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A Problem about Finite Integrability of a System of Vekua's Complex Differential Equations

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Abstract

In this work, a system of complex differential Vekua equations is solved. The idea of operational integrability is used and the system is solved applying composition of Vekua's and Fempl's differential operators.

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1 Introduction

In his famous monograph [9], I. Vekua studied very closely the elliptic system of partial equations

$$\begin{aligned} u'_x - v'_y &= a(x,y)\,u + b(x,y)\,v + c(x,y) \\ u'_y + v'_x &= b(x,y)\,u - a(x,y)\,v + d(x,y)\,, \end{aligned} \tag{1.1}$$

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where $a\left(x,y\right)$, $b\left(x,y\right), c\left(x,y\right)$ and $d\left(x,y\right)$ are given continuous functions in some simply-connected domain. This system has a very large theoretical and practical meaning as well as many applications in various parts of mechanic. If the second equation in (1.1) is multiplied by i and added to the first one, we get a complex differential equation of Vekua type

$$w'_{\bar{z}} = A\left(z,\bar{z}\right)\bar{w} + B\left(z,\bar{z}\right) \tag{1.2}$$

in canonical form, where

$$A\left(z,\bar{z}\right) = \left(a+ib\right)/2,$$
$$B\left(z,\bar{z}\right) = \left(c+id\right)/2.$$

Vekua has shown that the general solution of the differential equation (1.2) has the form

$$\begin{aligned} w\left(z,\bar{z}\right) = \phi\left(z\right) + \iint_T \Gamma_1\left(z,t\right)\phi\left(t\right)dT + \iint_T \Gamma_2\left(z,t\right)\overline{\phi\left(t\right)}dT \\ -\frac{1}{\pi}\iint_T \Omega_1\left(z,t\right)\mathrm{B}\left(t\right)dT - \frac{1}{\pi}\iint_T \Omega_2\left(z,t\right)\overline{\mathrm{B}\left(t\right)}dT, \end{aligned} \tag{1.3}$$

where

$$\Gamma_1(z,t) = \sum_{j=1}^{\infty} K_{2j}(z,t), \qquad \Gamma_2(z,t) = \sum_{j=1}^{\infty} K_{2j+1}(z,t),$$

$$\mathrm{K}_1(z,t) = -\frac{A(T)}{\pi(t-z)}, \qquad \mathrm{K}_n(z,t) = \iint\limits_T \mathrm{K}_1(z,\sigma)\overline{\mathrm{K}_{n-1}\left(\sigma,t\right)}dT_\sigma,$$

$$\Omega_1(z,t) = \frac{1}{t-z} + \iint\limits_T \frac{\Gamma_1(z,\sigma)}{t-\sigma}dT_\sigma, \qquad \Omega_2(z,t) = \iint\limits_T \frac{\Gamma_2(z,\sigma)}{\bar{t}-\bar{\sigma}}dT_\sigma.$$

Here $\Phi(z)$ is an arbitrary analytical function.

The solution (1.3) cannot be used in practice because besides infinite series and recurrent relations it also has Cauchy's double singular integrals which are very difficult to solve. In many applications, the real as well as the imaginary part of the solution have certain physical meaning. This is why finding finitely integrable Vekua equations is of importance. Their solutions can be expressed in a closed and explicit shape as

$$w = w\left(z,\bar{z},Q\left(z\right),\overline{Q\left(z\right)}\right), \tag{1.4}$$

where $Q\left(z\right)$ is an arbitrary analytical function. For these solutions, it is possible to separate the real from the imaginary part.

