)}$ and $\tilde{\alpha}^{(j)}$, and applying (4.9) instead of (4.8), we conclude that $\tilde{\alpha} = \lim_{j\to\infty} \tilde{\alpha}^{(j)}$ exists and satisfies the condition in (3.12) of Example 3.5 (b). Finally $A \in (w_\infty, w_\infty)$ and (3.11) and (3.12) imply $A \in (w, w)$ by Example 3.5 (b). Thus we have shown that (w, w) is complete. This completes the proof of (i).

Now we show that $A, B \in (w, w)$ implies $B \cdot A \in (w, w)$. Since $A, B \in (w, w)$, by Example 3.5 (b), there are complex numbers $\alpha_k, \tilde{\alpha}$ that satisfy (3.11) and (3.12), and complex numbers $\beta_k, \tilde{\beta}$ that satisfy (3.11) and (3.12) with b_{nk}, $\tilde{\beta}_k$ and $\tilde{\beta}$ instead of a_{nk}, $\tilde{\alpha}_k$ and $\tilde{\alpha}$. Let $x \in w$ be given and ξ be the strong limit of x. We put

$$\zeta = \left(\tilde{\beta} - \sum_{n=0}^{\infty} \beta_n\right) \cdot \left(\left(\tilde{\alpha} - \sum_{k=0}^{\infty} \alpha_k\right) \cdot \xi + \sum_{k=0}^{\infty} \alpha_k x_k\right) + \sum_{n=0}^{\infty} \beta_n A_n x.$$

We observe that $(\alpha_k)_{k=0}^{\infty}, (\beta_n)_{n=0}^{\infty} \in \mathcal{M}$ by the proof of Theorem 3.6, and also trivially $\mathcal{M} \subset \ell_1 \subset cs$. Therefore all the series in the definition of ζ converge. We write $C = B \cdot A$, $y = Ax$, η for the strong limit of the sequence y, and ζ' for the strong limit of the sequence $z = By$. Since $Cx = B(Ax)$ by Lemma 4.1 (b), we obtain by (3.13) in Theorem 3.6

$$\begin{aligned}
|C_m x - \zeta| &= |B_m y - \zeta| \\
&= \left| B_m y - \sum_{n=0}^{\infty} \beta_n y_n - \left(\tilde{\beta} - \sum_{n=0}^{\infty} \beta_n\right) \cdot \left(\left(\tilde{\alpha} - \sum_{k=0}^{\infty} \alpha_k\right) \cdot \xi + \sum_{k=0}^{\infty} \alpha_k x_k\right)\right| \\
&= \left| B_m y - \sum_{n=0}^{\infty} \beta_n y_n - \left(\tilde{\beta} - \sum_{n=0}^{\infty} \beta_n\right) \cdot \eta \right| \\
&= \left| z_m - \left(\sum_{n=0}^{\infty} \beta_n (y_n - \eta) + \eta\tilde{\beta}\right)\right| = |z_m - \zeta'| \text{ for all } m,
\end{aligned}$$

hence

$$\lim_{\mu \to \infty} \left(\frac{1}{2^\mu} \sum_{m \in M^{<\mu>}} |C_m x - \zeta|\right) = \lim_{\mu \to \infty} \left(\frac{1}{2^\mu} \sum_{m \in M^{<\mu>}} |z_m - \zeta'|\right) = 0.$$

This shows that $Cx \in w$, and completes the proof of (ii). □

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