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Some Integral Inequalities with Maximum of the Unknown Functions

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Abstract

This paper deals with a special type of delayed integral inequalities of Bihary type for scalar functions of two variables. The integrals involve the maximum of the unknown function over a past time interval. Several nonlinear types of integral inequalities are solved. The importance of these integral inequalities is based on their wide applications to the qualitative investigations of various properties of solutions of partial differential equations with "maxima" and it is illustrated by some direct applications.

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1 Introduction

In the last few decades, great attention has been paid to automatic control systems and their applications to computational mathematics and modeling. Many problems in control theory correspond to the maximal deviation of the regulated quantity and they are adequately modeled by differential equations with "maxima" [13]. The qualitative investigation of properties of differential equations with "maxima" [2, 4, 6, 8–10] requires building of an appropriate mathematical apparatus. One of the main mathematical tools, employed successfully for studying existence, uniqueness, continuous dependence, comparison, perturbation, boundedness and stability of solutions of differential and integral equations, is the method of integral inequalities [1, 5, 14–17]. The

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development of the theory of partial differential equations with “maxima” [3, 7, 12] requires solving linear and nonlinear integral inequalities that involve the maximum of the unknown scalar function of two variables [11].

This paper deals with nonlinear integral inequalities which involve the maximum of the unknown scalar function of two variables. Various cases of nonlinear integral inequalities are solved. The form of the solution depends significantly on the type of nonlinear function in the integral. These results generalize the classical integral inequalities of Gronwall–Belman and Bihari type. The importance of the solved integral inequalities is illustrated on some direct applications to partial differential equations with “maxima”.

2 Main Results

Let $h > 0$ be a constant, $x_0,\ y_0,\ X,\ Y$ be fixed points such that $0 \le x_0 < X \le \infty$ and $0 \le y_0 < Y \le \infty$.

Definition 2.1. We will say that the function $\alpha \in C^1([x_0, X), \mathbb{R}_+)$ is from the class $\mathcal{F}$ if it is a nondecreasing function and $\alpha(x) \le x$ for $x \in [x_0, X)$.

Let the functions $\alpha_i,\ \beta_j \in \mathcal{F}$ for $i = 1,\ 2, \dots,\ n,\ j = 1,\ 2, \dots,\ m$. Denote

$$J = \min\left(\min_{1\le i\le n} \alpha_i(x_0), \min_{1\le j\le m} \beta_j(x_0)\right).$$

Consider the sets $G,\ \Psi,\ \Lambda$ defined by

$$\begin{aligned} G &= \left\{(x,y) \in \mathbb{R}^2 : x \in [x_0, X),\ y \in [y_0, Y)\right\}, \\ \Psi &= \left\{(x,y) \in \mathbb{R}^2 : x \in [J-h, x_0],\ y \in [y_0, Y)\right\}, \\ \Lambda &= \left\{(x,y) \in \mathbb{R}^2 : x \in [J-h, X),\ y \in [y_0, Y)\right\} = G \cup \Psi. \end{aligned}$$

Theorem 2.2. *Let the following conditions be fulfilled:*

1. *The functions $\alpha_i,\ \beta_j \in \mathcal{F}$ for $i = 1,\ 2,\ \dots,\ n,\ \ j = 1,\ 2,\ \dots,\ m$.*

2. *The functions $f_i,\ g_j \in C([J, X) \times [y_0, Y), \mathbb{R}_+)$ for $i = 1,\ 2,\ \dots,\ n,\ j = 1,\ 2,\ \dots,\ m$.*

3. *The function $\phi \in C(\Psi, [0, k])$ where $k = const > 0$.*

4. *The functions $\omega_i,\ \tilde{\omega}_j \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing, $\omega_i(x) > 0$, $\tilde{\omega}_j(x) > 0$ for $x > 0$, $i = 1,\ 2,\ \dots,\ n,\ \ j = 1,\ 2,\ \dots,\ m$.*

