Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 6, Number 1, pp. 121–128 (2011)
http://campus.mst.edu/adsa

Carleman Boundary Value Problems for Polyanalytic Functions

Liljana Stefanovska
Faculty of Technology and Metallurgy
Skopje, Macedonia
liljana@tmf.ukim.edu.mk

Miloš Čanak
University of Novi Pazar, Serbia
miloscanak12@yahoo.com

Abstract

In this paper, the authors solve a boundary value problem of Carleman type for $\mathcal{F}$-polyanalytic functions and prove a theorem on existence of a unique solution.

AMS Subject Classifications: 30G20, 34M99, 35J55.
Keywords: Carleman boundary value problem, elliptic system of PDE, polyanalytic function, polyanalytic differential equation.

1 Introduction

In [7], the following Carleman boundary value problem for analytic functions is given:
Problem C_0**.** Find a function $\Phi(z)$ in the domain D^+ bounded by a simple and closed contour Γ that satisfies the boundary condition

$$\Phi[\alpha(t)] = G(t)\Phi(t) + g(t), \tag{1.1}$$

where $\alpha(t)$ satisfies the Carleman condition

$$\alpha[\alpha(t)] = t \tag{1.2}$$

and $G(t), g(t), \alpha'(t)$ satisfy the Hölder condition on Γ and $G(t) \neq 0$, $\alpha'(t) \neq 0$.

The complete solution to Problem C_0 for a bounded and simple connected domain was given by D. A. Kveselava [6] by using the method of integral equations. In [7], the following theorem is proved.

Received September 30, 2010; Accepted January 20, 2011
Communicated by Mališa Žižović

Theorem 1.1. *Let* $\chi = \dfrac{1}{2\pi}\{\arg G(t)\}_\Gamma$ *be an index of the boundary problem* (1.1)*, and* m^- *be a number of the fixed points of the movement* $\alpha(t)$*. If* $\chi + m^- \leq 0$*, then the number* l *of solutions of the homogeneous Carleman boundary value problem is*

$$l = 1 - \frac{\chi + m^-}{2},$$

and the corresponding nonhomogeneous problem is unconditionally solvable. If $\chi + m^- > 0$*, then the homogeneous problem has no nontrivial solutions* $(l = 0)$ *and*

$$p = \frac{\chi + m^-}{2} - 1$$

is the number of conditions that are necessary and sufficient for the solvability of the nonhomogeneous problem.

A corollary of this theorem is the case when $\chi+m^- = 2$, when the nonhomogeneous problem (1.1) has a unique solution $(l = p = 0)$.

Remark 1.2. In [7], an important special case of the Problem C_0 is given: the homogeneous boundary value problem

$$\Phi[\alpha(t)] = \lambda \cdot \Phi(t), \quad \lambda = \pm 1,$$

for $\lambda = 1$ in D^+ has only a constant solution, but for $\lambda = -1$, the constant is 0.

Let us consider the elliptical system of partial equations

$$\begin{aligned} u'_x - v'_y &= au - bv + c, \\ u'_y + v'_x &= bu + av + d, \end{aligned} \tag{1.3}$$

where $a = a(x,y)$, $b = b(x,y)$, $c = c(x,y)$, $d = d(x,y)$ are continuous functions in some closed domain D and $w(z,\bar z) = u(x,y) + iv(x,y)$ is a determining function. This system plays an important role in different problems in mechanics and it leads to complex differential equation

$$w'_{\bar z} + A(z,\bar z)w + B(z,\bar z) = 0, \tag{1.4}$$

where $A(z,\bar z) = -\dfrac{a+ib}{2}$, $B(z,\bar z) = \dfrac{c+id}{2}$. It is shown by S. Fempl [3] that the general solution of (1.4) is

$$w(z,\bar z) = \left[Q(z) - \int B(z,\bar z)e^{\int A(z,\bar z)d\bar z}d\bar z\right] e^{-\int A(z,\bar z)d\bar z}, \tag{1.5}$$

where $Q(z)$ is an arbitrary analytic function.

