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Matrix Transformations and Statistical Convergence II

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Abstract

In this paper we extend some of our recent results given in [15], so we consider a matrix transformation $A = (a_{nk})_{n,k\geq 1}$ and say that a sequence $X = (x_n)_{n\geq 1}$ is A-statistically convergent to $L \in V$ with respect to the intuitionistic fuzzy normed space (IFNS) V if

$$\lim_{n\to\infty} \frac{1}{n} \left|\{k \leq n : \nu\left([AX]_k - L, t\right) \geq \varepsilon \text{ or } 1 - \mu\left([AX]_k - L, t\right) \geq \varepsilon\}\right| = 0$$

for any $\varepsilon > 0$. The aim of this paper is to give conditions on X to have A-statistical convergence on IFNS V. Then, among other things, we consider the cases when A is either of the matrices $\widetilde{N}_{q,\chi}$, $\overline{N}_q$, $\overline{N}_p\overline{N}_q$, $D_{1/\tau}\Delta\left(\lambda\right)$, or $C\left(\lambda\right)$.

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1 Introduction

Fuzzy set theory was introduced in 1965 by Zadeh [22] and used to obtain many results in set theory. *Fuzzy logic* was used among other things in the study of nonlinear dynamical systems [12], in *control of chaos* [6], in population dynamics [1]. The *fuzzy topology* has many applications in *quantum particle physics*, see [4].

In this paper we consider the notion of *intuitionistic fuzzy normed space* (briefly *IFNS*), and we define and deal with A-*statistical convergence on IFNS*, where $A = (a_{nk})_{n,k\geq 1}$ is an infinite matrix. Then we give applications to the cases when A is either

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of the matrices $\widetilde{N}_{q,\chi}$, $\overline{N}_q$, $\overline{N}_p\overline{N}_q$, $D_{1/\tau}\Delta(\lambda)$, or $C(\lambda)$. These results extend in a certain sense those given in [15].

First recall some notations and definitions used in this paper. In the following we will write $I = [0, 1]$.

Definition 1.1. The map $T : I \times I \to I$ is said to be of type (N) if it satisfies the conditions

a) T is associative,

b) T is continuous and commutative,

c) for every a, b, c, $d \in I$, the condition $a \leq c$ and $b \leq d$ implies

$$aTb \leq cTd.$$

Definition 1.2 (See [20]). The map $* : I \times I \to I$ is said to be a continuous t-norm if it is of type (N) with $a * 1 = a$ for all $a \in I$.

Definition 1.3 (See [20]). The map $\diamondsuit : I \times I \to I$ is said to be a continuous t-conorm if it is of type (N) with $a\diamondsuit 0 = a$ for all $a \in I$.

There are some examples of maps that are continuous t-norms or continuous t-conorms. Consider the case when $V = \mathbb{R}$, $a * b = ab$, $a * b = \min\{a, b\}$, $a\diamondsuit b = \min\{a + b, 1\}$ and $a\diamondsuit b = \max\{a, b\}$ for all a, $b \in I$.

Definition 1.4. Let V be a vector space, $*$ be a continuous t-norm and $\diamondsuit$ a continuous t-conorm and μ, ν be fuzzy sets on $V \times (0, \infty)$ (that is, μ, $\nu : V \times (0, \infty) \to [0, 1]$). We say that $(V, \mu, \nu, *, \diamondsuit)$ is an intuitionistic fuzzy normed space (IFNS) if for every x, $y \in V$ and s, $t > 0$ we have

a) $\mu(x, t) + \nu(x, t) \leq 1$,

b) $\mu(x, t) > 0$,

c) $\mu(x, t) = 1$ if and only if $x = 0$,

d) $\mu(\alpha x, t) = \mu(x, t/|\alpha|)$ for all $\alpha \neq 0$,

e) $\mu(x, t) * \mu(y, s) \leq \mu(x + y, t + s)$,

f) $\mu(x, .) : (0, \infty) \to I$ is continuous,

g) $\lim_{t\to\infty} \mu(x, t) = 1$ and $\lim_{t\to 0} \mu(x, t) = 0$,

h) $\nu(x, t) \leq 1$,

i) $\nu(x,t) = 0$ if and only if $x = 0$,

j) $\nu(\alpha x,t) = \nu(x,t/|\alpha|)$ for all $\alpha \neq 0$,

k) $\nu(x,t) \diamondsuit \nu(y,s) \geq \nu(x+y,t+s)$,

l) $\nu(x,.) : (0,\infty) \to I$ is continuous,

m) $\lim_{t\to\infty} \nu(x,t) = 0$ and $\lim_{t\to 0} \nu(x,t) = 1$.

In this case (μ,ν) is called an intuitionistic fuzzy norm (briefly IFN). We give a standard example.

Example 1.5. Let V be a vector space with norm $\|\cdot\|$ and let $a * b = ab$ and $a \diamondsuit b = \min\{a+b,1\}$ for all $a, b \in I$. Put

$$\mu(x,t) = \frac{t}{t+\|x\|} \text{ and } \nu(x,t) = \frac{\|x\|}{t+\|x\|} \text{ for all } x \in V \text{ and } t > 0.$$

It can be easily be shown that $(V,\mu,\nu,*,\diamondsuit)$ is an IFNS.

2 Convergence with Respect to the IFN (μ,ν)

2.1 Convergence on IFNS

We write $s(V)$ for the set of all complex sequences on V, and write s for the set of all complex or real sequences.

Definition 2.1 (See [20]). Let $(V,\mu,\nu,*,\diamondsuit)$ be an IFNS. A sequence $X = (x_n)_{n\geq 1} \in s(V)$ is said to be convergent to $L \in V$ with respect to the IFN (μ,ν) if for every ε, $t > 0$ there is $k_0 \in \mathbb{N}$ depending on ε and t such that

$$\mu(x_k - L,t) > 1-\varepsilon \text{ and } \nu(x_k - L,t) < \varepsilon \text{ for all } k \geq k_0.$$

Then we write $(\mu,\nu) - \lim X = L$ or $x_k \overset{(\mu,\nu)}{\to} L \ (k\to\infty)$.

