Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 6, Number 1, pp. 49–56 (2011)
http://campus.mst.edu/adsa

A Result on the Neutrix Composition of the Delta Function

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Abstract

The neutrix composition $F(f(x))$ of a distribution $F(x)$ and a locally summable function $f(x)$ is said to exist and be equal to the distribution $h(x)$ if the neutrix limit of the sequence $\{F_n(f(x))\}$ is equal to $h(x)$, where $F_n(x) = F(x) * \delta_n(x)$ and $\{\delta_n(x)\}$ is a certain sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. It is proved that the neutrix composition $\delta^{(rsm-pm-1)}(x_+^{1/rm}/(1+x_+^{1/m}))$ exists for $r, s, m = 1, 2, 3\ldots$ and $p = 0, 1, 2, \ldots, rs-1$.

AMS Subject Classifications: 46F10.
Keywords: Distribution, Dirac-delta function, composition of distributions, neutrix, neutrix limit.

1 Introduction

In distribution theory, no meaning can be generally given to expressions of the form $F(f(x))$, where F and f are distributions. However, the compositions $\delta(f)$ and $\delta'(f)$, where $f = 0$ is a surface in three-dimensional space, appear in wave propagation problems, see [2, 3]. Furthermore, in physics, one finds the need to evaluate δ^2 when calculating the transition rates of certain particle interaction (see [8]).

The technique of neglecting appropriately defined infinite quantities was devised by Hadamard, and the resulting finite value extracted from a divergent integral is referred to as the Hadamard finite part. In fact, his method can be regarded as a particular application of the neutrix calculus developed by van der Corput (see [1]).

Received October 31, 2010; Accepted November 11, 2010
Communicated by Dragan Djurčić

Using the concepts of a neutrix and neutrix limit, the first author gave a general principle for the discarding of unwanted infinite quantities from asymptotic expansions, and this has been exploited particularly in connection with multiplication, convolution and composition of distributions, see [4, 5]. Using Fisher's definition, Koh and Li gave meaning to δ^r and $(\delta')^r$ for $r = 2, 3, \dots$, see [10], and the more general form $(\delta^{(s)}(x))^r$ was considered by Kou and Fisher in [11]. Recently the rth powers of the Dirac distribution and the Heaviside function for negative integers have been defined in [13] and [14], respectively.

In the following, we let $\mathcal{D}$ be the space of infinitely differentiable functions φ with compact support and let $\mathcal{D}[a, b]$ be the space of infinitely differentiable functions with support contained in the interval $[a, b]$. We let $\mathcal{D}'$ be the space of distributions defined on $\mathcal{D}$ and let $\mathcal{D}'[a, b]$ be the space of distributions defined on $\mathcal{D}[a, b]$. Now let $\rho(x)$ be a function in $\mathcal{D}$ having the following properties:

(i) $\rho(x) = 0$ for $|x| \geq 1$,

(ii) $\rho(x) \geq 0$,

(iii) $\rho(x) = \rho(-x)$,

(iv) $\int_{-1}^{1} \rho(x)\, dx = 1$.

Putting $\delta_n(x) = n\rho(nx)$ for $n = 1, 2, \dots$, it follows that $\{\delta_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to the Dirac delta-function $\delta(x)$. Further, if F is a distribution in $\mathcal{D}'$ and $F_n(x) = \langle F(x-t), \delta_n(x)\rangle$, then $\{F_n(x)\}$ is a regular sequence of infinitely differentiable functions converging to $F(x)$.

Now let $f(x)$ be an infinitely differentiable function having a single simple root at the point $x = x_0$. Gel'fand and Shilov defined the distribution $\delta^{(r)}(f(x))$ by the equation

$$\delta^{(r)}(f(x)) = \frac{1}{|f'(x_0)|}\left[\frac{1}{|f'(x)|}\frac{d}{dx}\right]^r \delta(x - x_0),$$

for $r = 0, 1, 2, \dots$, see [9].

In order to give a more general definition for the composition of distributions, the following definition for the neutrix composition of distributions was given in [4] and was originally called the composition of distributions.

