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The Theory of Convergence and the Set of Statistical Cluster Points

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Abstract

In this paper we develop a method suggested by Pehlivan and Mamedov [29]. We study some problems concerning the set of λ-statistical cluster points (briefly, λ-s.c. points) in finite-dimensional Banach spaces. We also give some results related to the set of λ-statistical cluster points Γ_x^λ.

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1 Introduction

The theory of statistical convergence is an active area of research. Over the years, and under different names, statistical convergence has been discussed in number theory [25], trigonometric series [36], and summability theory [15, 17]. Statistical convergence was introduced by H. Fast [13] and H. Steinhaus [33], as a generalization of ordinary convergence for real sequences. Since then it has been studied by many authors [2, 4, 5, 7, 10, 15, 26, 28, 30, 32]. Important applications of the theory can be found in turnpike theory [21, 22, 29]. Statistical convergence has been discussed in more general abstract spaces as finite-dimensional spaces [27], locally convex spaces [20], Banach spaces [19] and probabilistic metric spaces [34]. Recently, the concept of statistical convergence is

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investigated in topological and uniform spaces.The development and detailed history of this subject can be found in [9].

The concept of statistical convergence is based on the idea of asymptotic density of sets $B \subseteq \mathbb{N} = \{1, 2, \ldots, n, \ldots\}$ [3, 11, 12]. In [14] an axiomatic approach is given for introducing the notion of density of sets $B \subseteq \mathbb{N}$. The results make it possible to extend the concept of statistical convergence. Applications of such a regular method of summability of sequences can be found in [15, 17, 18, 23, 31]. In [16], Fridy introduced the definitions of statistical limit point and statistical cluster point and using classical techniques, established some basic results. The concepts of statistical limit point and statistical cluster point are extended to the T-statistical limit point (briefly, T-s.l.p) and T-statistical cluster point (briefly, T-s.c.p.), where T is an arbitrary nonnegative regular matrix summability method [6, 8]. The idea of λ-statistical convergence was introduced in [24] for a number sequence $x = (x_n)$. In [1], Armitage and Maddox introduced the summability method C_λ defined by deleting a set of rows from the Cesáro matrix. They gave some inclusion theorems for C_λ methods [35].

Our work can be outlined as follows: In the second section, we recall some of the basic notions related to statistical convergence, s.c.p., regular summability matrix, T-statistical convergence, T-s.c.p., λ-statistical limit point (briefly, λ-s.l.p.) and λ-statistical cluster point (briefly, λ-s.c.p.). The main results are given in the third section. We prove some properties of the set of λ-statistical cluster points in finite-dimensional Banach spaces. We also study some results related with the set of λ-statistical cluster points Γ_x^λ in finite-dimensional Banach spaces.

2 Definitions and Properties

First, we introduce some basic notions related to statistically convergent, T-statistically convergent and λ-statistically convergent sequences.

Let K be a subset of the set of natural numbers $\mathbb{N}$. We denote by $K(n)$ the number of elements of the set K which are less or equal to $n \in \mathbb{N}$. The natural (asymptotic) density of K is defined by $d(K) = \lim_n \dfrac{K(n)}{n}$ whenever the limit exists. We recall also that $d(\mathbb{N} \setminus K) = 1 - d(K)$. A sequence $x = (x_k)$ is statistically convergent to ξ if for every $\epsilon > 0$,

$$d(\{k \in \mathbb{N} : |x_k - \xi| \geq \epsilon\}) = 0.$$

In this case, we write $x_k \longrightarrow \xi(S)$ and S denotes the set of all statistically convergent sequences. A number $\xi \in \mathbb{R}$ is called a statistical limit point of a sequence $x = (x_k)$ if there is a set $\{k_1 < k_2 < \ldots < k_n < \ldots\} \subseteq \mathbb{N}$, the asymptotic density of which is not zero (i.e., it is greater than zero or does not exist), such that $\lim_{n\to\infty} x_{k_n} = \xi$. Let Λ_x denote the set of statistical limit points of x. ξ is an ordinary limit point of a sequence $x = (x_k)$ if there is a subsequence of x that converges to ξ, and L_x denotes the set of ordinary limit points of x.

Now we recall the well-known concept of regular summability matrix $T = (t_{nk})$. An $\mathbb{N} \times \mathbb{N}$ matrix $T = (t_{nk})$ is a regular summability matrix if, for any convergent sequence x with limit ξ, $\lim_n \sum_{k=1}^{\infty} t_{nk} x_k = \xi$, and T is nonnegative if $t_{nk} \geq 0$ for all n and k. For a nonnegative regular summability matrix T, we define

$$d_T(K) = \lim_n \sum_{k=1}^{\infty} t_{nk} \chi_K(k)$$

when the limit exists. We say that a set $K \subset \mathbb{N}$ is T-nonthin if $d_T(K) \neq 0$ or $d_T(K)$ is undefined. Given a sequence $x = (x_k)$ and a scalar ξ, let $K_{\epsilon,\xi} = \{k \in \mathbb{N} : |x_k - \xi| < \epsilon\}$. Throughout this note, we let $|K|$ denote the cardinality of the set K.