One important but degenerate case when this separation is possible is for the equation

$$w'_{\bar{z}} = Mw + L. \tag{1.5}$$

This equation was solved by S. Fempl (see [5]). He gave its general solution in the form

$$w\left(z,\bar{z}\right) = e^{J(M)}\left[Q\left(z\right) + J\left(Le^{-J(M)}\right)\right], \tag{1.6}$$

where $Q(z)$ is an arbitrary analytical function. Here J is an integral operator, inverse to the differential operator $\dfrac{\partial}{\partial \bar{z}}$, which often becomes simple complex integration with respect to $\bar{z}$. Unfortunately in the equation (1.5), the part with $\bar{w}$ is missing, which is characteristic for the Vekua equation. Čanak (see [1]) has found classes of finitely integrable Vekua equations (1.2) wide enough as well as the corresponding homogeneous equations whose solution has the shape (1.4). D. Dimitrovski, M. Rajović and J. Mitevska (see [4]) have studied a system of two differential Vekua equations with two unknown complex functions and solved it with iterations.

Research for systems of Vekua equations is much more difficult and much more complicated than solving a single Vekua equation. In this work, a system of two differential Vekua equations with two unknown complex functions is taken into consideration. The conditions when it is finitely integrated are determined. The notion of operational integrability is introduced, and the problem is solved with the method of composition of the differential operators of Vekua and Fempl.

2 Composition of Vekua's and Fempl's Differential Operators

In the theory of differential equations, an important role is played by Vekua's and Fempl's differential operators

$$W_A(w) = w'_{\bar{z}} - A\bar{w}, \qquad F_B(w) = w'_{\bar{z}} - Bw, \tag{2.1}$$

where $A = A(z, \bar{z})$ and $B = B(z, \bar{z})$ are given differentiable functions with respect to $\bar{z}$ in some simply-connected domain T. These operators and their properties are studied in [9]. If we make a composition of these two operators and if we equate it with zero, we get

$$\begin{aligned} F_B\{W_A(w)\} &= F_B\{w'_{\bar{z}} - A\bar{w}\} = \{w'_{\bar{z}} - A\bar{w}\}'_{\bar{z}} - B\{w'_{\bar{z}} - A\bar{w}\} \\ &= w''_{\bar{z}\bar{z}} - A'_{\bar{z}}\bar{w} - A\bar{w'}_{\bar{z}} - Bw'_{\bar{z}} + AB\bar{w} = 0. \end{aligned} \tag{2.2}$$

If we want to solve the complex differential equation (2.2), we may use a substitution

$$\Omega = w'_{z} - A\bar{w}, \tag{2.3}$$

so we get

$$\Omega'_{\bar{z}} - B\Omega = 0. \tag{2.4}$$

The general solution of the differential equation (2.4) based on the formula (1.6) is

$$\Omega = \varphi(z)\, e^{JB}, \tag{2.5}$$

where $\varphi(z)$ is an arbitrary analytical function. Using (2.3), we get a new equation

$$w'_z = A\bar{w} + \varphi(z)\, e^{JB}. \tag{2.6}$$

Hence the question of finite integrability of the complex differential equation (2.2) is based on the finite integrability of the differential Vekua equation (2.6), see [1].

Remark 2.1. The complex differential equation of the form

$$w''_{\bar{z}\bar{z}} + \alpha(z,\bar{z})\, w'_{\bar{z}} + \beta(z,\bar{z})\, \bar{w}'_{\bar{z}} + \gamma(z,\bar{z})\, \bar{w} = 0 \tag{2.7}$$

is said that to be operationally integrable if its left side can be expressed as a composition of Vekua's and Fempl's operators (2.1). If we equate the coefficients in the equations (2.2) and (2.7), we arrive at the condition

$$\gamma = \alpha\beta + \beta_{\bar{z}} \tag{2.8}$$

which ensures operational integrability of the equation (2.7).