Definition 1.1. Let F be a distribution in $\mathcal{D}'$ and let f be a locally summable function. We say that the neutrix composition $F(f(x))$ exists and is equal to h on the open interval (a, b) if

$$\underset{n\to\infty}{\mathrm{N-lim}} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$, where $F_n(x) = F(x) * \delta_n(x)$ for $n = 1, 2, \ldots$ and N is the neutrix, see [1], having domain N' the positive integers and range N'' the real numbers, with negligible functions which are finite linear sums of the functions

$$n^\lambda \ln^{r-1} n,\ \ln^r n : \lambda > 0,\ r = 1, 2, \ldots$$

and all functions which converge to zero in the usual sense as n tends to infinity.

In particular, we say that the composition $F(f(x))$ exists and is equal to h on the open interval (a,b) if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} F_n(f(x))\varphi(x)dx = \langle h(x), \varphi(x)\rangle$$

for all φ in $\mathcal{D}[a,b]$.

Note that taking the neutrix limit of a function $f(n)$ is equivalent to taking the usual limit of Hadamard's finite part of $f(n)$.

The following theorems were proved in [6, 7, 12] respectively.

Theorem 1.2. *The neutrix composition $\delta^{(s)}(\operatorname{sgn} x|x|^\lambda)$ exists and*

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = 0$$

for $s = 0, 1, 2, \ldots$ and $(s+1)\lambda = 1, 3, \ldots$ and

$$\delta^{(s)}(\operatorname{sgn} x|x|^\lambda) = \frac{(-1)^{(s+1)(\lambda+1)} s!}{\lambda[(s+1)\lambda - 1]!}\delta^{((s+1)\lambda-1)}(x)$$

for $s = 0, 1, 2, \ldots$ and $(s+1)\lambda = 2, 4, \ldots$.

Theorem 1.3. *The neutrix composition $\delta^{(rs-m)}[x_+^{1/r}/(1 + x_+^{1/r})]$ exists and*

$$\delta^{(rs-m)}[x_+^{1/r}/(1 + x_+^{1/r})] = \sum_{k=0}^{s-1} \frac{(-1)^{rs-m+k}(rs-m)!}{2k!}\binom{rs-m+1}{rk+r}\delta^{(k)}(x),$$

for $r, s = 1, 2, \ldots$ and $m = 1, 2, \ldots, rs$, where

$$\binom{rs-m+1}{rk+r} = 0$$

if $rs - m + 1 < rk + r$. In particular, we have

$$\delta[x_+^{1/r}/(1 + x_+^{1/r})] = 0$$

for $r = 2, 3, \ldots$ and

$$\delta[x_+/(1 + x_+)] = \frac{1}{2}\delta(x).$$

Theorem 1.4. *The neutrix composition $\delta^{(s)}[\ln^r(1+x_+^{1/r})]$ exists and*

$$\delta^{(s)}[\ln^r(1+x_+^{1/r})] = \sum_{k=0}^{s}\sum_{i=0}^{kr+r-1}\binom{kr+r-1}{i}\frac{(-1)^{(r+1)k+r+s+i-1}s!(i+1)^{sr+r-1}}{2(sr+r-1)!k!}\delta^{(k)}(x)$$

for $s=0,1,2,\ldots$ and $r=1,2,\ldots$.

2 Main Results

We now need the following lemma, which can be easily proved by induction.

Lemma 2.1. *We have*

$$\int_{-1}^{1} t^i\rho^{(s)}(t)\,dt = \begin{cases} 0, & 0 \le i < s, \\ (-1)^s s!, & i = s \end{cases}$$

and

$$\int_0^1 t^s\rho^{(s)}(t)\,dt = \frac{1}{2}(-1)^s s!$$

for $s=0,1,2,\ldots$.

We now prove the following theorem.

Theorem 2.2. *The neutrix composition $\delta^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/r})^{1/m}]$ exists and*

$$\delta^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/r})^{1/m}] = \sum_{k=0}^{s-1}\frac{(-1)^{rsm-pm+k-1}rm(rsm-pm-1)!}{2k!}\binom{rs-p}{rk+r}\delta^{(k)}(x), \quad (2.1)$$

for $r,s,m=1,2,\ldots$ and $p=0,1,\ldots,rs-1$.