Let $x = (x_k)$ be a sequence and T be a nonnegative regular summability matrix. Then we have the following.

- A sequence x is T-statistically convergent to ξ provided that, for every $\epsilon > 0$, $d_T(K_{\epsilon,\xi}) = 1$;
- ξ is a T-s.l.p. of x provided that there is a set $K \subseteq \mathbb{N}$ which is T-nonthin and such that x converges to ξ along K;
- ξ is T-s.c.p. of x provided that for every $\epsilon > 0$, the set $K_{\epsilon,\xi}$ is T-nonthin [6].

If $T = (t_{nk})$ is defined by $t_{nk} = \dfrac{1}{n}$ for $k \leq n$ and by $t_{nk} = 0$ otherwise, then T is called the Cesáro matrix.

We discuss methods generated by λ sequences later in this note. Let $\lambda = (\lambda_n)$ be a nondecreasing sequence of positive numbers tending to infinity such that $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$ and $I_n = [n - \lambda_n + 1, n]$. The concept of λ-s.c.p. is a special case of T-s.c.p.; for T one has to take the matrix (t_{nk}) with $t_{nk} = 1/\lambda_n$ if $k \in I_n$ and 0 elsewhere. If

$$\lim_n {\lambda_n}^{-1} |\{k \in I_n : |x_k - L| \geq \epsilon\}| = \lim_n {\lambda_n}^{-1} |\{k \in (n - \lambda_n, n] : |x_k - L| \geq \epsilon\}| = 0$$

for every $\epsilon > 0$, then x is said to be λ-statistically convergent to L [24]. In this case, we write $x_k \to L\,(S_\lambda)$ and S_λ denotes all λ-statistically convergent sequences. We can give λ-density of the set $B \subseteq \mathbb{N}$ by $d_\lambda(B) = \lim_n {\lambda_n}^{-1} |\{k \in I_n : k \in B\}|$ whenever the limit exists. The definition of λ-density of a set gives natural density in case $\lambda_n = n$. If a set has λ-density, then $d_\lambda(B) \leq 1$ for every $B \subseteq \mathbb{N}$.

Let X be a finite-dimensional Banach space, let $x = (x_k)$ be an X-valued sequence, and $\mu \in X$. The sequence (x_k) is norm statistically convergent to μ provided that $d(\{k : \|x_k - \mu\| \geq \epsilon\}) = 0$ for all $\epsilon > 0$ [7]. Let M be any closed subset of X. Let $\rho(M, \mu)$ stand for the distance from a point μ to the closed set M, where $\rho(M, \mu) = \min_{m \in M} \|m - \mu\|$.

We formulate the definition of statistical cluster point and some of its properties proved in [16, 29]. Let X be a finite-dimensional Banach space, let $x = (x_k)$ be an X-valued sequence. A point $\xi \in X$ is called a s.c.p. if for every $\epsilon > 0$

$$\limsup_{n\to\infty} \frac{1}{n} \left|\{k \in \mathbb{N} : \|x_k - \xi\| < \epsilon\}\right| > 0.$$

We will denote the set of statistical cluster points of the sequence x by Γ_x. It is clear that $\Gamma_x \subseteq L_x$ for every sequence x.

Example 2.1. Define the sequence $x = (x_k)$ by

$$x_k = \begin{cases} 1, & \text{if } k \text{ is prime,} \\ 0, & \text{otherwise.} \end{cases}$$

Then $\Gamma_x = \{0\}$ and $L_x = \{0, 1\}$. So,

$$d(\{k : \rho\left(\Gamma_x, x_k\right) \geq \varepsilon\}) = \lim_{n\to\infty} n^{-1} \left|\{k \leq n \,:\, \rho\left(\{0\}, x_k\right) \geq \varepsilon\}\right| = 0.$$

Note that if the sequence x has a bounded nonthin subsequence, then the set Γ_x is nonempty.

Throughout this paper, let X be a finite-dimensional Banach space. The purpose of this paper is to investigate the λ-s.c.p. and to prove some properties of the set of λ-s.c. points. We also study the some results related to the Γ_x^λ defined by this new method.

3 Some Properties of λ-s.c. Points

In this section we give some properties of the set of λ-s.c. points in finite-dimensional X Banach spaces. Firstly, we will give definition of λ-s.c. points. Let Ω denote the set of all nondecreasing sequences $\lambda = (\lambda_n)$ of positive numbers tending to infinity such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$.

Definition 3.1. Let $\lambda \in \Omega$. A sequence $x = (x_k)$ in X is λ-statistically convergent to $\mu \in X$ if for every $\epsilon > 0$,

$$\lim_n {\lambda_n}^{-1} |\{k \in I_n \,:\, \|x_k - \mu\| \geq \epsilon\}| = 0,$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 3.2. Let $\lambda \in \Omega$. A point $\mu \in X$ is a λ-s.c.p. of a sequence $x = (x_k)$ in X, if for every $\epsilon > 0$

$$\limsup_{n\to\infty} {\lambda_n}^{-1} |\{k \in I_n : \|x_k - \mu\| < \epsilon\}| > 0.$$

By Γ_x^λ we denote the set of all λ-s.c. points of the sequence x.

