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About the Use of the Classical Lagrange Method for Solving a Class of Vekua Equations

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Abstract

In this paper, the classical Lagrange method for variation of constants from the theory of ordinary linear differential equations on the so-called canonical Vekua equation is used. This method leads us to a problem that is equivalent to the problem for solving the nonhomogeneous Vekua equation. The result is formulated in a theorem. Eight special, but general enough, cases are solved, when the mentioned method enables solving the corresponding nonhomogeneous Vekua equation.

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1 Introduction

G. V. Kolosov in 1909 [6], while working on a problem from the theory of elasticity, introduced the expressions

$$\frac{1}{2}\left[\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\right]=\frac{\hat{d}W}{dz} \tag{1.1}$$

and

$$\frac{1}{2}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}+i\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]=\frac{\hat{d}W}{d\bar{z}}, \tag{1.2}$$

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known as operational derivatives of a complex function $W = W(z) = u(x,y) + iv(x,y)$, $z = x+iy$, $\bar{z} = x-iy$. The operational rules for this derivatives are completely given in the monograph of G. N. Polozij [9, pp. 18–31]. In the mentioned monograph, so-called operational integrals are defined

$$\int^{\wedge} f(z)dz \qquad \text{and} \qquad \int^{\wedge} f(z)d\bar{z},$$

$z = x + iy$, $\bar{z} = x - iy$ (see [9, pp. 32–41]). The equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F, \tag{1.3}$$

where $A = A(z)$, $B = B(z)$ and $F = F(z)$ are given complex functions of a complex variable $z \in D \subseteq \mathbb{C}$, is the well-known Vekua equation [10]. Here the unknown function is $W = W(z) = u + iv$. In the case when $A \equiv 0$, $B \equiv 0$ and $F \equiv 0$ in $D \subseteq \mathbb{C}$, the equation (1.3) takes the expression

$$\frac{\hat{d}W}{d\bar{z}} = 0. \tag{1.3$'$}$$

This equation, in the class of the functions $W = u(x,y) + iv(x,y)$ whose real and imaginary parts have continuous partial derivatives u'_x, u'_y, v'_x and v'_y in D, is a complex variant of the Cauchy–Riemann conditions. In other words, (1.3′) defines an analytical function in the sense of the classical theory of analytical functions. In the case when $B \equiv 0$ in D, we get the equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + F, \tag{1.3$''$}$$

which is the so-called areolar linear differential equation [9, pp. 39–40]. It can be solved with quadratures by the formula

$$W = e^{\int^{\wedge} A(z)d\bar{z}} \left[\Phi(z) + \int^{\wedge} F(z)\, e^{-\int^{\wedge} A(z)d\bar{z}} d\bar{z} \right]. \tag{1.4}$$

Here $\Phi = \Phi(z)$ is an arbitrary analytical function in the role of an integral constant. In the case when in the Vekua equation (1.3) the unknown function $W = W(z)$ is under the sign of complex conjugation, which is equivalent to the fact that $B = B(z)$ is not identically equal to zero in D, then for (1.3), quadratures stop existing. Depending on the coefficients A, B and F, the equation (1.3) defines different classes of generalized analytical functions. For example, for $F = F(z) \equiv 0$ in D, the equation (1.3), i.e., $\dfrac{\hat{d}W}{d\bar{z}} = AW + B\overline{W}$, defines so-called generalized analytical functions of fourth class; and for $A \equiv 0$ and $F \equiv 0$ in D, the equation (1.3), i.e., the equation

$$\frac{\hat{d}W}{d\bar{z}} = B\overline{W},$$

defines so-called generalized analytical functions of third class or the $(r+is)$-analytical functions [1, 9].