In the same way we will say that $\mu - \lim X = L$ or $x_k \overset{\mu}{\to} L \ (k\to\infty)$ if for every ε, $t > 0$ there is $k_0 \in \mathbb{N}$ depending on ε and t such that

$$\mu(x_k - L,t) > 1-\varepsilon \text{ for all } k \geq k_0.$$

For a given sequence $X \in s(V)$, $L \in V$ and ε, $t > 0$, put

$$\Gamma_{\mu,\nu}(X,\varepsilon,t) = \{k \in \mathbb{N} : \mu(x_k - L,t) > 1-\varepsilon \text{ and } \nu(x_k - L,t) < \varepsilon\},$$
$$\Gamma_{\nu}(X,\varepsilon,t) = \{k \in \mathbb{N} : \nu(x_k - L,t) < \varepsilon\}$$

and

$$\Gamma_{\mu}(X,\varepsilon,t) = \{k \in \mathbb{N} : \mu(x_k - L, t) > 1 - \varepsilon\}.$$

It can easily be seen that $x_k \overset{(\mu,\nu)}{\to} L$ $(k \to \infty)$ if for every ε, $t > 0$ there is k_0 such that

$$[k_0, +\infty[\subset \Gamma_{\mu,\nu}(X,\varepsilon,t).$$

Similarly, $x_k \overset{\mu}{\to} L$ $(k \to \infty)$ if for all $\varepsilon, t > 0$ there is k_0 with $[k_0, +\infty[\subset \Gamma_{\mu}(X,\varepsilon,t)$. To simplify we write $\Gamma_{\mu,\nu}(X)$, $\Gamma_{\nu}(X)$ and $\Gamma_{\mu}(X)$ instead of $\Gamma_{\mu,\nu}(X,\varepsilon,t)$, $\Gamma_{\nu}(X,\varepsilon,t)$ and $\Gamma_{\mu}(X,\varepsilon,t)$. We state the following elementary lemma.

Lemma 2.2. *Let $(V,\mu,\nu,*,\diamondsuit)$ be an IFNS. Then for given $X \in s(V)$, $L \in V$ and ε, $t > 0$ we have*

$$\Gamma_{\mu,\nu}(X) = \Gamma_{\mu}(X) \text{ and } \Gamma_{\mu}(X) \subset \Gamma_{\nu}(X).$$

Proof. Trivially $\Gamma_{\mu,\nu}(X) \subset \Gamma_{\mu}(X)$. Now take $k \in \Gamma_{\mu}(X)$. Then $\mu(x_k - L, t) > 1 - \varepsilon$ and from Definition 1.4 a) we have

$$\nu(x_k - L, t) \leq 1 - \mu(x_k - L, t) < \varepsilon.$$

This shows that $k \in \Gamma_{\mu,\nu}(X)$ and $\Gamma_{\mu}(X) \subset \Gamma_{\mu,\nu}(X)$. We conclude $\Gamma_{\mu,\nu}(X) = \Gamma_{\mu}(X)$. Then trivially $\Gamma_{\mu,\nu}(X) = \Gamma_{\mu}(X) \subset \Gamma_{\nu}(X)$. This completes the proof. □

As a direct consequence of Lemma 2.2, we obtain the next lemma.

Lemma 2.3. *Let $(V,\mu,\nu,*,\diamondsuit)$ be an IFNS. Then $x_k \overset{(\mu,\nu)}{\to} L$ $(k \to \infty)$ if and only if $x_k \overset{\mu}{\to} L$ $(k \to \infty)$.*

Proof. If $x_k \overset{(\mu,\nu)}{\to} L$ $(k \to \infty)$ for every ε, $t > 0$, then there is k_0 such that $[k_0, +\infty[\subset \Gamma_{\mu,\nu}(X)$, and since $\Gamma_{\mu,\nu}(X) = \Gamma_{\mu}(X)$, we deduce $x_k \overset{\mu}{\to} L$. The converse can be shown similarly. □

Remark 2.4. The convergence $(\mu,\nu) - \lim X = L$ is equivalent to the convergence $\mu - \lim X = L$, and it is not necessary to introduce the first convergence. In this paper, we introduce the first convergence, mainly to keep the conventional approach. Furthermore, in the special case, in Example 1.5, (μ,ν)-convergence and norm convergence for the sequence $(x_n)_{n\geq 1}$ are equivalent.

Remark 2.5. Let us remark that the notion of intuitionistic fuzzy metric spaces was introduced by Park in [18]. In Gregori, Romaguera, Veeramani [11] (see also Saadati, Sedghi, Shobe [19]), it was shown that Park's definition of intuitionistic fuzzy metric spaces contains extra conditions and can be derived, in an equivalent manner, from the definition of fuzzy metric spaces.

2.2 Statistical Convergence on IFNS

2.2.1 Statistical Convergence

The notion of *statistical convergence* was first introduced by Steinhaus in 1949, see [21], and studied by several authors such as Fast, Fridy [5, 7–10] and Connor. The idea of statistical convergence suggests many other possible lines of investigation. We mention here that recently Di Maio and Kočinac [2] introduced and investigated statistical convergence in topological and uniform spaces and showed how this convergence can be applied to selection principles theory, function spaces and hyperspaces. Di Maio, Djurčić, Kočinac and Žižović [3] considered the set of sequences of positive real numbers in the context of statistical convergence and showed that some of its subclasses have certain nice selection and game-theoretic properties.

A sequence $X = (x_n)_{n\geq 1}$ tends statistically to L if for each $\varepsilon > 0$ we have

$$\frac{1}{n}\left|K\left(X, n, \varepsilon\right)\right| = o\left(1\right) \quad \left(n \to \infty\right),$$

where $K\left(X, n, \varepsilon\right) = \{k \leq n : |x_k - L| \geq \varepsilon\}$ and the symbol $|\cdot|$ denotes the number of elements in the enclosed set. In this case we write $x_k \to L\left(S\right)$. To simplify we put $K\left(X, n\right) = K\left(X, n, \varepsilon\right)$.

For the convenience of the reader, we recall the following well-known lemma.