3 A System of Complex Differential Vekua Equations

We are considering the system of complex differential equations of Vekua type

$$\begin{aligned} w'_{\bar{z}} &= A\bar{w} + B\Omega, \\ \Omega'_{\bar{z}} &= C\bar{w} + D\Omega \end{aligned} \tag{3.1}$$

with two unknown functions $w = w(z,\bar{z})$ and $\Omega = \Omega(z,\bar{z})$, where the coefficients $A(z,\bar{z})$, $B(z,\bar{z})$, $C(z,\bar{z})$ and $D(z,\bar{z})$ are given differentiable functions with respect to the variable $\bar{z}$ in a simply-connected domain. From the first equation of (3.1), we find

$$\Omega = \frac{w'_{\bar{z}} - A\bar{w}}{B},$$

and replacing this together with its derivative in the second equation of (3.1), we get the complex differential equation of second order

$$w''_{\bar{z}\bar{z}} - \left(\frac{B'_{\bar{z}}}{B} + D\right) w'_{\bar{z}} - A\bar{w}'_{\bar{z}} + \left(DA - CB + \frac{B'_{\bar{z}}A}{B} - A'_{\bar{z}}\right) \bar{w} = 0. \tag{3.2}$$

To solve the differential equation (3.2) in general is much more difficult than to solve the Vekua equation (1.2). That is why it is of our interest to determine under what conditions the equation (3.2) is operationally integrable, i.e., when its solution is the same as the solution of the Vekua equation.

We will equate the coefficients of the equation (3.2) with the coefficients of the operationally integrable equation

$$w''_{\bar{z}\bar{z}} + \alpha w'_{\bar{z}} + \beta\bar{w}'_{\bar{z}} + (\alpha\beta + \beta'_{\bar{z}})\, \bar{w} = 0. \tag{3.3}$$

Eliminating the values α and β from the condition

$$-\frac{B'_{\bar{z}}}{B} - D = \alpha, \quad -A = \beta, \quad DA - CB + \frac{AB'_{\bar{z}}}{B} - A'_{\bar{z}} = \alpha\beta + \beta'_{\bar{z}},$$

we get the condition

$$CB = 0. \tag{3.4}$$

So, we obtain the following theorem.

Theorem 3.1. *A sufficient condition for the system of two complex differential Vekua equations* (3.1) *to be operationally integrable is the condition* (3.4).

Remark 3.2. According to Theorem 3.1, we can distinguish two cases.

First case: For $C = 0$, we have

$$\begin{aligned} w'_{\bar{z}} &= A\bar{w} + B\Omega, \\ \Omega'_{\bar{z}} &= D\Omega. \end{aligned} \tag{3.5}$$

The general solution of the second equation in (3.5) is $\Omega = \varphi(z)\, e^{J(D)}$. Replacing this value in the first equation, we get

$$w'_{\bar{z}} A\bar{w} + B\varphi(z)\, e^{J(D)}. \tag{3.6}$$

So the problem is equivalent to the problem of finite integrability of a single Vekua equation (see [1]).

Second case: For $B = 0$, we have

$$\begin{aligned} w'_{\bar{z}} &= A\bar{w}, \\ \Omega'_{\bar{z}} &= C\bar{w} + D\Omega. \end{aligned} \tag{3.7}$$

Now, the problem is equivalent to the problem of finite integrability of the first equation in (3.7). If it is finitely integrable, then its solution is replaced in the other equation in (3.7), and again it can be solved with the formula (1.6).

Example 3.3. Consider the system of Vekua equations

$$\begin{aligned} w'_{\bar{z}} &= -\frac{1}{z+\bar{z}}\bar{w} + \frac{3(z+\bar{z})}{\bar{z}}\Omega, \\ \Omega'_{\bar{z}} &= \frac{1}{\bar{z}}\Omega. \end{aligned} \tag{3.8}$$

According to the formula (3.2), this system can be reduced to a complex differential equation of second order

$$w''_{\bar{z}\bar{z}} - \frac{1}{z+\bar{z}}w'_{\bar{z}} + \frac{1}{z+\bar{z}}\bar{w}'_{\bar{z}} - \frac{2}{(z+\bar{z})^2}\bar{w} = 0. \tag{3.9}$$