5. *The function* $u \in C(\Lambda, \mathbb{R}_+)$ *and satisfies the inequalities*

$$\begin{aligned} u(x,y) \le k &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\Big(u(s,t)\Big)dt\,ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\,s]} u(\xi,t)\Big)dt\,ds \quad \textit{for } (x,y)\in G, \end{aligned} \tag{2.1}$$

$$u(x,y) \le \phi(x,y), \qquad \textit{for } (x,y) \in \Psi. \tag{2.2}$$

Then for $(x,y) \in G_1$*, the inequality*

$$\begin{aligned} u(x,y) \le W^{-1}\Bigg(W(k) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)dt\,ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)dt\,ds\Bigg) \end{aligned} \tag{2.3}$$

holds, where W^{-1} *is the inverse function of*

$$W(r) = \int_{r_0}^{r} \frac{ds}{\omega(s)} < \infty\,, \quad 0 \le r_0 < k \le r, \tag{2.4}$$

$$\omega(t) = \max\Big(\max_{1\le i\le n} \omega_i(t),\, \max_{1\le j\le m} \tilde{\omega}_j(t)\Big) \tag{2.5}$$

$$\begin{aligned} G_1 = \Bigg\{(x,y)\in G : W(k) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)dt\,ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)dt\,ds \quad \in \mathrm{Dom}\left(W^{-1}\right)\Bigg\}. \end{aligned}$$

Proof. Define a function $z : \Lambda \to \mathbb{R}_+$ by the equalities

$$z(x,y) = \begin{cases} k + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\Big(u(s,t)\Big)dt\,ds & \\ \quad + \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\,s]} u(\xi,t)\Big)dt\,ds, & (x,y)\in G, \\ k, & (x,y)\in\Psi. \end{cases}$$

The function $z(x,y)$ is nondecreasing in its both arguments, $z(x_0,y) = k$ for $y \in [y_0, Y)$, and $u(x,y) \le z(x,y)$ for $(x,y) \in \Lambda$. Note that $\max_{\xi\in[s-h,\,s]} z(\xi,y) = z(s,y)$ for

$s \in [\beta_j(x_0), \beta_j(X)),\ j = 1,\, 2,\, \dots,\, m$ and $y \in [y_0, Y)$. Then from inequality (2.1) and the definition of the function $\omega(t)$, we get for $(x,y) \in G$

$$\begin{aligned} z(x,y) \le k &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega\Big(z(s,t)\Big)dt\,ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\omega\Big(z(s,t)\Big)dt\,ds. \end{aligned} \tag{2.6}$$

Define a function $K : G \to [k, \infty)$ by the equality

$$\begin{aligned} K(x,y) = k &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega\Big(z(s,t)\Big)dt\,ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\omega\Big(z(s,t)\Big)dt\,ds. \end{aligned} \tag{2.7}$$

Note that $K(x,y)$ is an increasing function and the inequality $z(x,y) \le K(x,y)$ holds for $(x,y) \in G$. From (2.7) and condition 1 of Theorem 2.2, we obtain for the partial derivative $K'_x(x,y)$ with respect to x

$$\begin{aligned} K'_x(x,y) &= \sum_{i=1}^{n} \int_{y_0}^{y} f_i\big(\alpha_i(x),t\big)\omega\Big(z(\alpha_i(x),t)\Big)\big(\alpha_i(x)\big)'dt \\ &\quad + \sum_{j=1}^{m} \int_{y_0}^{y} g_j\big(\beta_j(x),t\big)\omega\Big(z(\beta_j(x),t)\Big)\big(\beta_j(x)\big)'dt \\ &\le \sum_{i=1}^{n} \int_{y_0}^{y} f_i\big(\alpha_i(x),t\big)\omega\Big(K(\alpha_i(x),t)\Big)\big(\alpha_i(x)\big)'dt \\ &\quad + \sum_{j=1}^{m} \int_{y_0}^{y} g_j\big(\beta_j(x),t\big)\omega\Big(K(\beta_j(x),t)\Big)\big(\beta_j(x)\big)'dt \\ &\le \omega\big(K(x,y)\big)\Bigg(\sum_{i=1}^{n} \int_{y_0}^{y} f_i(\alpha_i(x),t)(\alpha_i(x))'dt \\ &\qquad + \sum_{j=1}^{m} \int_{y_0}^{y} g_j(\beta_j(x),t)(\beta_j(x))'dt\Bigg). \end{aligned} \tag{2.8}$$