Lemma 2.6. *Let K_1 and K_2 be two subsets of $\mathbb{N}$ with $K_1 \subset K_2$ then*

$$\lim_{n\to\infty} \frac{1}{n}\left|\{k \leq n : k \in K_2\}\right| = 0 \Rightarrow \lim_{n\to\infty} \frac{1}{n}\left|\{k \leq n : k \in K_1\}\right| = 0.$$

Remark 2.7. It is to be noted (see [10]) that every convergent sequence is statistically convergent with the same limit so that statistical convergence is a natural generalization of the usual convergence of sequences. A sequence which is statistically convergent may neither be convergent nor bounded. This is also demonstrated by the following example. Let us consider the sequence $(a_i)_{i\geq 1}$ whose terms are i when $i = n^2$ for all $n = 1, 2, 3,$ and $a_i = 1/i$ otherwise. Then, the sequence $(a_i)_{i\geq 1}$ is divergent in the ordinary sense, while 0 is the statistical limit of $(a_i)_{i\geq 1}$. Not all properties of convergent sequences are true for statistical convergence. It is well known that a subsequence of a convergent sequence is convergent, however, for statistical convergence this is not true. The sequence $(b_i)_{i\geq 1}$ whose terms are i, $i = 1, 2, 3,$ is a subsequence of the statistically convergent sequence $(a_i)_{i\geq 1}$, and $(b_i)_{i\geq 1}$ is statistically divergent.

2.2.2 Statistical Convergence with respect to the IFN

Let $\varepsilon, t > 0$, $n \in \mathbb{N}$ and let $(V, \mu, \nu, *, \diamondsuit)$ be an IFNS. For given $X \in s$ and $L \in V$, put

$$K_\mu\left(X, n, \varepsilon, t\right) = \{k \leq n : 1 - \mu\left(x_k - L, t\right) \geq \varepsilon\},$$
$$K_\nu\left(X, n, \varepsilon, t\right) = \{k \leq n : \nu\left(x_k - L, t\right) \geq \varepsilon\}$$

and

$$K_{\mu,\nu}(X,n,\varepsilon,t)=\{k\leq n:\nu(x_k-L,t)\geq\varepsilon \text{ or } 1-\mu(x_k-L,t)\geq\varepsilon\}.$$

To simplify we write $K_\mu(X,n)$, $K_\nu(X,n)$ and $K_{\mu,\nu}(X,n)$ instead of $K_\mu(X,n,\varepsilon,t)$, $K_\nu(X,n,\varepsilon,t)$ and $K_{\mu,\nu}(X,n,\varepsilon,t)$.

Definition 2.8. Let $(V,\mu,\nu,*,\diamondsuit)$ be an IFNS. A sequence X is said to be statistically convergent to $L\in V$ with respect to the IFN (μ,ν) provided that for every $\varepsilon,t>0$ we have

$$\frac{1}{n}|K_{\mu,\nu}(X,n)|=o(1)\quad(n\to\infty).$$

In this case we write $x_k\to L_{\mu,\nu}(S)$.

Definition 2.9. We say that $x_k\to L_\mu(S)$ if for each $\varepsilon,t>0$ we have

$$\frac{1}{n}|K_\mu(X,n)|=o(1)\quad(n\to\infty).$$

2.2.3 Condition (φ) on IFNS

In the case when V is a set of scalars $\mathbb{R}$ or $\mathbb{C}$, we say that an IFN (μ,ν) satisfies condition (φ) if there is a function $\varphi:]0,\infty[\,\to\,]0,\infty[$ such that for every $x\in V$ and $t>0$

$$\mu(x,t)\geq 1-\varphi(t)|x|. \tag{2.1}$$

Consider the classical example where $V=\mathbb{R}$, $a*b=ab$, $a\diamondsuit b=\min\{a+b,1\}$ and

$$\mu(x,t)=\frac{t}{t+|x|},\ \nu(x,t)=1-\mu(x,t)=\frac{|x|}{t+|x|} \text{ for all } x\in\mathbb{R} \text{ and } t>0.$$

We have

$$\nu(x,t)=\frac{|x|}{t+|x|}\leq\frac{1}{t}|x|$$

and so $\varphi(t)=1/t$.

Lemma 2.10. *Let $(V,\mu,\nu,*,\diamondsuit)$ be an IFNS and let $X\in s$, $L\in V$, $\varepsilon,t>0$ and $n\in\mathbb{N}$. Then*

i) *a)* $K_{\mu,\nu}(X,n)=K_\mu(X,n)$ *and* $K_\nu(X,n)\subset K_\mu(X,n)$*;*

b) $x_k\to L_{\mu,\nu}(S)$ *if and only if* $x_k\to L_\mu(S)$*;*

c) $x_k\to L_{\mu,\nu}(S)$ *implies* $x_k\to L_\nu(S)$.

ii) *If V is a set of scalars and (μ,ν) satisfies condition (φ), then*

$$K_\mu(X,n,\varepsilon\varphi(t),t)\subset K(X,n,\varepsilon).$$

Proof. We first show i). Take $k \in K_{\mu,\nu}(X,n)$. By Definition 1.4 a) we have

$$1 - \mu(x_k - L, t) \geq \nu(x_k - L, t). \tag{2.2}$$

Then for any given $\varepsilon, t > 0$, $\nu(x_k - L, t) \geq \varepsilon$ implies $1 - \mu(x_k - L, t) \geq \varepsilon$ and $K_{\mu,\nu}(X,n) \subset K_\mu(X,n)$. By definition of $K_{\mu,\nu}(X,n)$, it is obvious that $K_\mu(X,n) \subset K_{\mu,\nu}(X,n)$. Thus, we conclude $K_{\mu,\nu}(X,n) = K_\mu(X,n)$. Now we show $K_\nu(X,n) \subset K_\mu(X,n)$. For this we take $k \in K_\nu(X,n)$. We have $\nu(x_k - L, t) \geq \varepsilon$, and since (2.2) holds we conclude $k \in K_{\mu,\nu}(X,n) = K_\mu(X,n)$. This proves a), while b) and c) are direct consequences of a) and Lemma 2.6.

Next we show ii). For each integer k and $t > 0$ we have $1 - \mu(x_k - L, t) \leq \varphi(t)|x_k - L|$ with $\varphi :]0,\infty[\to]0,\infty[$. So $k \in K_\mu(X, n, \varepsilon\varphi(t), t)$ means that $1 - \mu(x_k - L, t) \geq \varepsilon\varphi(t)$ and by (2.1)

$$|x_k - L| \geq [1 - \mu(x_k - L, t)]/\varphi(t) \geq \varepsilon\varphi(t)/\varphi(t) = \varepsilon,$$

that is, $k \in K(X, n, \varepsilon)$. □

3 A-Statistical Convergence with Respect to IFN (μ, ν)

In this section we deal with A-statistical convergence with respect to the IFN (μ, ν). For this we recall some results on matrix transformations and w^0. Then we give conditions ensuring $x_k \to L_{\mu,\nu}(S(A))$.