2 Main Result

In the paper [4], with the method of areolar series $W = W(z) = \sum_{m,n=0}^{\infty} a_{mn} z^m \bar{z}^n$, a solution of the basic Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = \overline{W} \tag{2.1}$$

is found in the form

$$\begin{aligned} W = \Phi(z) &+ \int^{\wedge} \left[\sum_{k=1}^{\infty} \frac{\bar{z}^k (z-\varsigma)^{k-1}}{k!\,(k-1)!} \right] \Phi(\varsigma)\, d\varsigma \\ &+ \int^{\wedge} \left[\sum_{k=0}^{\infty} \frac{z^k (\bar{z}-\bar{\varsigma})^k}{(k!)^2} \right] \overline{\Phi}(\varsigma)\, d\bar{\varsigma}. \end{aligned} \tag{2.2}$$

Here, Φ is an analytical function of its variable and plays the role of an integral constant. If for the second sum in (2.2) we put

$$S = S(z,\varsigma) = \sum_{k=0}^{\infty} \frac{z^k (\bar{z}-\bar{\varsigma})^k}{(k!)^2}, \tag{2.3}$$

then

$$\frac{\hat{d}S}{d\bar{z}} = \sum_{k=1}^{\infty} \frac{k z^k (\bar{z}-\bar{\varsigma})^{k-1}}{(k!)^2},$$

from where we get

$$\overline{\left(\frac{\hat{d}S}{d\bar{z}}\right)} = \sum_{k=1}^{\infty} \frac{\bar{z}^k (z-\varsigma)^{k-1}}{k!\,(k-1)!}. \tag{2.4}$$

The last expression, (2.4), is nothing else but the first sum in (2.2). According to (2.3) and (2.4), the general solution (2.2) of the basic Vekua equation (2.1) assumes the form

$$W = \Phi(z) + \int^{\wedge} \overline{\left(\frac{\hat{d}S}{d\bar{z}}\right)} \Phi(\varsigma)\, d\varsigma + \int^{\wedge} S\overline{\Phi}(\varsigma)\, d\bar{\varsigma}. \tag{2.2$'$}$$

Now it is natural to state the following problem: Can the classical Lagrange method for variation of constants, from the theory of ordinary differential equations, solve the nonhomogeneous Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = \overline{W} + F(z), \tag{2.5}$$

knowing the solution (2.2′) of the basic Vekua equation (2.1)? In other words, if we suppose that Φ in (2.2′) is not an analytical function of its variable z, i.e., $\dfrac{\hat{d}\Phi}{d\bar{z}} \equiv 0$ in D (which is equivalent to the fact that in Φ besides z, there is $\bar{z} = x - iy$, too), but it is a function for which there exist the operational derivatives of that variable and its complex conjugated value, then can Φ be determined in the way that the function

$$W = \Phi(z,\bar{z}) + \int^{\wedge} \overline{\left(\frac{\hat{d}S}{d\bar{z}}\right)} \Phi(\varsigma,\bar{\varsigma})\, d\varsigma + \int^{\wedge} S\overline{\Phi}(\varsigma,\bar{\varsigma})\, d\bar{\varsigma} \tag{2.6}$$

is a solution of the nonhomogeneous equation (2.5)?

Remark 2.1. Here we discuss the use of the expression

$$\Phi = \Phi(z,\bar{z}) = u(x,y) + iv(x,y), \quad z = x + iy$$

for complex functions of a complex variable which are not analytical functions in the sense of the classical theory of complex functions instead of the expression $\Phi = \Phi(z) = u(x,y) + iv(x,y)$ (see [8, page 249]). Substituting the function (2.6) and its operational derivative by $\bar{z}$

$$\begin{aligned}
\frac{\hat{d}W}{d\bar{z}} &= \frac{\hat{d}\Phi(z,\bar{z})}{d\bar{z}} + \frac{\hat{d}}{d\bar{z}} \int^{\wedge} \overline{\left(\frac{\hat{d}S}{d\bar{z}}\right)} \Phi(\varsigma,\bar{\varsigma})\, d\varsigma + \frac{\hat{d}}{d\bar{z}} \int^{\wedge} S\overline{\Phi}(\varsigma,\bar{\varsigma})\, d\bar{\varsigma} \\
&= \frac{\hat{d}\Phi(z,\bar{z})}{d\bar{z}} + \int^{\wedge} \overline{\frac{\hat{d}}{d\bar{z}}\left(\frac{\hat{d}S}{d\bar{z}}\right)} \Phi(\varsigma,\bar{\varsigma})\, d\varsigma + \int^{\wedge} \frac{\hat{d}S}{d\bar{z}} \overline{\Phi}(\varsigma,\bar{\varsigma})\, d\bar{\varsigma}
\end{aligned}$$