From inequality (2.8), we get

$$\begin{aligned} \frac{K'_x(x,y)}{\omega\big(K(x,y)\big)} &\le \sum_{i=1}^{n} \big(\alpha_i(x)\big)' \int_{y_0}^{y} f_i\big(\alpha_i(x),t\big)dt \\ &\quad + \sum_{j=1}^{m} \big(\beta_j(x)\big)' \int_{y_0}^{y} g_j\big(\beta_j(x),t\big)dt. \end{aligned} \tag{2.9}$$

Integrate inequality (2.9) with respect to x from x_0 to x, $x \in [x_0, X)$, change the variable $\eta = \alpha_i(s)$ $(i = 1, \ldots, n)$ in the first sum of integrals, and $\eta = \beta_j(s)$ $(j = 1, \ldots, m)$ in the second sum of integrals, use the definition (2.4), the equalities

$$\int_{x_0}^{x} \frac{K_x(s,y)\, ds}{\omega\big(K(x,y)\big)} = \int_{k}^{K(x,y)} \frac{du}{\omega(u)} = W(K(x,y)) - W(k)$$

and the inequality $z(x,y) \le K(x,y)$, and obtain for $(x,y) \in G$

$$\begin{aligned} W\Big(K(x,y)\Big) \le W(k) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(\eta,t)dt\, d\eta \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(\eta,t)dt\, d\eta. \end{aligned} \tag{2.10}$$

Since W^{-1} is increasing, from inequalities (2.10) and $u(x,y) \le z(x,y) \le K(x,y)$ we obtain the required inequality (2.3) for $(x,y) \in G_1$. $\square$

Define the following class of functions.

Definition 2.3. The function $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is said to be from the class Ω_1 if

(i) $\omega(x) > 0$ for $x > 0$ and it is a nondecreasing function;

(ii) $\omega(tx) \ge t\omega(x)$ for $0 \le t \le 1$.

In the case when the nonlinear functions in the integrals, additionally to the conditions of Theorem 2.2, are submultiplicative, then the constant k in inequality (2.1) may be substituted by an increasing function.

Theorem 2.4. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 2 of Theorem 2.2 are satisfied.*
2. *The function $k \in C(G, [1,\infty))$ is nondecreasing in its both arguments.*
3. *The function $\phi \in C(\Psi, [0, \tilde{k}])$, where $\tilde{k} = k(x_0, y_0)$.*
4. *The functions $\omega_i,\ \tilde{\omega}_j \in \Omega_1$, $i = 1, 2, \ldots, n,\ \ j = 1, 2, \ldots, m$.*
5. *The function $u \in C(\Lambda, \mathbb{R}_+)$ and satisfies the inequalities*

$$\begin{aligned} u(x,y) \le k(x,y) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\Big(u(s,t)\Big)dt\, ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\Big(\max_{\xi \in [s-h,\, s]} u(\xi,t)\Big)dt\, ds \ \ \textit{for } (x,y) \in G, \end{aligned} \tag{2.11}$$

$$u(x,y) \le \phi(x,y), \quad \textit{for } (x,y) \in \Psi. \tag{2.12}$$

Then for $(x,y) \in G_2$*, the inequality*

$$\begin{aligned} u(x,y) \le k(x,y) \times W^{-1}\Big(W(1) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)dtds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)dtds \Big) \end{aligned} \tag{2.13}$$

holds, where the functions W, ω *are defined by* (2.4) *and* (2.5)*, respectively,*

$$\begin{aligned} G_2 = \Big\{ (x,y) \in G : W(1) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)dt\,ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)dt\,ds \ \in \mathrm{Dom}\left(W^{-1}\right) \Big\}. \end{aligned} \tag{2.14}$$

Proof. Define the continuous nondecreasing function $K : \Lambda \to [1,\infty)$ by

$$K(x,y) = \begin{cases} k(x,y), & \text{for} \quad (x,y) \in G, \\ k(x_0,y), & \text{for} \quad (x,y) \in \Psi. \end{cases}$$