The regular solutions of the elliptic system of partial equations (1.3) define a class of so-called $\mathcal{F}$-analytic functions that is an important connection between the set of analytic functions and the set of Vekua generalized analytic functions.

On the basis of (1.4), a differential operator of Fempl type can be introduced by

$$\mathcal{F}_{A,B}w = w'_{\bar{z}} + A(z,\bar{z})w + B(z,\bar{z}), \tag{1.6}$$

that for given continuous characteristics $A(z,\bar{z})$ and $B(z,\bar{z})$ to every differentiable complex function w corresponds to different continuous complex functions $\Phi(z,\bar{z})$.

Now an auxiliary Carleman boundary value problem for $\mathcal{F}$-analytic functions can be formulated and solved, because its solution is necessary for solving the Carleman boundary value problem for $\mathcal{F}$-polyanalytic functions.

Boundary value problem F_0. Let the domain D^+ be bounded by a simple and smooth closed curve L and let $G(t), g(t), \alpha(t)$ be functions from the contour $(t \in L)$. Suppose that the functions $G(t)$ and $g(t)$ satisfy on L the Hölder condition, the function $\alpha(t)$ satisfies the Carleman condition (1.2) and $\alpha'(t) \neq 0$. Let $A(z,\bar{z})$ and $B(z,\bar{z})$ be given continuous functions on $D^+ \cup L$. The problem is to find a solution of the differential equation (1.4) that on L satisfies the Carleman boundary value condition

$$w[\alpha(t)] = G(t)w(t) + g(t). \tag{1.7}$$

We now find this solution. If we introduce the notations

$$e^{-\int A(z,\bar{z})d\bar{z}} = \mathcal{A}_1 + i\mathcal{A}_2 = \mathcal{A},$$
$$Q(z) = Q_1 + iQ_2,$$
$$-\int B(z,\bar{z})e^{\int A(z,\bar{z})d\bar{z}}d\bar{z} = \mathcal{B}_1 + i\mathcal{B}_2 = \mathcal{B},$$

then according to (1.5), the solution of the equation (1.4) is

$$w = \mathcal{A}(Q + \mathcal{B}). \tag{1.8}$$

By the substitution of (1.8) in (1.7), we obtain

$$\mathcal{A}[\alpha(t)]\,\{Q[\,\alpha(t)] + \mathcal{B}[\alpha(t)]\} = G(t)\mathcal{A}(t)[Q(t) + \mathcal{B}(t)] + g(t),$$

and after some arranging

$$Q[\alpha(t)] = \frac{G(t)\mathcal{A}(t)}{\mathcal{A}[\alpha(t)]}Q(t) + \frac{G(t)\mathcal{A}(t)\mathcal{B}(t) + g(t) - \mathcal{A}[\alpha(t)]\mathcal{B}[\alpha(t)]}{\mathcal{A}[\alpha(t)]}. \tag{1.9}$$

The relation (1.9) is the Carleman boundary value problem for determining an analytic function $Q(z)$. When such a function is determined through the procedure from [7], by its substitution in (1.8), the solution of the boundary problem (1.7) is obtained. Let the index of boundary value problem (1.9) be denoted as

$$\chi = \frac{1}{2\pi}\left\{\arg\frac{G(t)\mathcal{A}(t)}{\mathcal{A}[\alpha(t)]}\right\}_L.$$

Because of Theorem 1.1, in the case when $\chi + m^- = 2$, the boundary value problem F_0 has a unique solution.