Lemma 3.3. *Let $\lambda \in \Omega$ and $\Gamma_x^\lambda \neq \emptyset$. Then Γ_x^λ is a closed set.*

Proof. Assume that $\mu_n \in \Gamma_x^\lambda$ and $\mu_n \to \mu$. Then there exists a number n' such that $\|\mu_{n'} - \mu\| < \epsilon/2$ for a given positive number $\epsilon > 0$. Since $\mu_{n'} \in \Gamma_x^\lambda$, by Definition 3.2 we have

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : \|x_k - \mu_{n'}\| < \epsilon/2\}| > 0$$

which implies

$$\{k \in I_n : \|x_k - \mu\| < \epsilon\} \supset \{k \in I_n : \|x_k - \mu_{n'}\| < \epsilon/2\}.$$

Hence,

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : \|x_k - \mu\| < \epsilon\}| > 0$$

and $\mu \in \Gamma_x^\lambda$. □

Lemma 3.4. *Let $\lambda \in \Omega$ and let (x_k) be a sequence in X. Γ_x^λ be a set of λ-s.c. points of the sequence x. Let D be a nonempty compact subset of X. If*

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : x_k \in D\}| > 0, \tag{3.1}$$

then $D \cap \Gamma_x^\lambda \neq \emptyset$.

Proof. We assume that $D \cap \Gamma_x^\lambda = \emptyset$. In this case no point $\mu \in D$ is λ-s.c.p, that is for each $\mu \in D$ there is a positive number $\epsilon = \epsilon(\mu) > 0$ such that

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : \|x_k - \mu\| < \epsilon\}| = 0.$$

Let $N_\epsilon(\mu) = \{z \in X : \|z - \mu\| < \epsilon\}$. Then the sets $N_{\epsilon_i}(\mu_i)$ form an open covering of D, and because D is compact, there are finitely many $\mu_1, \ldots, \mu_m$ in D such that $D \subset \bigcup_{i=1}^{m} N_{\epsilon_i}(\mu_i)$. It is clear that

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : \|x_k - \mu_i\| < \epsilon_i\}| = 0$$

for every i. Then we have

$$|\{k \in I_n : x_k \in D\}| \le \sum_{i=1}^{m} |\{k \in I_n : \|x_k - \mu_i\| < \epsilon_i\}|, \tag{3.2}$$

and from (3.2) we obtain

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : x_k \in D\}| = 0$$

which contradicts (3.1). □

Theorem 3.5. *Let $\lambda \in \Omega$ and let $x = (x_k)$ be a bounded sequence in X. Then for every $\epsilon > 0$*

$$\lim_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n \,:\, \rho(\Gamma_x^\lambda, x_k) \geq \epsilon\}| = 0.$$

Proof. Let x be a bounded sequence. Then we can find a closed set $D \subset X$ such that $x_k \in D$ for every k. We will denote the ϵ-neighbourhood of D by $N_\epsilon(D) = \{z \in X \,:\, \rho(D, z) < \epsilon\}$. Now we assume that

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n \,:\, \rho(\Gamma_x^\lambda, x_k) \geq \epsilon\}| > 0.$$

Then for the set $\tilde{D} = D \setminus \overset{\circ}{N}_\epsilon(\Gamma_x^\lambda)$ we have

$$\limsup_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : x_k \in \tilde{D} \subset X\}| > 0.$$

By Lemma 3.4, $\Gamma_x^\lambda \cap \tilde{D} \neq \emptyset$, which is a contradiction. The proof is complete. $\square$

Corollary 3.6. *Let $\lambda \in \Omega$ and let $D \subset X$ be a compact set such that $D \cap \Gamma_x^\lambda = \emptyset$. Then for every point $\mu \in D$ there is $\epsilon = \epsilon(\mu) > 0$ with $\lim_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : x_k \in D\}| = 0$.*

Proof. Let $N_\epsilon(\mu) = \{y \in X : \|y - \mu\| < \epsilon\}$. The open sets $N_\epsilon(\mu)$, $\mu \in D$ form an open covering of D. But D is a compact set and so there exists a finite subcover of D, say $N_j = N_{\epsilon_j}(\mu_j),\ \ j = 1, 2, \ldots, m$. Clearly $D \subset \cup_j N_j$ and $\lim_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : \|x_k - \mu_j\| < \epsilon_j\}| = 0$ for every j. We have

$$|\{k \in I_n : x_k \in D\}| \leq \sum_{j=1}^{m} |\{k \in I_n : \|x_k - \mu_j\| < \epsilon_j\}|$$

and therefore

$$\lim_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : x_k \in D\}| \leq \sum_{j=1}^{m} \lim_{n\to\infty} {\lambda_n}^{-1}|\{k \in I_n : \|x_k - \mu_j\| < \epsilon_j\}| = 0,$$

and the proof is complete. $\square$

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