From inequalities (2.11), (2.12), and $\dfrac{1}{k(x,y)} \le 1$, and condition 4 of Theorem 2.4, we obtain

$$\begin{aligned} \frac{u(x,y)}{k(x,y)} &\le 1 + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\left(\frac{u(s,t)}{k(x,y)}\right)dt\,ds \\ &\quad + \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\left(\frac{\max_{\xi\in[s-h,\,s]} u(\xi,t)}{k(x,y)}\right)dt\,ds, \\ &\le 1 + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\left(\frac{u(s,t)}{K(s,t)}\right)dt\,ds \\ &\quad + \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\left(\frac{\max_{\xi\in[s-h,\,s]} u(\xi,t)}{K(s,t)}\right)dt\,ds, \quad (x,y) \in G \end{aligned} \tag{2.15}$$

and

$$\frac{u(x,y)}{k(x_0,y_0)} \le \frac{\phi(x,y)}{k(x_0,y_0)} \le 1, \qquad (x,y) \in \Psi. \tag{2.16}$$

For every $j :\ 1 \le j \le m$ and $t \in [y_0, Y)$, $s \in [\beta_j(x_0), \beta_j(X))$, the inequalities

$$\frac{\max\limits_{\xi\in[s-h,s]} u(\xi,t)}{K(s,t)} = \frac{u(\xi_1,t)}{K(s,t)} \le \frac{u(\xi_1,t)}{K(\xi_1,t)} \le \max_{\xi\in[s-h,s]} \frac{u(\xi,t)}{K(\xi,t)} \le \max_{\xi\in[s-h,s]} Z(\xi,t) \tag{2.17}$$

hold, where $\xi_1 \in [s-h, s]$ and $Z(x,y) = \dfrac{u(x,y)}{K(x,y)}$. From inequality (2.17) it follows that the inequalities (2.15), (2.16) may be written in the form

$$\begin{aligned} Z(x,y) \le 1 &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\Big(Z(s,t)\Big)dt\,ds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\,s]} Z(\xi,t)\Big)dt\,ds,\ (x,y)\in G, \end{aligned} \tag{2.18}$$

$$Z(x,y) \le 1, \qquad (x,y) \in \Psi. \tag{2.19}$$

From inequalities (2.18) and (2.19) according to Theorem 2.2, we obtain for $(x,y) \in G_2$

$$\begin{aligned} Z(x,y) \le W^{-1}\Big(W(1) &+ \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)dtds \\ &+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)dtds\Big). \end{aligned} \tag{2.20}$$

Inequality (2.20) and definitions of the functions $Z(x,y)$, $K(x,y)$ imply the validity of inequality (2.13). □

Define the following class of functions.

Definition 2.5. We will say that the function $\omega \in C(\mathbb{R}_+, \mathbb{R}_+)$ is from the class Ω_2 if it satisfies the following conditions:

(i) $\omega(x) > 0$ for $x > 0$ and it is a nondecreasing function;

(ii) $\omega(tx) \ge t\omega(x)$ for $0 \le t \le 1$;

(iii) $\omega(x) + \omega(y) \ge \omega(x+y)$.

Remark 2.6. For example, the functions $\omega(x) = \sqrt{x}$ and $\omega(x) = x$ are from the class Ω_2.

In the case when the nonlinear functions in the integrals are from the class Ω_2, the following result holds.

Theorem 2.7. *Let the following conditions be fulfilled:*

1. *The conditions 1 and 2 of Theorem 2.2 are satisfied.*
2. *The functions* $\omega_i,\ \tilde{\omega}_j \in \Omega_2$, $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$.
3. *The function* $k \in C(\Lambda, \mathbb{R}_+)$.

4. *The function* $\mu \in C(G, [1, \infty))$.

5. *The function* $u \in C(\Lambda, \mathbb{R}_+)$ *and satisfies the inequalities*

$$
\begin{aligned}
u(x,y) \le k(x,y) + \mu(x,y)\Bigg(& \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\Big(u(s,t)\Big)dt\,ds \\
& + \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\,s]} u(\xi,t)\Big)dt\,ds\Bigg) \text{ for } (x,y) \in G,
\end{aligned}
\tag{2.21}
$$