The condition (2.8) is fulfilled, so this equation is operationally integrable and can be solved using a composition of Vekua's and Fempl's operators

$$\begin{aligned} F_B\left\{W_A\left(w\right)\right\} &= F_B\left\{w'_{\bar{z}} - A\bar{w}\right\} = \left\{w'_{\bar{z}} - A\bar{w}\right\}'_{\bar{z}} - B\left\{w'_{\bar{z}} - A\bar{w}\right\} \\ &= \left\{w'_{\bar{z}} + \frac{1}{z+\bar{z}}\bar{w}\right\}'_{\bar{z}} - \frac{1}{z+\bar{z}}\left\{w'_{\bar{z}} + \frac{1}{z+\bar{z}}\bar{w}\right\} = 0. \end{aligned} \tag{3.10}$$

With the substitution

$$w'_{\bar{z}} + \frac{1}{z+\bar{z}}\bar{w} = V, \tag{3.11}$$

where $V = V\left(z, \bar{z}\right)$ is a new unknown complex function, the equation is

$$V'_{\bar{z}} + \frac{1}{z+\bar{z}}V = 0. \tag{3.12}$$

The general solution of the equation (3.12) according to the formula (1.6) is $V = \left(z+\bar{z}\right)\theta\left(z\right)$, where $\theta\left(z\right)$ is an arbitrary analytical function. Replacing these values in (3.11), we get the complex differential Vekua equation

$$w'_{\bar{z}} + \frac{1}{z+\bar{z}}\bar{w} - \left(z+\bar{z}\right)\theta\left(z\right) = 0. \tag{3.13}$$

We will look for the general solution of the equation (3.13) in the form $w = w_h + w_p$, where w_h is the general solution of the corresponding homogeneous equation

$$w'_{\bar{z}} + \frac{1}{z+\bar{z}}\bar{w} = 0 \tag{3.14}$$

and w_p is a singular solution of the nonhomogeneous equation. The general solution of the homogeneous equation (see [1]) is

$$w_h = Q'\left(z\right) - \left(Q + \bar{Q}\right)\frac{1}{z+\bar{z}}. \tag{3.15}$$

Here the arbitrary analytical function $Q(z)$ is not quite arbitrary, but it belongs to a wide enough class of analytical functions which fulfill the condition $\overline{Q(z)} = Q\left(\bar{z}\right)$. This means that its Taylor development has only real coefficients which provides conjugating part by part. So, we have

$$\overline{Q\left(z\right)} = \overline{\sum_{k=0}^{\infty} c_k z^k} = \sum_{k=0}^{\infty} c_k \bar{z}^k = Q\left(\bar{z}\right)$$

and

$$\overline{Q'\left(z\right)} = \overline{\sum_{k=0}^{\infty} k c_k z^{k-1}} = \sum_{k=0}^{\infty} k c_k \bar{z}^{k-1} = \bar{Q}'_{\bar{z}}.$$

For determining a particular solution of the equation (3.13), M. Čanak in his earlier works was using the method of Mitrinović (see [6]). We will use the relation

$$\bar{w} = R(z,\bar{z})\,w + S(z,\bar{z}), \tag{3.16}$$

where the coefficients $R(z,\bar{z})$ and $S(z,\bar{z})$ are chosen according to the coefficients in the Vekua equation. Based on (3.16), the equation (3.13) becomes

$$w'_{\bar{z}} + \frac{1}{z+\bar{z}}R(z,\bar{z})\,w + \frac{1}{z+\bar{z}}S(z,\bar{z}) - (z+\bar{z})\,\theta(z) = 0. \tag{3.17}$$

The function $S(z,\bar{z})$ is chosen so that $S(z,\bar{z}) = (z+\bar{z})^2\,\theta(z)$. We get

$$w'_{\bar{z}} + \frac{R(z,\bar{z})}{z+\bar{z}}w = 0. \tag{3.18}$$

If the function $R(z,\bar{z})$ is chosen to be a real constant, then the general solution of the equation (3.18) is

$$w = P(z)\,(z+\bar{z})^{-R}, \tag{3.19}$$

where $P(z)$ for now is an arbitrary analytical function. After replacing the values R, S and (3.19) in the relation (3.16), we get