Proof. We will first of all prove equation (2.1) on the interval $[-1,1]$. To do this, we need to evaluate

$$\begin{aligned}
&\int_{-1}^{1} x^k\delta_n^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/r})^{1/m}]\,dx \\
&= \int_0^1 x^k\delta_n^{(rsm-pm-1)}[(x^{1/r}/(1+x^{1/r}))^{1/m}]\,dx + \int_{-1}^{0} x^k\delta_n^{(rsm-pm-1)}(0)\,dx \\
&= n^{rsm-pm}\int_0^1 x^k\rho^{(rsm-pm-1)}[nx^{1/rm}/(1+x^{1/r})^{1/m}]\,dx \\
&\quad + n^{rsm-pm}\int_{-1}^{0} x^k\rho^{(rsm-pm-1)}(0)\,dx \\
&= I_1 + I_2.
\end{aligned}$$

It is seen immediately that

$$\operatorname*{N-lim}_{n\to\infty} I_2 = 0 \tag{2.2}$$

for $r, s, m = 1, 2, \dots$ and $k = 0, 1, 2, \dots$. Substituting $nx^{1/rm}/(1 + x^{1/r})^{1/m} = t$, i.e.,

$$x = \frac{t^{rm}}{n^{rm}(1 - t^m/n^m)^r},$$

we have

$$dx = \frac{rmt^{rm-1}\,dt}{n^{rm}(1 - t^m/n^m)^{r+1}}.$$

Then for $n > 1$, we have

$$\begin{aligned} I_1 &= rmn^{rm(s-k-1)-pm} \int_0^1 \frac{t^{rm(k+1)-1}}{(1 - t^m/n^m)^{r(k+1)+1}} \rho^{(rsm-pm-1)}(t)\,dt \\ &= rm \sum_{i=0}^{\infty} \int_0^1 \binom{rk + r + 1}{i} \frac{t^{rm(k+1)+mi-1}}{n^{-rm(s-k-1)+m(i+p)}} \rho^{(rsm-pm-1)}(t)\,dt. \end{aligned}$$

Hence

$$\begin{aligned} \operatorname*{N-lim}_{n\to\infty} I_1 &= \operatorname*{N-lim}_{n\to\infty} \int_0^1 x^k \delta_n^{(rsm-pm-1)}[x^{1/mr}/(1 + x^{1/r})^{1/m}]\,dx \\ &= \frac{(-1)^{rsm-pm-1} rm(rsm - pm - 1)!}{2} \binom{rs - p}{rk + r}, \end{aligned} \tag{2.3}$$

on using Lemma 2.1, for $k = 0, 1, 2, \dots, s - 1$ and $r, s, m = 1, 2, \dots$ and $p = 0, 1, 2, \dots, rs - 1$. Next, we have

$$\begin{aligned} &\int_0^1 \left| x^s \delta_n^{(rsm-pm-1)}[x^{1/rm}/(1 + x^{1/r})^{1/m}] \right| dx \\ &\le\ rmn^{-rm-pm} \int_0^1 \left| \frac{t^{rm(s+1)-1}}{(1 - t^m/n^n)^{r(s+1)+1}} \rho^{(rsm-pm-1)}(t) \right| dt, \\ &=\ O(n^{-rm-pm}) \end{aligned}$$

and so if $\psi(x)$ is an arbitrary continuous function, then

$$\lim_{n\to\infty} \int_0^1 x^s \delta_n^{(rsm-pm-1)}[x^{1/rm}/(1 + x^{1/r})^{1/m}]\psi(x)\,dx = 0, \tag{2.4}$$

for $r, s, m = 1, 2, \dots$. Further,

$$\begin{aligned} &\operatorname*{N-lim}_{n\to\infty} \int_{-1}^0 x^s \delta_n^{(rsm-pm-1)}(0)\psi(x)\,dx \\ &= \operatorname*{N-lim}_{n\to\infty} n^{rsm-pm} \int_{-1}^0 x^s \rho^{(rsm-pm-1)}(0)\psi(x)\,dx \\ &= 0, \end{aligned} \tag{2.5}$$

for $r, s, m = 1, 2, \ldots$. Now let φ be an arbitrary function in $\mathcal{D}[-1, 1]$. By Taylor's theorem, we have

$$\varphi(x) = \sum_{k=0}^{s-1} \frac{x^k \varphi^{(k)}(0)}{k!} + \frac{x^s \varphi^{(s)}(\xi x)}{s!},$$