$$
u(x,y) \le k(x,y) \qquad \text{for} \quad (x,y) \in \Psi. \tag{2.22}
$$

Then for $(x,y) \in G_3$*, the inequality*

$$
u(x,y) \le k(x,y) + M(x,y)P(x,y)W^{-1}\Big(W(1) + A(x,y)\Big) \tag{2.23}
$$

holds, where the functions $W,\ w$ *are defined by* (2.4) *and* (2.5)*, respectively,* W^{-1} *is the inverse of* W*,*

$$
G_3 = \Big\{(x,y) \in G : \quad W(1) + A(x,y) \ \in \mathrm{Dom}\left(W^{-1}\right)\Big\},
$$

$$
\begin{aligned}
P(x,y) = 1 + & \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\Big(k(s,t)\Big)dt\,ds \\
& + \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\,s]} k(\xi,t)\Big)dt\,ds, \ (x,y) \in G,
\end{aligned}
\tag{2.24}
$$

$$
\begin{aligned}
A(x,y) = & \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)M(s,t)dt\,ds \\
& + \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)M(s,t)dt\,ds,
\end{aligned}
\tag{2.25}
$$

$$
M(x,y) = \begin{cases} \mu(x,y) & \text{for } (x,y) \in G, \\ \mu(x_0,y) & \text{for } (x,y) \in \Psi. \end{cases} \tag{2.26}
$$

Proof. Define a function $z : \ \Lambda \to \mathbb{R}_+$ by the equalities

$$
z(x,y) = \begin{cases} \displaystyle\sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s,t)\omega_i\big(u(s,t)\big)dt\,ds \\ \quad \displaystyle + \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s,t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\,s]} u(\xi,t)\Big)dt\,ds \\ \hfill \text{for } (x,y) \in G, \\ 0 \hfill \text{for } (x,y) \in \Psi. \end{cases} \tag{2.27}
$$

From inequality (2.21) and the definition of the function $z(x, y)$, we obtain

$$u(x, y) \le k(x, y) + M(x, y)z(x, y), \qquad (x, y) \in \Lambda. \tag{2.28}$$

Since the function $M(x, y)$ is nondecreasing in its both arguments and for $s \in [J, X)$, $y \in [y_0, Y)$ we obtain

$$\max_{\xi\in[s-h,\, s]} u(\xi, y) \le \max_{\xi\in[s-h,\, s]} k(\xi, y) + M(s, y) \max_{\xi\in[s-h,\, s]} z(\xi, y). \tag{2.29}$$

From inequality (2.28), the definition of the function $z(x, y)$ and conditions 1 and 2 we get for $(x, y) \in G$

$$\begin{aligned}
&\int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s, t)\omega_i\Big(u(s, t)\Big)dt\, ds \\
&\le \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s, t)\omega_i\Big(k(s, t) + M(s, t)z(s, t)\Big)dt\, ds \\
&\le \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s, t)\omega_i\Big(k(s, t)\Big)dt\, ds \\
&\quad + \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s, t)M(s, t)\omega_i\Big(z(s, t)\Big)dt\, ds
\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}
&\int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s, t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\, s]} u(\xi, t)\Big)dt\, ds \\
&\le \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s, t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\, s]} k(\xi, t)\Big)dt\, ds \\
&\quad + \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s, t)M(s, t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\, s]} z(\xi, t)\Big)dt\, ds.
\end{aligned} \tag{2.31}$$

From the definition of the function $z(x, y)$ and inequalities (2.28), (2.30), (2.31), it follows that

$$\begin{aligned}
z(x, y) &\le P(x, y) + \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} \int_{y_0}^{y} f_i(s, t)M(s, t)\omega_i\Big(z(s, t)\Big)dt\, ds \\
&+ \sum_{j=1}^{m} \int_{\beta_j(x_0)}^{\beta_j(x)} \int_{y_0}^{y} g_j(s, t)M(s, t)\tilde{\omega}_j\Big(\max_{\xi\in[s-h,\, s]} z(\xi, t)\Big)dt\, ds, \ \text{ for } (x, y) \in G,
\end{aligned} \tag{2.32}$$

$$z(x, y) \le 0, \qquad \text{for } \ (x, y) \in \Psi. \tag{2.33}$$

From inequalities (2.32), (2.33), according to Theorem 2.4, we obtain the inequality (2.23). □