3.1 Matrix Transformations

For a given infinite matrix $A = (a_{nk})_{n,k\geq 1}$ we define the operators A_n for any $n \in \mathbb{N}$ by

$$A_n(X) = \sum_{k=1}^{\infty} a_{nk}x_k, \tag{3.1}$$

where $X = (x_n)_{n\geq 1}$, the series intervening in the second member being convergent. So we are led to the study of the infinite linear system

$$A_n(X) = b_n \quad n \in \mathbb{N}, \tag{3.2}$$

where $b = (b_n)_{n\geq 1}$ is a one-column matrix and X is the unknown. System (3.2) can be written in the form $AX = b$, where $AX = (A_n(X))_{n\geq 1}$. In this paper we also consider A as an operator from a sequence space into another sequence space. We write c_0 for the sets of null sequences. For $E, F \subset s$ we will denote by (E, F) the set of all matrix transformations $A = (a_{nk})_{n,k\geq 1}$ that map E to F, see [16].

A Banach space E of complex sequences with the norm $\|\cdot\|_E$ is a *BK space* if each projection $P_n : X \mapsto P_nX = x_n$ is continuous. A BK space $E \subset s$ is said to have *AK* if every sequence $X = (x_n)_{n\geq 1} \in E$ has a unique representation $X = \sum_{n=1}^{\infty} x_n e_n$, where e_n is the sequence with 1 in the n-th position, and 0 otherwise. For given $F \subset s$ and a given matrix A, we will put

$$F(A) = \{X \in s : AX \in F\}.$$

The matrix T is a triangle if $[T]_{nn} \neq 0$ for all n and $[T]_{nk} = 0$ for $k > n$. Here we use triangles represented by $C(\lambda)$ and $\Delta(\lambda)$ for a given sequence λ with $\lambda_n \neq 0$ for all n. We define $C(\lambda) = (c_{nk})_{n,k\geq 1}$ by

$$c_{nk} = \begin{cases} \dfrac{1}{\lambda_n} & \text{if } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

It was proved in [13] that the matrix $\Delta(\lambda) = (c'_{nm})_{n,m\geq 1}$ with

$$c'_{nk} = \begin{cases} \lambda_n & \text{if } k = n, \\ -\lambda_{n-1} & \text{if } k = n-1 \text{ and } n \geq 2, \\ 0 & \text{otherwise} \end{cases}$$

is the inverse of $C(\lambda)$. Using the notation $e = (1, 1, \ldots)$, we write $\Delta = \Delta(e)$ and $\Sigma = C(e)$.

In the following we use the operators represented by $C(\lambda)$ and $\Delta(\lambda)$. Let U be the set of all sequences $(u_n)_{n\geq 1}$ with $u_n \neq 0$ for all n. We define the triangle $C(\lambda) = (c_{nm})_{n,m\geq 1}$ for $\lambda = (\lambda_n)_{n\geq 1} \in U$ by $c_{nk} = 1/\lambda_n$ for $k \leq n$. It can be proved that the triangle $\Delta(\lambda)$ with $[\Delta(\lambda)]_{nn} = \lambda_n$ for all n, $[\Delta(\lambda)]_{n,n-1} = -\lambda_{n-1}$ for $n \geq 2$, is the inverse of $C(\lambda)$, see [13]. Denote by U^+ the set of all sequences $X = (x_n)_n$ such that $x_n > 0$ for all n. We also use the set Γ of all sequences $X \in U^+$ with $\overline{\lim}_{n\to\infty} (x_{n-1}/x_n) < 1$.

When $\lambda_n = n$ for all n, we get the well known space w^0 (or w_0) studied by Maddox (see e.g., []), i.e.,

$$w^0 = \left\{ X = (x_n)_n : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |x_k| = 0 \right\}.$$

It is well known that w^0 normed by

$$\|X\| = \sup_n \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)$$

is a BK space with AK. It can easily be deduced that the class $\left(w^0, w^0\right)$ is a *Banach algebra* (see [15]) normed by

$$\|A\|^* = \sup_{X\neq 0} \left(\frac{\|AX\|}{\|X\|} \right). \tag{3.3}$$

In the following we will apply these results.

Obviously, $c_0 \subset w^0$. Actually, c_0 is a proper subset of w^0. For example, the following sequence $(x_n)_{n\geq 1}$ is in w^0 but not in c_0: $x_n = 1$ if n is prime number, and $x_n = 1/n$ otherwise.

3.2 A-Statistical Convergence on IFNS

In this subsection we extend some results obtained in [15] where we dealt with A-statistical convergence.

First recall some definitions used in [15]. Let L be a scalar and $A \in (E, F)$. For $\varepsilon > 0$, we will use the notation

$$K(X, n, A) = \{k \leq n : |[AX]_k - L| \geq \varepsilon\}$$

(where we assume that every series $[AX]_k = A_k(X) = \sum_{m=1}^{\infty} a_{km} x_m$ for $k \geq 1$ is convergent). We say that $X = (x_n)_{n\geq 1} \in s$ *A-statistically convergent to L* if for every $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |K(X, n, A)| = 0.$$

We then write $x_k \to L(S(A))$.

Now let $X \in s$ and let $(V, \mu, \nu, *, \diamondsuit)$ be an IFNS. For given $L \in V$ and $\varepsilon > 0$, put

$$K_{\mu,\nu}(X, n, A) = \{k \leq n : \nu([AX]_k - L, t) \geq \varepsilon \text{ or } 1 - \mu([AX]_k - L, t) \geq \varepsilon\}$$

We say that $X = (x_n)_{n\geq 1}$ is *A-statistically convergent to L with respect to the IFN* (μ, ν) if for every $\varepsilon, t > 0$, we have

$$\frac{1}{n} |K_{\mu,\nu}(X, n, A)| = o(1) \quad (n \to \infty).$$

We then write $x_k \to L_{\mu,\nu}(S(A))$. As an immediate consequence of Lemma 2.2, we have

$$K_{\mu,\nu}(X, n, A) = K_\mu(X, n, A) \text{ and } K_\nu(X, n, A) \subset K_\mu(X, n, A).$$

Now we state a lemma where for a given subvector space $\Phi \subset s$ and for $L \in V$, we write $L + \Phi$ for the set of all sequences of the form $X = Le + Y$ with $Y \in \Phi$. Let us recall that $w^0(A)$ is defined in Subsection 3.1.