2 On $\mathcal{F}$-Polyanalytic Functions

The polyanalytic function of the n-th order is defined by Teodorescu [8] as a solution of the polyanalytic differential equation

$$\frac{\partial^n w}{\partial \bar{z}^n} = 0. \tag{2.1}$$

It was shown by Teodorescu that the general solution of (2.1) is

$$w(z, \bar{z}) = \sum_{k=0}^{n-1} \bar{z}^k \varphi_k(z), \tag{2.2}$$

where $\varphi_k(z)$ are arbitrary analytic functions in some closed domain D. It is important to mention that Balk [1] developed a theory of polyanalytic functions by defining n generalized polyanalytic functions as solutions of the differential equation

$$\mathcal{D}_A^n f = 0 \quad (\mathcal{D}_A f = f'_{\bar{z}} + Af), \tag{2.3}$$

where $A(z, \bar{z})$ is a given n-times differentiable function in some domain D. Gabrinovich and Sokolov [5] as well gave a survey on different generalizations of polyanalytic functions and these boundary value problems, and Čanak [2] defined p-polyanalytic functions and solved the corresponding boundary value problem of the Riemann type.

In this article, a new generalization of ordinary polyanalytic functions is defined via $\mathcal{F}$-polyanalytic functions. Let us consider the differential equation

$$\mathcal{F}_{A,B}^{(n)} w = 0 \quad (\mathcal{F}_{A,B} w = w'_{\bar{z}} + A(z, \bar{z})w + B(z, \bar{z})), \tag{2.4}$$

where $A(z, \bar{z}), B(z, \bar{z})$ are given n-times differentiable functions in some closed domain D. For $n = 1$, the equation (2.4) becomes

$$w'_{\bar{z}} + A(z, \bar{z})w + B(z, \bar{z}) = 0. \tag{2.5}$$

Because of (1.5), its general solution is

$$w(z, \bar{z}) = Q(z)e^{-\int A(z,\bar{z})d\bar{z}} + S_1(z, \bar{z}), \tag{2.6}$$

where

$$S_1(z, \bar{z}) = -e^{-\int A(z,\bar{z})d\bar{z}} \int B(z, \bar{z})e^{\int A(z,\bar{z})d\bar{z}} d\bar{z} \tag{2.7}$$

and $Q(z)$ is an arbitrary analytic function. For $n = 2$, the equation (2.4) becomes $\mathcal{F}_{A,B}^{(1.2)} w = 0$ and

$$\mathcal{F}_{A,B} w = Q(z)e^{-\int A(z,\bar{z})d\bar{z}} + S_1(z, \bar{z}),$$

hence

$$w'_{\bar{z}} + A(z, \bar{z})w + \left[B(z, \bar{z}) - Q(z)e^{-\int A(z,\bar{z})d\bar{z}} - S_1(z, \bar{z})\right] = 0. \tag{2.8}$$

The general solution of (2.8) is

$$w(z,\bar{z}) = e^{-\int A(z,\bar{z})d\bar{z}}\left[Q_1(z) + \bar{z}Q_2(z)\right] + S_2(z,\bar{z}), \tag{2.9}$$

where

$$S_2(z,\bar{z}) = -e^{-\int A(z,\bar{z})d\bar{z}} \int \left[B(z,\bar{z}) - S_1(z,\bar{z})\right] e^{\int A(z,\bar{z})d\bar{z}} d\bar{z}. \tag{2.10}$$

By using mathematical induction, the general solution of a polyanalytic differential equation of the Fempl type is

$$w(z,\bar{z}) = e^{-\int A(z,\bar{z})d\bar{z}}\left[Q_1(z) + \bar{z}Q_2(z) + \ldots + \bar{z}^{n-1}Q_n(z)\right] + S_n(z,\bar{z}), \tag{2.11}$$

where $Q_1(z), Q_2(z), \ldots, Q_n(z)$ are arbitrary polyanalytic functions and the function $S_{n-1}(z,\bar{z})$ satisfies the recurrence formula

$$S_n(z,\bar{z}) = -e^{-\int A(z,\bar{z})d\bar{z}} \int \left[B(z,\bar{z}) - S_{n-1}(z,\bar{z})\right] e^{\int A(z,\bar{z})d\bar{z}} d\bar{z}. \tag{2.12}$$

The regular solutions of the differential equation (2.4) will be called $\mathcal{F}$-polyanalytic functions of the n-th order.