3 Applications

Consider the scalar partial differential equation with "maxima"

$$u''_{x\,y} = F\Big(x,\ y,\ u(x,y),\ \max_{s\in[\sigma(x),\,\tau(x)]} u(s,y)\Big)\ \text{ for }\ (x,y)\in G \tag{3.1}$$

with the initial conditions

$$\begin{aligned} u(x_0,y) &= \varphi_1(y) && \text{for }\ y\in[y_0,Y),\\ u(x,y_0) &= \varphi_2(x) && \text{for }\ x\in[x_0,X),\\ u(x,y) &= \psi(x,y) && \text{for }\ (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y), \end{aligned} \tag{3.2}$$

where $u\in\mathbb{R}$, $\varphi_1:[y_0,Y)\to\mathbb{R}$, $\varphi_2:[x_0,X)\to\mathbb{R}$, $\psi:[\tau(x_0)-h,x_0]\times[y_0,Y)\to\mathbb{R}$, $F:[x_0,X)\times[y_0,Y)\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$.

Theorem 3.1 (Upper bound). *Let the following conditions be fulfilled:*

1. *The functions $\tau\in\mathcal{F}$, $\sigma\in C([x_0,X),\mathbb{R}_+)$ and there exists a constant $h>0$: $0\le\tau(x)-\sigma(x)\le h$ for $x\in[x_0,X)$.*
2. *The function $F\in C([x_0,X)\times[y_0,Y)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$ and satisfies for $(x,y)\in G$ and $\gamma,\ \nu\in\mathbb{R}$, the condition*
$$\big|F(x,\ y,\ \gamma,\ \nu)\big|\le Q(x,y)\sqrt{|\gamma|}+R(x,y)\sqrt{|\nu|}\,,$$
where $Q,\ R\in C(G,\mathbb{R}_+)$.
3. *The function $\psi\in C([\tau(x_0)-h,x_0]\times[y_0,Y),\mathbb{R})$.*
4. *The functions $\varphi_1\in C([y_0,Y),\mathbb{R})$, $\varphi_2\in C([x_0,X),\mathbb{R})$ and $\varphi_1(y_0)=\varphi_2(x_0)$, $\varphi_1(y)=\psi(x_0,y)$ for $y\in[y_0,Y)$.*
5. *The function $u(x,y)$ is a solution of the initial value problem* (3.1), (3.2) *which is defined for $(x,y)\in[\tau(x_0)-h,X)\times[y_0,Y)$.*

Then the solution $u(x,y)$ of the partial differential equation with "maxima" (3.1), (3.2) *satisfies for $(x,y)\in G$ the inequality*

$$\big|u(x,y)\big|\le K(x,y)+P_1(x,y)\bigg(1+\frac{1}{2}\int_{x_0}^{x}\int_{y_0}^{y}\Big[Q(s,t)+R(s,t)\Big]dt\,ds\bigg)^2, \tag{3.3}$$

where

$$K(x,y)=\begin{cases}\big|\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)\big|, & (x,y)\in G\\ \big|\psi(x,y)\big|, & (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y),\end{cases} \tag{3.4}$$

$$\begin{aligned} P_1(x,y) = 1 &+ \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t)\sqrt{K(s,t)}\,dt\,ds\\ &+ \int_{x_0}^{x}\int_{y_0}^{y} R(s,t)\sqrt{\max_{\xi\in[\sigma(s),\,\tau(s)]} K(\xi,t)}\,dt\,ds. \end{aligned} \tag{3.5}$$

Proof. For the function $u(x,y)$ we obtain

$$\begin{aligned} \big|u(x,y)\big| &\le \big|\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)\big| \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y}\Big|F\Big(s,\ t,\ u(s,t),\ \max_{\xi\in[\sigma(s),\,\tau(s)]} u(\xi,t)\Big)\Big|dt\,ds \\ &\le\ \big|\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)\big| + \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t)\sqrt{|u(s,t)|}dt\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} R(s,t)\sqrt{\max_{\xi\in[\sigma(s),\,\tau(s)]}|u(\xi,t)|}dt\,ds \quad \text{for } (x,y)\in G, \end{aligned} \tag{3.6}$$

$$\big|u(x,y)\big| =\ \big|\psi(x,y)\big|, \quad \text{for } (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y). \tag{3.7}$$