in the nonhomogeneous Vekua equation (2.5), we get the identity

$$\begin{aligned}
&\frac{\hat{d}\Phi(z,\bar{z})}{d\bar{z}} + \int^{\wedge} \frac{\hat{d}}{d\bar{z}} \overline{\left(\frac{\hat{d}S}{d\bar{z}}\right)} \Phi(\varsigma,\bar{\varsigma})\, d\varsigma + \int^{\wedge} \frac{\hat{d}S}{d\bar{z}} \overline{\Phi}(\varsigma,\bar{\varsigma})\, d\bar{\varsigma} \\
&\equiv \overline{\Phi}(z,\bar{z}) + \overline{\int^{\wedge} \overline{\left(\frac{\hat{d}S}{d\bar{z}}\right)} \Phi(\varsigma,\bar{\varsigma})\, d\varsigma} + \overline{\int^{\wedge} S\overline{\Phi}(\varsigma,\bar{\varsigma})\, d\bar{\varsigma}} + F.
\end{aligned} \tag{2.7}$$

Making a conjugation of the integrals in the identity (2.7) according to the fact that

$$\frac{\hat{d}}{d\bar{z}} \frac{\hat{d}S}{d\bar{z}} = \bar{S},$$

we get

$$\begin{aligned}
&\frac{\hat{d}\Phi(z,\bar{z})}{d\bar{z}} + \int^{\wedge} \bar{S}\Phi(\varsigma,\bar{\varsigma})\, d\varsigma + \int^{\wedge} \frac{\hat{d}S}{d\bar{z}} \overline{\Phi}(\varsigma,\bar{\varsigma})\, d\bar{\varsigma} \\
&\equiv \overline{\Phi}(\varsigma,\bar{\varsigma}) + \int^{\wedge} \frac{\hat{d}S}{d\bar{z}} \bar{\Phi}(\varsigma,\bar{\varsigma})\, d\bar{\varsigma} + \int^{\wedge} \bar{S}\,\Phi(\varsigma,\bar{\varsigma})\, d\varsigma + F,
\end{aligned}$$

that is, after cancelling equal terms, we get

$$\frac{\hat{d}\Phi\left(z,\bar{z}\right)}{d\bar{z}} \equiv \overline{\Phi}\left(z,\bar{z}\right) + F. \tag{2.8}$$

So, the function $\Phi = \Phi\left(z,\bar{z}\right)$ in (2.6) is a solution of an equation which is the same as the initial nonhomogeneous Vekua equation (2.5), i.e., of the equation

$$\frac{\hat{d}\Phi}{d\bar{z}} \equiv \overline{\Phi} + F\left(z\right). \tag{2.9}$$

Hence we have proved the following theorem.

Theorem 2.2. *The classical Lagrange method for variation of constants in the general solution* (2.2′) *of the nonhomogeneous Vekua equation* (2.5)*, in which* $\Phi = \Phi\left(z\right)$ *is not an arbitrary analytical function of* z*, i.e.,* $\frac{\hat{d}\Phi}{d\bar{z}}$ *is not identically equal to zero in* D*, leads us to an equation* (2.9) *for determining the function* $\Phi = \Phi\left(z,\bar{z}\right)$ *in* (2.6)*, which is of the same type as the initial nonhomogeneous Vekua equation* (2.5)*.*

Remark 2.3. Theorem 2.2 states that the problem for solving the function $\Phi = \Phi\left(z,\bar{z}\right)$ so that (2.6) is a solution of a nonhomogeneous equation (2.5) is equivalent to the problem for solving the same nonhomogeneous equation (2.5). This does not mean that in some special cases, i.e., for some special forms of the free term $F = F\left(z\right)$, the equation (2.5) cannot be solved with the Lagrange method for variation of constants.