Remark 2.1. In the case when $B(z,\bar{z}) = 0$, we have $S_1 = S_2 = \cdots = S_n = 0$, and the equation (2.4) is the Balk polyanalytic differential equation (2.3). When $A(z,\bar{z}) = A(z)$ and $B(z,\bar{z}) = B(z)$, we obtain

$$\begin{aligned} S_n &= -\frac{B}{A} + \frac{S_{n-1}}{A} = -\frac{B}{A} - \frac{B}{A^2} + \frac{S_{n-2}}{A^2} \\ &= -\frac{B}{A} - \frac{B}{A^2} - \cdots - \frac{B}{A^n} = -\frac{B(z)\left[A^n(z) - 1\right]}{A^n(z)\left[A(z) - 1\right]}, \end{aligned} \tag{2.13}$$

and the solution (2.11) is called an areolar exponential equation of the Fempl type [4]. Finally, when $A(z,\bar{z}) = B(z,\bar{z}) = 0$, the ordinary polyanalytic functions are obtained.

3 Carleman Boundary Value Problem

Let L be a simple, smooth and closed contour that bounds the finite domain D^+ and let $G_k(t), g_k(t)$ ($k = 0,1,\ldots,n-1$) and $\alpha(t)$ be given functions on the contour L. Let us suppose that the functions $G_k(t), g_k(t)$ satisfy the Hölder condition, the function $\alpha(t)$ satisfies the Carleman condition (1.2) and $\alpha'(t) \neq 0$. Let $A(z,\bar{z})$ and $B(z,\bar{z})$ be given continuous functions on $D^+ \cup L$. The problem is to find a regular solution of the differential equation (2.4) that on L satisfies n boundary conditions of the Carleman type

$$\mathcal{F}^{(k)}_{A,B}w[\alpha(t)] = G_k(t)\mathcal{F}^{(k)}_{A,B}w(t) + g_k(t); \quad k = 0,1,\ldots,n-1. \tag{3.1}$$

The general solution of the equation (2.4) is given via (2.11), from where we obtain

$$
\begin{aligned}
\mathcal{F}_{A,B}w &= e^{-\int A(z,\bar{z})d\bar{z}}\big[Q_2(z) + 2\bar{z}Q_3(z) + \cdots + \\
&\qquad + (n-1)\bar{z}^{n-2}Q_n(z)\big] + S_{n-1}(z,\bar{z}) \\
\mathcal{F}^{(1.2)}_{A,B}w &= e^{-\int A(z,\bar{z})d\bar{z}}\big[2Q_3(z) + 3\cdot 2\bar{z}Q_4(z) + \cdots + \\
&\qquad + (n-1)(n-2)\bar{z}^{n-3}Q_n(z)\big] + S_{n-2}(z,\bar{z}) \\
&\vdots \\
\mathcal{F}^{(n-2)}_{A,B}w &= e^{-\int A(z,\bar{z})d\bar{z}}\,[(n-2)!Q_{n-1}(z) + (n-1)!\bar{z}Q_n(z)] + S_2(z,\bar{z}) \\
\mathcal{F}^{(n-1)}_{A,B}w &= e^{-\int A(z,\bar{z})d\bar{z}}\,[(n-1)!Q_n(z)] + S_1(z,\bar{z}).
\end{aligned}
\tag{3.2}
$$