Denote $\big|u(x,y)\big| = U(x,y)$ for $(x,y)\in[\tau(x_0)-h,X)\times[y_0,Y)$. From (3.6) and (3.7), we have

$$\begin{aligned} U(x,y) &\le\ \big|\varphi_1(y)+\varphi_2(x)-\varphi_2(x_0)\big| + \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t)\sqrt{U(s,t)}\,dt\,ds \\ &\quad + \int_{x_0}^{x}\int_{y_0}^{y} R(s,t)\sqrt{\max_{\xi\in[\sigma(s),\,\tau(s)]} U(\xi,t)}\,dt\,ds, \quad (x,y)\in G, \end{aligned} \tag{3.8}$$

$$U(x,y) = \big|\psi(x,y)\big| = K(x,y), \quad \text{for } (x,y)\in[\tau(x_0)-h,x_0]\times[y_0,Y). \tag{3.9}$$

Change the variable $s=\tau^{-1}(\eta)$ in the second integral of (3.8), use the inequality $\max_{\xi\in[\sigma(x),\,\tau(x)]} U(\xi,y) \le \max_{\xi\in[\tau(x)-h,\,\tau(x)]} U(\xi,y)$ for $y\in[y_0,Y)$ and $x\in[x_0,X)$ that follows from condition 1 of Theorem 3.1 and obtain

$$\begin{aligned} U(x,y) &\le K(x,y) + \int_{x_0}^{x}\int_{y_0}^{y} Q(s,t)\sqrt{U(s,t)}dtds \\ &\quad + \int_{\tau(x_0)}^{\tau(x)}\int_{y_0}^{y} R(\tau^{-1}(\eta),t)(\tau^{-1}(\eta))'\sqrt{\max_{\xi\in[\eta-h,\,\eta]} U(\xi,t)}\,dtd\eta,\ (x,y)\in G. \end{aligned} \tag{3.10}$$

Note that the conditions of Theorem 2.7 are satisfied for $n=m=1$, $\alpha(x)\equiv x$, $\beta(x)\equiv\tau(x)$, $k(x,y)\equiv K(x,y)$, where the function $K(x,y)$ is defined by (3.4), $f(x,y)\equiv Q(x,y)$ for $(x,y)\in G$, $g(x,y)\equiv R(\tau^{-1}(x),y)(\tau^{-1}(x))'$ for $x\in[\tau(x_0),X)$, $y\in[y_0,Y)$, $\omega(v)=\tilde\omega(v)=\sqrt{v}$, $W(v)=2\sqrt{v}$, $W^{-1}(v)=\dfrac{1}{4}v^2$ for $v\in\mathbb{R}_+$, and $\mathrm{Dom}(W^{-1})=\mathbb{R}_+$. According to Theorem 2.7, from inequalities (3.10), (3.9), we obtain for $(x,y)\in G$ the bound

$$U(x,y) \le K(x,y) + P_1(x,y)\frac{1}{4}\bigg(2+\int_{x_0}^{x}\int_{y_0}^{y}\Big[Q(s,t)+R(s,t)\Big]dt\,ds\bigg)^2, \tag{3.11}$$

where the function $P_1(x,y)$ is defined by equality (3.5). Inequality (3.11) and the definition of the function $U(x,y)$ imply the validity of the required inequality (3.3). $\square$

In the case when the initial conditions in problem (3.1), (3.2) are constants and Lipschitz functions for the function F are constants, we may apply Theorem 2.2 instead of Theorem 2.7 and we obtain a better bound for the solution.

Theorem 3.2. *Let the conditions of Theorem 3.1 be satisfied, where* $\varphi_1(y) \equiv C, \varphi_2(x) \equiv C, \psi(x,y) \equiv C$*,* $Q(x,y) \equiv Q, R(x,y) \equiv R$*,* C, Q, R *are constants. Then the solution* $u(x,y)$ *of the partial differential equation with "maxima"* (3.1), (3.2) *satisfies for* $(x,y) \in G$ *the inequality*

$$\big|u(x,y)\big| \leq \left(\sqrt{|C|} + \frac{1}{2}(Q+R)(x-x_0)(y-y_0)\right)^2. \tag{3.12}$$

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