Example 2.4. We consider the following eight cases.

Case 1. Let Φ be an analytical function of z, i.e., $\frac{\hat{d}\Phi}{d\bar{z}} = 0$ in D. According to the condition (2.9) for determining Φ, we get

$$0 = \overline{\Phi}\left(z\right) + F\left(z\right),$$

i.e.,

$$F\left(z\right) = -\overline{\Phi}\left(z\right).$$

According to this, we have a class of functions $F\left(z\right) = -\overline{\Phi}\left(z\right)$, and for them, the nonhomogeneous equation (2.5), i.e., the equation

$$\frac{\hat{d}W}{d\bar{z}} = \overline{W} - \bar{\Phi}\left(z\right), \tag{2.5a}$$

according to (2.6), has the solution

$$W\left(z\right) = \Phi\left(z\right) + \int^{\wedge} \frac{\hat{d}S\left(z,\varsigma\right)}{d\bar{z}}\Phi\left(\varsigma\right)d\varsigma + \int^{\wedge} S\left(z,\varsigma\right)\overline{\Phi}\left(\varsigma\right)d\bar{\varsigma}. \tag{2.6a}$$

Here, $\Phi = \Phi\left(z\right)$ is a given analytical function of z.

Case 2. Let Φ be a product of an analytical function of z and its complex conjugated value, i.e.,

$$\Phi\left(z,\bar{z}\right)=\left|f\left(z\right)\right|^{2}=f\left(z\right)\cdot\bar{f}\left(z\right).$$

According to (2.9) we have

$$f\left(z\right)\frac{\hat{d}\bar{f}\left(z\right)}{d\bar{z}}=\bar{f}\left(z\right)\cdot f\left(z\right)+F\left(z\right),$$

from where we get

$$F\left(z\right)=f\left(z\right)\left(\frac{\hat{d}\bar{f}\left(z\right)}{d\bar{z}}-\bar{f}\left(z\right)\right).$$

According to this, the nonhomogeneous Vekua equation (2.5), i.e., the equation

$$\frac{\hat{d}W}{d\bar{z}}=\overline{W}+f\left(z\right)\left(\frac{\hat{d}\bar{f}\left(z\right)}{d\bar{z}}-\bar{f}(z)\right) \tag{2.5b}$$

has a solution

$$W\left(z\right)=f\left(z\right)\cdot\bar{f}\left(z\right)+\int^{\wedge}S\left(z,\varsigma\right)\bar{f}\left(\varsigma\right)\,f\left(\varsigma\right)d\bar{\varsigma}+\int^{\wedge}\frac{\hat{d}S}{d\bar{z}}f\left(\varsigma\right)\bar{f}\left(\varsigma\right)d\varsigma. \tag{2.6b}$$

Case 3. Let Φ be a product of an analytical and an antianalytical function, i.e.,

$$\Phi=f\left(z\right)\cdot\bar{g}\left(z\right),\quad\frac{\hat{d}f}{d\bar{z}}=\frac{\hat{d}g}{d\bar{z}}=0\text{ in }D.$$

The equation (2.9), in this case, becomes

$$f\left(z\right)\frac{\hat{d}\overline{g}\left(z\right)}{d\bar{z}}=\bar{f}\left(z\right)\cdot g\left(z\right)+F\left(z\right),$$

from where we get

$$F\left(z\right)=f\left(z\right)\overline{\left(\frac{\hat{d}g\left(z\right)}{d\bar{z}}\right)}-\bar{f}\left(z\right)\cdot g\left(z\right).$$

This means that the nonhomogeneous Vekua equation (2.5), i.e., the equation

$$\frac{\hat{d}W}{d\bar{z}}=\overline{W}+f\left(z\right)\overline{\left(\frac{\hat{d}g\left(z\right)}{d\bar{z}}\right)}-\bar{f}\left(z\right)\cdot g\left(z\right), \tag{2.5c}$$

has a solution

$$W\left(z\right)=f\left(z\right)\cdot\bar{g}\left(z\right)+\int^{\wedge}S\left(z,\varsigma\right)\bar{f}\left(\varsigma\right)\,g\left(\varsigma\right)d\bar{\varsigma}+\int^{\wedge}\frac{\hat{d}S\left(z,\varsigma\right)}{d\bar{z}}f\left(\varsigma\right)\bar{g}\left(\varsigma\right)d\varsigma. \tag{2.6c}$$

Here, $f=f\left(z\right)$ and $g=g\left(z\right)$ are given analytical functions in $D\subseteq\mathbb{C}$.