By substitution of (3.2) in (2.11) and (3.1), we obtain

$$
\begin{aligned}
&\mathcal{A}[\alpha(t)]\{Q_1[\alpha(t)] + \overline{\alpha(t)}\cdot Q_2[\alpha(t)] + \cdots + \overline{\alpha(t)}^{n-1}Q_n[\alpha(t)]\} + S_n[\alpha(t)] \\
&\quad = G_0(t)\big\{\mathcal{A}(t)[Q_1(t) + \bar{t}Q_2(t) + \cdots + \bar{t}^{n-1}Q_n(t)] + S_n(t)\big\} + g_0(t) \\
&\vdots \\
&\mathcal{A}[\alpha(t)]\{(n-2)!Q_{n-1}[\alpha(t)] + (n-1)!\overline{\alpha(t)}\,Q_n[\alpha(t)]\} + S_2[\alpha(t)] \\
&\quad = G_{n-2}(t)\big\{\mathcal{A}(t)[(n-2)!Q_{n-1}(t) + (n-1)!\bar{t}Q_n(t)] + S_2(t)\big\} + g_{n-2}(t) \\
&\mathcal{A}[\alpha(t)](n-1)!Q_n[\alpha(t)] + S_1[\alpha(t)] \\
&\quad = G_{n-1}(t)[\mathcal{A}(t)(n-1)!Q_n(t) + S_1(t)] + g_{n-1}(t),
\end{aligned}
\tag{3.3}
$$

where $\mathcal{A}(z,\bar{z}) = e^{-\int A(z,\bar{z})d\bar{z}}$. The last condition (3.3) is the Carleman boundary value problem for determining the analytic function $Q_n(z)$. It can be written as

$$
Q_n[\alpha(t)] = \frac{G_{n-1}(t)\mathcal{A}(t)}{\mathcal{A}[\alpha(t)]}Q_n(t) + \frac{G_{n-1}(t)S_1(t) + g_{n-1}(t) - S_1[\alpha(t)]}{(n-1)!\mathcal{A}[\alpha(t)]}
\tag{3.4}
$$

that corresponds to the boundary condition (1.9). If the sum of indexes of the boundary value problem

$$
\chi_{n-1} = \frac{1}{2\pi}\left\{\arg\frac{G_{n-1}(t)\mathcal{A}(t)}{\mathcal{A}[\alpha(t)]}\right\}_L
$$

and the number of fixed points m^- of the Carleman movement is $\chi_{n-1} + m^- = 2$, then this problem has a unique solution $Q_n(z)$ that can be determined by the technique developed by Litvinchuk. By the substitution of this functions $Q_n(z)$ in (3.3), the Carleman boundary value problem is obtained as well for determining the function $Q_{n-1}(z)$. This problem has a unique solution if its coefficients satisfy the Hölder condition, the free coefficient is not annulated and $\chi_{n-1} + m^- = 2$ holds. By repeating this procedure n times, all the unknown functions $Q_n(z), Q_{n-1}(z), \ldots, Q_1(z)$ are determined, and by these substitutions in (2.11), the solution of the boundary problem is determined. Hence the following result holds.

Theorem 3.1 (Existence and uniqueness of solutions of the Carleman boundary value problem). *If the indexes*

$$\chi_i = \frac{1}{2\pi}\left\{\arg\frac{G_i(t)\mathcal{A}(t)}{\mathcal{A}[\alpha(t)]}\right\}_L; \quad i = 0, 1, 2, \ldots, n-1$$

of the boundary value problem (2.4), (3.1) *satisfy the condition*

$$\chi_i + m^- = 2$$

and if all the free coefficients of the successive boundary value problems are not zero, then the boundary value problem (2.4), (3.1) *has a unique solution.*

Example 3.2. Find an $\mathcal{F}$-bianalytic function with characteristics $A = 1$, $B = 1$ that on the unit circle $\bar{t} = \dfrac{1}{t}$ satisfies the Carleman boundary value conditions

$$\begin{aligned} w_{1,1}(1/t) &= e^{\frac{1}{t}-t}w_{1,1}(t) + 2e^{\frac{1}{t}-t} - 2, \\ \mathcal{F}_{1,1}w(1/t) &= -e^{\frac{1}{t}-t}\mathcal{F}_{1,1}w(t) + \frac{t + 1/t - e^t - e^{\frac{1}{t}}}{e^t}. \end{aligned} \tag{3.5}$$