where $0 < \xi < 1$. Then

$$\begin{aligned}
&\mathrm{N}\!-\!\lim_{n\to\infty} \langle \delta_n^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/r})^{1/m}], \varphi(x)\rangle \\
&= \mathrm{N}\!-\!\lim_{n\to\infty} \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{1} x^k \delta_n^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/r})^{1/m}]\, dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty} \int_{-1}^{1} \frac{x^s}{s!} \delta_n^{(rsm-pm-1)}[x_+^{1/rm}/(1+x_+^{1/r})^{1/m}] \varphi^{(s)}(\xi x)\, dx \\
&= \mathrm{N}\!-\!\lim_{n\to\infty} \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} \int_{0}^{1} x^k \delta_n^{(rsm-pm-1)}[x^{1/r}/(1+x^{1/r})]\, dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty} \sum_{k=0}^{s-1} \frac{\varphi^{(k)}(0)}{k!} \int_{-1}^{0} x^k \delta_n^{(rsm-pm-1)}(0)\, dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty} \int_{0}^{1} \frac{x^s}{s!} \delta_n^{(rsm-pm-1)}[x^{1/r}/(1+x^{1/r})] \varphi^{(s)}(\xi x)\, dx \\
&\quad + \mathrm{N}\!-\!\lim_{n\to\infty} \int_{-1}^{0} \frac{x^s}{s!} \delta_n^{(rsm-pm-1)}(0) \varphi^{(s)}(\xi x)\, dx \\
&= \sum_{k=0}^{s-1} \frac{(-1)^{rsm-pm-1} rm(rsm-pm-1)!}{2k!} \binom{rs-p}{rk+r} \varphi^{(k)}(0) \\
&= \sum_{k=0}^{s-1} \frac{(-1)^{rsm-pm+k-1} rm(rsm-pm-1)!}{2k!} \binom{rs-p}{rk+r} \langle \delta^{(k)}(x), \varphi(x)\rangle,
\end{aligned}$$

on using equations (2.2), (2.3), (2.4) and (2.5). This proves that the neutrix composition $\delta^{(rsm-pm-1)}[x_+^{1/r}/(1+x_+^{1/r})]$ exists and

$$\begin{aligned}
&\delta^{(rsm-pm-1)}[x_+^{1/r}/(1+x_+^{1/r})] \\
&\qquad = \sum_{k=0}^{s-1} \frac{(-1)^{rsm-pm+k-1} rm(rsm-pm-1)!}{2k!} \binom{rs-p}{rk+r} \delta^{(k)}(x),
\end{aligned}$$

on the interval $[-1, 1]$ for $r, s, m = 1, 2, 3, \ldots$ and $p = 0, 1, 2, \ldots, rs-1$. $\square$

Replacing x by $-x$ in Theorem 1.4, we get the following corollary.

Corollary 2.3. *The neutrix composition* $\delta^{(rsm-pm-1)}[x_-^{1/rm}/(1+x_-^{1/r})^{1/m}]$ *exists and*

$$\delta^{(rsm-pm-1)}[x_-^{1/rm}/(1+x_-^{1/r})^{1/m}] = \sum_{k=0}^{s-1} \frac{(-1)^{rsm-pm-1} rm(rsm-pm-1)!}{2k!} \binom{rs-p}{rk+r} \delta^{(k)}(x),$$

for $r, s, m = 1, 2, \dots$ *and* $p = 0, 1, \dots, rs-1$.

Corollary 2.4. *The neutrix composition* $\delta^{(rsm-pm-1)}[|x|^{1/rm}/(1+|x|^{1/r})^{1/m}]$ *exists and*

$$\delta^{(rsm-pm-1)}[|x|^{1/rm}/(1+|x|^{1/r})^{1/m}] = \sum_{k=0}^{s-1} \frac{(-1)^{rsm-pm-1}[1+(-1)^k] rm(rsm-pm-1)!}{2k!} \binom{rs-p}{rk+r} \delta^{(k)}(x), \quad (2.6)$$

for $r, s, m = 1, 2, \dots$ *and* $p = 0, 1, \dots, rs-1$.

Proof. Noting that

$$\int_{-1}^{1} x^k \delta^{(rsm-pm-1)}[|x|^{1/rm}/(1+|x|^{1/r})^{1/m}]\, dx = [1+(-1)^k] \int_0^1 x^k \delta^{(rsm-pm-1)}[|x|^{1/rm}/(1+|x|^{1/r})^{1/m}]\, dx,$$

we see that equation (2.6) follows. □

Acknowledgment

The first author is supported by Tübitak. The second author acknowledges support under Research Project Number 10 G 702001, awarded by the Hacettepe University Scientific Research Unit.

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