Lemma 3.1. *Let V be a set of scalars and let $(V, \mu, \nu, *, \diamondsuit)$ be an IFNS and assume (μ, ν) satisfies condition (φ).*

i) If $x_k \to L(S)$, then $x_k \to L_{\mu,\nu}(S)$.

ii) Let A be an infinite matrix. If $x_k \to L(S(A))$, then $x_k \to L_{\mu,\nu}(S(A))$.

iii) Let A be a triangle. If $X \in LA^{-1}e + w^0(A)$, then

$$x_k \to L_{\mu,\nu}(S(A)).$$

Proof. Since $K_\mu(X,n) \subset K(X,n)$ and $x_k \to L(S)$, we have $x_k \to L_{\mu,\nu}(S)$. Thus i) holds.

For $\varepsilon > 0$, the inequalities

$$\varphi(t)\left|[AX]_k - L\right| \geq 1 - \mu\left([AX]_k - L, t\right) \geq \varepsilon$$

imply

$$K_{\mu,\nu}(X,n,A) = K_\mu(X,n,A) \subset K(X,n,A).$$

Now $x_k \to L(S(A))$ means $\lim_{n\to\infty} |K(X,n,A)|/n = 0$, and by Lemma 2.6, we have $\lim_{n\to\infty} |K_{\mu,\nu}(X,n,A)|/n = 0$ and $x_k \to L_{\mu,\nu}(S(A))$. Thus ii) holds.

Since A is a triangle, we put $\widetilde{l} = A^{-1}e$. Putting $\lambda = (n)_{n\geq 1}$, we obtain

$$[C(\lambda)|AX - Le|]_n = \frac{1}{n}\sum_{k=1}^{n}\left|[AX]_k - L\right|$$

and $C(\lambda)|AX - Le| = C(\lambda)\left|A\left(X - L\widetilde{l}\right)\right|$. Thus

$$\begin{aligned} [C(\lambda)|AX - Le|]_n &= \frac{1}{n}\sum_{k=1}^{n}\left|\left[A\left(X - L\widetilde{l}\right)\right]_k\right| \\ &= \frac{1}{n}\sum_{k=1}^{n}\left|[AX]_k - L\right| \\ &\geq \frac{1}{n}\sum_{k\in K(X,n,A)}\left|[AX]_k - L\right| \\ &\geq \frac{\varepsilon}{n}|K(X,n,A)|. \end{aligned}$$

Then $X \in L\widetilde{l} + w^0(A)$ means

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left|\left[A\left(X - L\widetilde{l}\right)\right]_k\right| = 0,$$

and we deduce from the preceding that

$$\lim_{n\to\infty}\frac{1}{n}|K(X,n,A)| = 0$$

and $x_k \to L(S(A))$. By part ii), we conclude $x_k \to L_{\mu,\nu}(S(A))$. Thus iii) holds. $\square$

4 $\widetilde{N}_{q,\chi}$-, $\overline{N}_q$- and $\overline{N}_p\overline{N}_q$-Statistical Convergence on IFNS

In this section we give conditions on the sequence $X = (x_n)_{n\geq 1}$ guaranteeing $x_k \to L_{\mu,\nu}\left(S\left(\widetilde{N}_{q,\chi}\right)\right)$, where $\widetilde{N}_{q,\chi}$ is a matrix band. Similarly we obtain conditions on X to successively have $x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_q\right)\right)$ and $x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_p\overline{N}_q\right)\right)$, where $\overline{N}_q$ is the matrix of weighted means.

4.1 $\widetilde{N}_{q,\chi}$-Statistical Convergence on IFNS

To state the next result, we will use an infinite matrix band defined for given $\chi \in \mathbb{N}$ and $q \in U^+$ by

$$\widetilde{N}_{q,\chi} = \begin{pmatrix} \frac{q_1}{Q} & . & \frac{q_\chi}{Q} & & \mathbf{0} \\ & . & . & . & \\ & & \frac{q_1}{Q} & . & \frac{q_\chi}{Q} \\ \mathbf{0} & & & . & . \\ & & & & \end{pmatrix}$$

with $Q = \sum_{k=1}^{\chi} q_k$. Then we obtain the next result.

Proposition 4.1. *Let $(V, \mu, \nu, *, \diamondsuit)$ be an IFNS with $\nu = 1 - \mu$. If*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \nu\left(x_k - L, t\right) \diamondsuit \cdots \diamondsuit \nu\left(x_{k+\chi-1} - L, t\right) = 0,$$

then $x_k \to L_{\mu,\nu}\left(S\left(\widetilde{N}_{q,\chi}\right)\right)$, that is, for every $\varepsilon, t > 0$,

$$\lim_{n\to\infty} \frac{1}{n} \left|\left\{k \leq n : \nu\left(\frac{1}{Q}\sum_{i=1}^{\chi} q_i x_{k+i-1} - L, t\right) \geq \varepsilon\right\}\right| = 0.$$

Proof. We have

$$\left[\widetilde{N}_{q,\chi}X\right]_k = \frac{1}{Q}\sum_{i=1}^{\chi} q_i x_{k+i-1} \text{ for all } X \in s,$$

and since $Q = \sum_{i=1}^{\chi} q_i$, we deduce

$$\left[\widetilde{N}_{q,\chi}X\right]_k - L = \frac{1}{Q}\left(\sum_{i=1}^{\chi} q_i x_{k+i-1} - QL\right) = \frac{1}{Q}\sum_{i=1}^{\chi} q_i\left(x_{k+i-1} - L\right), \tag{4.1}$$

and by Definition 1.4 j), we get

$$\nu\left(\frac{1}{Q}\sum_{i=1}^{\chi} q_i\left(x_{k+i-1}-L\right),t\right)=\nu\left(\sum_{i=1}^{\chi} q_i\left(x_{k+i-1}-L\right),Qt\right). \tag{4.2}$$