Solution. By (2.11), we have

$$\begin{aligned} w_{1,1}(z,\bar{z}) &= e^{-\bar{z}}[Q_1(z) + \bar{z}Q_2(z)] - 2, \\ \mathcal{F}_{1,1}w &= e^{-\bar{z}}Q_2(z) - 1 \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} w_{1,1}(1/t) &= e^{-(1/\bar{t})}[Q_1(1/t) + \frac{1}{\bar{t}}Q_2(1/t)] - 2, \\ w_{1,1}(t) &= e^{-\bar{t}}[Q_1(t) + \bar{t}Q_2(t)] - 2, \\ \mathcal{F}_{1,1}w(1/t) &= e^{-(1/\bar{t})}Q_2(1/t) - 1, \\ \mathcal{F}_{1,1}w(t) &= e^{-\bar{t}}Q_2(t) - 1. \end{aligned} \tag{3.7}$$

By substitution of (3.7) in (3.5) and observing that $\bar{t} = \dfrac{1}{t}$, we obtain

$$\begin{aligned} Q_1(1/t) + tQ_2(1/t) &= Q_1(t) + \frac{1}{t}Q_2(t), \\ Q_2(1/t) &= -Q_2(t) + t + \frac{1}{t}. \end{aligned} \tag{3.8}$$

The second relation from (3.8) is the Carleman boundary value problem for determining the analytic function $Q_2(z)$. As the index of the boundary value problem $\chi = 0$ and the number of fixed points $m^- = 2$ for the Carleman movement $\alpha(t) = \dfrac{1}{t}$, we have $\chi + m^- = 2$, and the problem has a unique solution. By using Litvinchuk's technique and after lengthy calculations, we obtain that the unique solution is $Q_2(z) = z$. If this is substituted in the first boundary condition in (3.8), then

$$Q_1(1/t) = Q_1(t) \tag{3.9}$$

is obtained. The relation (3.9) is the Carleman homogeneous boundary value problem that has only a constant solution $Q_1(z) = c$. So, the general solution of the problem (3.5) is

$$w(z, \bar{z}) = e^{-\bar{z}}[c + z\bar{z}] - 2. \tag{3.10}$$

This solves the problem $\square$

References

[1] M. B. Balk. Polyanalytic functions. In *Complex analysis*, volume 61 of *Math. Lehrbücher Monogr. II. Abt. Math. Monogr.*, pages 68–84. Akademie-Verlag, Berlin, 1983.

[2] Miloš Čanak. Randwertaufgabe von Riemann-Typus für die p-polyanalytischen Funktionen auf der spiralformigen Kontur. *Mat. Vesnik*, 40(3-4):197–203, 1988. Third International Symposium on Complex Analysis and Applications (Herceg-novi, 1988).

[3] Stanimir Fempl. Reguläre Lösungen eines Systems partieller Differentialgleichungen. *Publ. Inst. Math. (Beograd) (N.S.)*, 4 (18):115–120, 1964.

[4] Stanimir Fempl. Areoläre Exponentialfunktion als Lösung einer Klasse Differentialgleichungen. *Publ. Inst. Math. (Beograd) (N.S.)*, 8 (22):138–142, 1968.

[5] V. A. Gabrinovich and I. A. Sokolov. Studies in boundary value problems for polyanalytic functions. In *Proceedings of a commemorative seminar on boundary value problems (Russian) (Minsk, 1981)*, pages 43–47, Minsk, 1985. "Universitet·skoe".

[6] D. A. Kveselava. Rešenie odnoi graničnoi zadači T. Karlemana. *Dokl. Akad. Nauk SSSR*, 55(8):683–686, 1947.

[7] G. S. Litvinchuk. *Kraevye zadachi i singulyarnye integralnye uravneniya so sdvigom*. Izdat. "Nauka", Moscow, 1977.

[8] N. Teodorescu. *La dérivée aréolaire et ses applications à la physique mathématique*. PhD thesis, Sorbonne, Paris.