Case 4. The function Φ is a sum of an analytical and an antianalytical function of z, i.e.,

$$\Phi = f(z) + \bar{g}(z).$$

From the equation (2.9), we get

$$\begin{aligned} F(z) &= \frac{\hat{d}\Phi}{d\bar{z}} - \bar{\Phi} = \frac{\hat{d}f}{d\bar{z}} + \frac{\hat{d}\bar{g}(z)}{d\bar{z}} - \overline{(f(z) + \bar{g}(z))} \\ &= \overline{\left(\frac{\hat{d}g}{d\bar{z}}\right)} - \bar{f}(z) - g(z), \end{aligned}$$

which is why the nonhomogeneous Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = \bar{W} + \overline{\left(\frac{\hat{d}g}{d\bar{z}}\right)} - \bar{f}(z) - g(z) \tag{2.5d}$$

according to (2.6) has a solution

$$\begin{aligned} W(z) = f(z) + \bar{g}(z) &+ \int^{\wedge} S(z,\varsigma)\left(\bar{f}(\varsigma) - g(\varsigma)\right)d\bar{\varsigma} \\ &+ \int^{\wedge} \frac{\hat{d}S(z,\varsigma)}{d\bar{z}}\left(f(\varsigma) + \bar{g}(\varsigma)\right)d\varsigma. \end{aligned} \tag{2.6d}$$

Case 5. Φ is a bianalytical Goursat function, i.e., a function determined by the equation $\frac{\hat{d}^2\Phi}{d\bar{z}^2} = 0$. In other words, it is a function of the form

$$\Phi = f(z)\,\bar{z} + g(z),$$

where $f = f(z)$ and $g = g(z)$ are analytical functions of $z \in D \subseteq \mathbb{C}$. With a similar technique as in the previous two cases, from (2.9) we get

$$F(z) = \frac{\hat{d}\Phi}{d\bar{z}} - \bar{\Phi} = f(z) - \bar{f}(z)\,z - \bar{g}(z),$$

which is why the Vekua equation (2.5), i.e., the equation

$$\frac{\hat{d}W}{d\bar{z}} = \overline{W} + f(z) - \bar{f}(z)\,z - g(z) \tag{2.5e}$$

has a solution (according to (2.6)),

$$\begin{aligned} W = f(z)\,\bar{z} + g(z) &+ \int^{\wedge} S(z,\varsigma)\left(\varsigma\bar{f}(\varsigma) + \bar{g}(\varsigma)\right)d\bar{\varsigma} \\ &+ \int^{\wedge} \overline{\left(\frac{\hat{d}S(z,\varsigma)}{d\bar{z}}\right)}\left(\bar{\varsigma}f(\varsigma) + \bar{g}(\varsigma)\right)d\varsigma. \end{aligned} \tag{2.6e}$$

Case 6. The function Φ is an areolar polynomial whose biggest degree is n, i.e., it is a solution of the areolar equation $\frac{\hat{d}^{n+1}\Phi}{d\bar{z}^{n+1}} = 0$. In other words, Φ is a function of the form

$$\Phi = \sum_{k=0}^{\infty} a_k(z)\,\bar{z}^k, \qquad \frac{\hat{d}a_k(z)}{d\bar{z}} = 0, \quad \forall k = \overline{1,n}.$$

According to the relation (2.9) for the functions $F$and Φ, we get

$$F(z) = \frac{\hat{d}\Phi}{d\bar{z}} - \bar{\Phi} = \sum_{k=1}^{\infty} k a_k(z)\,\bar{z}^{k-1} - \sum_{k=0}^{\infty} \bar{a}_k(z)\,\bar{z}^k.$$