By Definition 1.4 k), we get

$$\begin{aligned}\nu\left(\sum_{i=1}^{\chi} q_i\left(x_{k+i-1}-L\right),Qt\right)&=\nu\left(\sum_{i=1}^{\chi} q_i\left(x_{k+i-1}-L\right),\sum_{i=1}^{\chi} tq_i\right)\\&\leq\nu\left(q_1\left(x_k-L\right),q_1t\right)\diamondsuit\cdots\diamondsuit\nu\left(q_\chi\left(x_{k+\chi-1}-L\right),q_\chi t\right),\end{aligned}$$

and by Definition 1.4 j), we have

$$\nu\left(q_i\left(x_k-L\right),q_it\right)=\nu\left(x_k-L,q_it/q_i\right)=\nu\left(x_k-L,t\right),\quad i=1,2,\ldots,\chi.$$

Then

$$\nu\left(\sum_{i=1}^{\chi} q_i\left(x_{k+i-1}-L\right),Qt\right)\leq\nu\left(x_k-L,t\right)\diamondsuit\cdots\diamondsuit\nu\left(x_{k+\chi-1}-L,t\right). \tag{4.3}$$

Now put $\diamondsuit_k=\nu\left(x_k-L,t\right)\diamondsuit\cdots\diamondsuit\nu\left(x_{k+\chi-1}-L,t\right)$. Since $K_\nu\left(X,n,\widetilde{N}_{q,\chi}\right)\subset[1,n]$, by (4.1), (4.2) and (4.3), we obtain

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}\diamondsuit_k&\geq\frac{1}{n}\sum_{k=1}^{n}\nu\left(\left[\widetilde{N}_{q,\chi}\right]_k-L,t\right)\\&\geq\frac{1}{n}\sum_{k\in K_\nu\left(x,\widetilde{N}_{q,\chi}\right)}\nu\left(\left[\widetilde{N}_{q,\chi}\right]_k-L,t\right)\\&\geq\frac{\varepsilon}{n}\left|K_\nu\left(X,n,\widetilde{N}_{q,\chi}\right)\right|.\end{aligned}$$

So the condition $\left(\diamondsuit_k\right)_{k\geq 1}\in w_0$ implies

$$\frac{1}{n}\left|K_\nu\left(X,n,N'_{q,\chi}\right)\right|\to 0\ \ (n\to\infty)$$

and $x_k\to L_{\mu,\nu}\left(S\left(N'_{q,\chi}\right)\right)$. $\square$

4.2 Applications to A-Statistical Convergence on IFNS

To study $\overline{N}_q$- and $\overline{N}_p\overline{N}_q$-statistical convergence, we need to state some results on the sets $W_\tau^0\left(C\left(\lambda\right)\right)$ and $W_\tau^0\left(\Delta\left(\lambda\right)\right)$.

4.2.1 The Sets $W_\tau^0(\Delta(\lambda))$ and $W_\tau^0(C(\lambda))$

For a given sequence $\tau = (\tau_n)_{n\geq 1} \in U^+$, we define the infinite diagonal matrix $D_\tau = (\tau_n \delta_{nm})_{n,m\geq 1}$. For any subset E of s, we will write $D_\tau E$ for the set of sequences $(x_n)_{n\geq 1}$ such that $(x_n/\tau_n)_{n\geq 1} \in E$and put $W_\tau^0 = D_\tau w^0$ for $\tau \in U^+$. The space

$$W_\tau = \left\{ X : \sup_n \sum_{k=1}^{n} \frac{|x_k|}{\tau_k} < \infty \right\}$$

is called the set of sequences that are strongly τ-convergent to zero. Then we will explicitly give the sets $W_\tau^0(C(\lambda))$ and $W_\tau^0(\Delta(\lambda))$. Note that we have

$$\begin{aligned} W_\tau^0(C(\lambda)) &= \left\{ X : \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1}{\lambda_k \tau_k} \sum_{i=1}^{k} |x_i| \right) < \infty \right\}, \\ W_\tau^0(\Delta(\lambda)) &= \left\{ X : \lim_{n\to\infty} \left(\frac{1}{n} \sum_{k=1}^{n} \frac{1}{\tau_k} |\lambda_k x_k - \lambda_{k-1} x_{k-1}| \right) < \infty \right\}. \end{aligned}$$

For a given sequence $\rho = (\rho_n)_{n\geq 1}$, we will consider the triangle Δ_ρ defined by $[\Delta_\rho]_{nn} = 1$ for all n, $[\Delta_\rho]_{n,n-1} = -\rho_n$ for all $n \geq 2$. Recall the next result which is a direct consequence of [14, Theorem 5.1 and Theorem 5.12].

Lemma 4.2. *If*

$$\overline{\lim_{n\to\infty}} |\rho_n| < 1, \tag{4.4}$$

then for any given $b \in w^0$, the equation $\Delta_\rho X = b$ has a unique solution in w^0.

We immediately deduce the following result.

Lemma 4.3. *Let $\lambda, \tau \in U^+$. Then*

(i) If $\tau \in \Gamma$, then the operators Δ and Σ are bijective from W_τ^0 into itself, and

$$W_\tau^0(\Delta) = W_\tau^0, \quad W_\tau^0(\Sigma) = W_\tau^0.$$

(ii) a) If $\lambda\tau \in \Gamma$, then $W_\tau^0(C(\lambda)) = W_{\lambda\tau}^0$.

b) If $\tau \in \Gamma$, then $W_\tau^0(\Delta(\lambda)) = W_{\tau/\lambda}^0$.

Proof. (i) By Lemma 2.2, where $\rho_n = \tau_{n-1}/\tau_n$ and $\lambda_n = n$ for all n, we easily see that if

$$\overline{\lim_{n\to\infty}} \frac{\tau_{n-1}}{\tau_n} < 1,$$

that is, $\tau \in \Gamma$, then $D_{1/\tau} \Delta D_\tau$ is bijective from w^0 to itself. This means that Δ is bijective from $D_\tau w^0$ to itself. Since Σ is also bijective from $D_\tau w^0$ to itself, this shows $W_\tau^0(\Delta) = W_\tau^0$ and $W_\tau^0(\Sigma) = W_\tau^0$. (ii) We have $X \in W_\tau^0(C(\lambda))$ if and only if $\Sigma X \in D_{\lambda\tau} w^0 = W_{\lambda\tau}^0$. This means that $X \in W_{\lambda\tau}^0(\Sigma)$ and by (i) the condition $\lambda\tau \in \Gamma$ implies $W_{\lambda\tau}^0(\Sigma) = W_{\lambda\tau}$. Then $W_\tau^0(C(\lambda)) = W_{\lambda\tau}^0$ and $C(\lambda)$ is bijective from $W_{\lambda\tau}^0$ to W_τ^0. Since $\Delta(\lambda) = C(\lambda)^{-1}$ we conclude $\Delta(\lambda)$ is bijective from W_τ^0 to $W_{\lambda\tau}^0$ and $W_{\lambda\tau}^0(\Delta(\lambda)) = W_\tau^0$. We deduce that for $\tau \in \Gamma$ we have $W_\tau^0(\Delta(\lambda)) = W_{\tau/\lambda}^0$. $\square$

4.2.2 $\overline{N}_q$- and $\overline{N}_p\overline{N}_q$-Statistical Convergence on IFNS

The operator of weighted means $\overline{N}_q$ is defined for $q \in U^+$ and $Q_n = \sum_{k=1}^{n} q_k$ for all n by

$$\left[\overline{N}_q\right]_{nk} = \begin{cases} \dfrac{q_k}{Q_n} & \text{for } k \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

First we state the following result.