$$2P + \bar{P} = \theta(z) \tag{3.20}$$

what is known as a problem of Mitrinović for determining an analytical function $P(z)$. This problem has a solution if $\theta(z) = \theta$ is a real constant. In that case, its solution is $P = \dfrac{\theta}{3}$ and $w_p = \dfrac{\theta}{3}(z+\bar{z})^2$. At the end, the solutions of the initial system of equations (3.8) are given by

$$\begin{aligned} w_h &= Q'(z) - \frac{Q+\bar{Q}}{z+\bar{z}} + \frac{\theta}{3}(z+\bar{z})^2, \\ \Omega &= \frac{\theta}{3}\bar{z}. \end{aligned} \tag{3.21}$$

4 Conclusion

In the paper, a system of complex differential equations of first order of Vekua type is discussed. This system as well as the Vekua equation itself brings a big theoretical interest, possibilities for generalization and application. Here the finitely or operationally integrable differential equations and systems are especially important and significant. Let us see some of those possibilities.

1) In its monograph [7], G. Polozij thoroughly explored the system of partial differential equations

$$u'_x = \frac{1}{p}v'_y, \qquad u'_y = -\frac{1}{p}v'_x, \tag{4.1}$$

defining so-called p-analytical functions with given characteristic $p = p(x, y)$. He pointed out a few important applications of this functions in mechanics (in filtration theory, torsion theory of rotating bodies, axial-symmetric theory of shells etc.) when the characteristic $p(x, y)$ has the form $p = x^k$. In all of these problems, the real and the imaginary part of the solution have an exact physical meaning. The system (4.1) can be written as a complex differential equation of Vekua type. If this equation is finitely integrable, then it is possible to separate the real and imaginary part of the solution. This is important for solving various real-world boundary problems for this class of functions. In the paper [2], M. Čanak, L. Stefanovska and L. Protić have shown that the system (4.1) can be written as a complex differential equation of Vekua type which is often finitely integrable. For example, for $p = x^2$, this equation has the form

$$w'_{\bar{z}} = -\frac{1}{z + \bar{z}}\bar{w},$$

and its solution is

$$w(z, \bar{z}) = Q'(z) - \frac{Q + \bar{Q}}{z + \bar{z}},$$

where $Q(z)$ is an arbitrary analytical function.

2) Various generalizations of this paper are possible. In the paper [3], D. Dimitrovski, B. Ilievski and M. Rajović studied the generalized differential Vekua equation

$$aw'_{\bar{z}} + bw'_z + c\bar{w}'_{\bar{z}} + d\bar{w}'_z = Aw + B\bar{w} + F, \tag{4.2}$$

where a, b, c, d, A, B, F are given analytical coefficients of $z, \bar{z}$ in a domain D. It is possible to discuss a system of two equations like (4.2) with two unknown functions $w(z, \bar{z})$ and $\Omega(z, \bar{z})$ and to examine the cases when a system like this is finitely or operationally integrable.

3) If the Vekua equation or the system of Vekua equations is neither finitely nor operationally integrable, then some approximate methods can be used to solve them, see [8].

4) An interesting special case with important applications is when the complex differential equation (3.2) becomes a real linear differential equation of second order with functional coefficients. The complex Vekua and Fempl differential operators become two real differential operators of the same type

$$D_a y = y' - a(x)\,y, \qquad D_b y = y' - b(x)\,y,$$

where $a(x)$ and $b(x)$ are given differentiable functions. If we form the composition of these operators and equate it with zero, we get

$$\begin{aligned} D_b\{D_a y\} &= (y' - a(x)\,y)' - b(x)\,(y' - a(x)\,y) \\ &= y'' - (a + b)\,y' + (ab - a')\,y = 0. \end{aligned} \tag{4.3}$$

It is easy to show that the general solution of this differential equation has the form

$$y = e^{\int a(x)dx}\left[C_2 + C_1 \int e^{\int [b(x)-a(x)]dx}dx\right], \tag{4.4}$$

where C_1 and C_2 are two arbitrary, independent, real constants.

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