This means that the nonhomogeneous Vekua equation (2.5), i.e., the equation

$$\frac{\hat{d}W}{d\bar{z}} = \overline{W} + \sum_{k=1}^{\infty} k a_k(z)\,\bar{z}^{k-1} - \sum_{k=0}^{\infty} \bar{a}_k(z)\,\bar{z}^k, \tag{2.5f}$$

has a solution

$$W = \sum_{k=0}^{\infty} a_k(z)\,\bar{z}^k + \int^{\wedge} S(z,\varsigma) \sum_{k=0}^{\infty} \bar{a}_k(\varsigma)\,\varsigma^k d\bar{\varsigma} + \int^{\wedge} \frac{\hat{d}S(z,\varsigma)}{d\bar{z}} \sum_{k=0}^{\infty} a_k(\varsigma)\,\bar{\varsigma}^k d\varsigma. \tag{2.6f}$$

Case 7. $\Phi = f\left(|z|^2\right)$, where $\Phi = \Phi(u)$ is a differentiable function of its variable $u = |z|^2$. From the equation (2.9), we get

$$F = \frac{\hat{d}\Phi}{d\bar{z}} - \bar{\Phi} = f'(u)\,\frac{\hat{d}u}{d\bar{z}} - \bar{f}(u) = f'\left(|z|^2\right) z - \bar{f}\left(|z|^2\right).$$

According to this, the nonhomogeneous Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = \overline{W} + f'\left(|z|^2\right) z - \bar{f}\left(|z|^2\right) \tag{2.5g}$$

has a solution

$$W = f\left(|z|^2\right) + \int^{\wedge} S(z,\varsigma)\,\bar{f}\left(|\varsigma|^2\right) d\bar{\varsigma} + \int^{\wedge} \frac{\hat{d}S(z,\varsigma)}{d\bar{z}} f\left(|\varsigma|^2\right) d\varsigma. \tag{2.6g}$$

Case 8. If $\Phi = \Phi(z,\bar{z})$ is an arbitrary complex function of z and the function $F = F(z)$ is determined by the condition (2.9), i.e.,

$$F = \frac{\hat{d}\Phi}{d\bar{z}} - \bar{\Phi},$$

then the nonhomogeneous Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = \bar{W} + \frac{\hat{d}\Phi}{d\bar{z}} - \bar{\Phi} \tag{2.5h}$$

has a solution

$$W = \Phi(z,\bar{z}) + \int^{\wedge} S(z,\varsigma)\,\bar{\Phi}(\varsigma,\bar{\varsigma})\,d\bar{\varsigma} + \int^{\wedge} \frac{\hat{d}S(z,\varsigma)}{d\bar{z}}\,\Phi(\varsigma,\bar{\varsigma})\,d\varsigma. \tag{2.6h}$$

3 Conclusion

The Lagrange method for variation of constants applies to linear differential equations of higher order. Analogue results as in the theory of the ordinary linear differential equations of higher order in the class of areolar linear differential equations are mentioned in the papers [2, 3, 5, 7]. The Vekua equation

$$\frac{\hat{d}W}{d\bar{z}} = AW + B\overline{W} + F,$$

which defines different classes of generalized analytical functions depending on the coefficients $A = A(z)$, $B = B(z)$ and $F = F(z)$, is not a linear equation, because the unknown function is under the sign of complex conjugation. For the canonical Vekua equation $\frac{\hat{d}W}{d\bar{z}} = \overline{W} + F(z)$, we observed that is not linear; so, we came to the idea to explore if it can be solved when we know the general solution of the basic Vekua equation. When working on this problem, we used the Lagrange method for variation of constants. It lead us to a problem that is equivalent to the problem for solving the canonical Vekua equation itself. In the paper, eight special cases are solved. They are special, but general enough. These results constitute a small contribution to the theory of generalized analytical functions. Of course, these results can be of some importance in some of the applied sciences. Here, we should mention that these areolar differential equations originated in the needs of the theory of elasticity [6].

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