Proposition 4.4. *Let V be a set of scalars and let (μ, ν) be an IFN satisfying condition (φ). If*

$$\left(\frac{1}{Q_k} \sum_{i=1}^{k} q_i (x_i - L) \right)_{k \geq 1} \in w^0, \tag{4.5}$$

then $x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_q\right)\right)$.

Proof. First we have

$$X \in L\overline{N}_q^{-1} e + w^0\left(\overline{N}_q\right). \tag{4.6}$$

Since $\overline{N}_q e = e$ implies that $\overline{N}_q^{-1} e = e$, condition (4.6) means that $\overline{N}_q (X - Le) \in w^0$ and (4.5) holds. We conclude by Lemma 3.1 iii) that $x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_q\right)\right)$. This concludes the proof. □

Example 4.5. Consider the classical example, where $V = \mathbb{R}$, $a * b = ab$, $a \diamondsuit b = \min\{a + b, 1\}$ and

$$\mu(x,t) = \frac{t}{t + |x|}, \quad \nu(x,t) = 1 - \mu(x,t) = \frac{|x|}{t + |x|} \text{ for all } x \in \mathbb{R} \text{ and } t > 0.$$

Assume for every $t > 0$ we have

$$\nu\left(q_i \frac{x_i - L}{Q_k}, t\right) = \frac{q_i \dfrac{|x_i - L|}{Q_k}}{t + q_i \dfrac{|x_i - L|}{Q_k}} \leq \frac{1}{tQ_k} q_i |x_i - L| \text{ for } i = 1, 2, \ldots, k,$$

and by Proposition 4.4, the condition

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \frac{1}{Q_k} \left| \sum_{i=1}^{k} q_i (x_i - L) \right| = 0$$

implies $x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_q\right)\right)$.

Now among other things we consider the case when (μ, ν) is an IFN satisfying condition (φ).

Theorem 4.6. *i) Let $Q = (Q_n)_{n \geq 1} \in \Gamma$. Then*

$$x_k \to L\left(S\left(\overline{N}_q\right)\right) \text{ for all } X \in L + D_{Q/q} w^0.$$

ii) Let V be a set of scalars and let (μ, ν) be an IFN satisfying condition (φ). Then

a)

$$x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_q\right)\right) \text{ for all } X \in L + D_{Q/q} w^0,$$

that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{q_k}{Q_k} |x_k - L| = 0.$$

Hence

$$\frac{1}{n}\left|\left\{k \leq n : \mu\left(\frac{1}{Q_k}\left(\sum_{i=1}^{k} q_i x_i\right) - L, t\right) \leq 1 - \varepsilon\right\}\right| = 0\,(1) \quad (n \to \infty).$$

b) Let P, $PQ/p \in \Gamma$. If

$$X \in L + D_{PQ/pq} w_0, \text{ then } x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_p \overline{N}_q\right)\right),$$

that is,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{p_k q_k}{P_k Q_k} |x_k - L| = 0.$$

Thus, for every $\varepsilon, t > 0$,

$$\frac{1}{n}\left|\left\{k \leq n : \mu\left(\frac{1}{P_k}\sum_{j=1}^{k} \frac{p_j}{Q_j}\left(\sum_{i=1}^{j} q_i x_i\right) - L, t\right) \leq 1 - \varepsilon\right\}\right|$$
$$= \frac{1}{n}\left|\left\{k \leq n : \mu\left(\left[\overline{N}_p \overline{N}_q X\right]_k - L, t\right) \leq 1 - \varepsilon\right\}\right| = 0\,(1) \quad (n \to \infty).$$

Proof. i) is a direct consequence of [15, Corollary 3, pp. 381–382]. ii) a) By Lemma 3.1 iv), if there is $\varphi :]0, \infty[\to]0, \infty[$ such that (2.1) holds, then $x_k \to L\left(S\left(\overline{N}_q\right)\right)$ implies $x_k \to L_{\mu,\nu}\left(S\left(\overline{N}_q\right)\right)$. b) We have

$$\overline{N}_p \overline{N}_q X - Le = \overline{N}_p \overline{N}_q \left[X - L\left(\overline{N}_p \overline{N}_q\right)^{-1} e\right] = \overline{N}_p \overline{N}_q (X - Le)$$

since $\left(\overline{N}_p \overline{N}_q\right)^{-1} e = \overline{N}_q^{-1}\left(\overline{N}_p^{-1} e\right) = \overline{N}_q^{-1} e = e$. Then $\overline{N}_p \overline{N}_q X - Le \in w^0$ if and only if

$$X - Le \in w^0\left(\overline{N}_p \overline{N}_q\right).$$

We can explicitly give $w^0\left(\overline{N}_p\overline{N}_q\right)$. For this note that

$$\overline{N}_q^{-1}=\left(D_{1/Q}\Sigma D_q\right)^{-1}=D_{1/q}\Sigma^{-1}D_Q$$

and since $\Sigma^{-1}=\Delta$ we deduce $\overline{N}_q^{-1}=D_{1/q}\Delta D_Q=D_{1/q}\Delta(Q)$. Similarly we obviously have $\overline{N}_p^{-1}=D_{1/p}\Delta D_P=D_{1/p}\Delta(P)$. By Lemma 4.3, since $P\in\Gamma$, we immediately get $\Delta W_P^0=W_P^0$ and

$$\overline{N}_p^{-1}w_0=D_{1/p}\Delta W_P^0=D_{1/p}W_P^0=W_{P/p}^0.$$

So

$$w^0\left(\overline{N}_p\overline{N}_q\right)=\overline{N}_q^{-1}\overline{N}_p^{-1}w^0=\overline{N}_q^{-1}W_{P/p}^0=D_{1/q}\Delta(Q)\,W_{P/p}^0.$$

Since $PQ/p\in\Gamma$ by Lemma 4.3 ii) b)

$$\Delta(Q)\,W_{P/p}^0=W_{P/p}^0\left(C(Q)\right)=W_{PQ/p}^0,$$

we conclude

$$w^0\left(\overline{N}_p\overline{N}_q\right)=W_{PQ/pq}^0.$$

Finally $X-Le\in w^0\left(\overline{N}_p\overline{N}_q\right)$ means that $X-Le\in W_{PQ/pq}^0$ and $x_k\to L\left(S\left(\overline{N}_p\overline{N}_q\right)\right)$. Because there is $\varphi:\,]0,\infty[\,\to\,]0,\infty[$ such that (2.1) holds, we conclude $x_k\to L_{\mu,\nu}\left(S\left(\overline{N}_p\overline{N}_q\right)\right)$. □

We also obtain the following result which is a direct consequence of the preceding.

Corollary 4.7. *Let* $(\mathbb{C},\mu,\nu,*,\diamondsuit)$ *be an IFNS satisfying condition* (φ). *Let* $1<a<b$ *and put*

$$y_k=\frac{1}{a^k-1}\left(\sum_{j=1}^{k}\frac{a^j}{b^j-1}\sum_{i=1}^{j}b^ix_i\right).$$

Then

$$\frac{1}{n}\left|\left\{k\le n:\mu\left(y_k-L\frac{ab}{(a-1)(b-1)},t\right)\le 1-\varepsilon\right\}\right|=0(1)\quad(n\to\infty).\tag{4.7}$$

Proof. It is enough to take $p=\left(a^k\right)_{k\ge1}$ and $q=\left(b^k\right)_{k\ge1}$. So trivially

$$\frac{P_kQ_k}{p_k}=\frac{1}{a^k}\frac{a^{k+1}-a}{a-1}\frac{b^{k+1}-b}{b-1}\sim\frac{(ab)^{k+1}}{(a-1)(b-1)a^k}=\frac{ab^{k+1}}{(a-1)(b-1)}$$

and $PQ/p\in\Gamma$. The calculation gives

$$y_k=\frac{ab}{(a-1)(b-1)}\left[\overline{N}_p\overline{N}_qX\right]_k.$$

Then putting $L'=Lab/(a-1)(b-1)$, by Theorem 4.6 ii) b), we conclude $x_k\to L'_{\mu,\nu}\left(S\left(\overline{N}_p\overline{N}_q\right)\right)$ and (4.7) holds. □

5 Other Applications

In this section we deal with $D_{1/\tau}\Delta(\lambda)$- and $C(\lambda)$-statistical convergence on IFNS.

Theorem 5.1. *Let V be a set of scalars and let (μ,ν) be an IFN satisfying condition (φ).*

i) Let $\tau\in\Gamma$. If

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\lambda_k}{\tau_k}\left|x_k-L\frac{\tau_1+\ldots+\tau_k}{\lambda_k}\right|\to 0\ \ (n\to\infty)\,, \tag{5.1}$$

then $x_k\to L_{\mu,\nu}\left(S\left(D_{1/\tau}\Delta(\lambda)\right)\right)$. Thus

$$\frac{1}{n}\left|\left\{k\le n:\mu\left(\frac{\lambda_k x_k-\lambda_{k-1}x_{k-1}}{\tau_k}-L,t\right)\le 1-\varepsilon\right\}\right|=0\,(1)\ \ (n\to\infty)\,.$$

ii) Let $\lambda\in\Gamma$. If

$$\frac{1}{n}\sum_{k=1}^{n}\left|\frac{x_k}{\lambda_k}-L\left(1-\frac{\lambda_{k-1}}{\lambda_k}\right)\right|\to 0\ \ (n\to\infty)\,, \tag{5.2}$$

then $x_k\to L_{\mu,\nu}\left(S\left(C(\lambda)\right)\right)$. Thus

$$\frac{1}{n}\left|\left\{k\le n:\mu\left(\frac{x_1+\ldots+x_k}{\lambda_k}-L,t\right)\le 1-\varepsilon\right\}\right|=0\,(1)\ \ (n\to\infty)\,.$$

Proof. i) We have $x_k\to L_{\mu,\nu}\left(S\left(D_{1/\tau}\Delta(\lambda)\right)\right)$ if

$$D_{1/\tau}\Delta(\lambda)\,X-Le\in w^0. \tag{5.3}$$

Further (5.3) holds if $D_{1/\tau}\Delta(\lambda)\left(X-LC(\lambda)\,D_\tau e\right)\in w^0$, that is, $X-LC(\lambda)\,D_\tau e\in w^0\left(D_{1/\tau}\Delta(\lambda)\right)$. But $w^0\left(D_{1/\tau}\Delta(\lambda)\right)=W_\tau^0\left(\Delta(\lambda)\right)$ and since $\tau\in\Gamma$, we deduce by Lemma 4.3 ii) that $w^0\left(D_{1/\tau}\Delta(\lambda)\right)=W^0_{\tau/\lambda}$. Then

$$\left[C(\lambda)\,D_\tau e\right]_n=\frac{\tau_1+\ldots+\tau_n}{\lambda_n}$$

and $X-LC(\lambda)\,D_\tau e\in W^0_{\tau/\lambda}$ is equivalent to (5.1). This concludes the proof of i). ii) Here $x_k\to L_{\mu,\nu}\left(S\left(C(\lambda)\right)\right)$ if

$$C(\lambda)\,X-Le=C(\lambda)\left(X-L\Delta(\lambda)\,e\right)\in w^0. \tag{5.4}$$

Since $\lambda\in\Gamma$ and from Lemma 4.3 ii) condition (5.4) is equivalent to $X-L\Delta(\lambda)\,e\in w^0\left(C(\lambda)\right)=W_\lambda^0$. Now $\left[\Delta(\lambda)\,e\right]_n=\lambda_n-\lambda_{n-1}$ and $X-L\Delta(\lambda)\,e\in W^0_\lambda$ means that (5.2) holds. This concludes the proof of ii). □

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