Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 5, Number 2, pp. 215–230 (2010)
http://campus.mst.edu/adsa

Determination of Unknown Coefficients in a Nonlocal Parabolic Problem from Neumann Type Boundary Measured Data

Burhan Pektaş
Izmir University
Department of Mathematics and Computer Sciences
35350 Uckuyular, Izmir, Turkey
burhan.pektas@izmir.edu.tr
burhanps@gmail.com

Abstract

This article develops an inverse polynomial method for determining the unknown coefficients $D = D(x)$ and $c = c(x)$ of the transport-diffusion operator $Au := u_t - (D(x)u_x)_x + \nu(c(x)u(x))_x$ from the boundary measurements given in the form of Dirichlet and Neumann type boundary conditions. The presented approach is based on the Lagrange–Hermite interpolation of the solution of the corresponding direct problem and use of the information about singular points. The main distinguished feature of the proposed approach is the possibility of reconstruction of the unknown coefficient in the neighborhood of the singular point. This guarantees the high accuracy of the reconstruction in whole interval $(-L, L)$. An efficiency and applicability of the method is demonstrated on various numerical examples with noisy free and noisy data.

AMS Subject Classifications: 35J65, 35K60, 35A15.
Keywords: Inverse coefficient problem, transport-diffusion equation, optimal control, nonlocal condition.

1 Introduction

The mathematical model of sludge particles settling in the water treatment plant (settler) is given by the transport-diffusion operator $Au := u_t - (D(x)u_x)_x + \nu(c(x)u(x))_x$. In the

Received April 9, 2010; Accepted May 6, 2010
Communicated by A. Okay Çelebi

case of the residence time of sludge particles in the settler, the model leads to a nonlinear age-dependent transport-diffusion with nonlocal additional condition. Identification of the diffusion coefficient $D(x)$ is considered. This problem is formulated as an identification/optimal control problem, where the boundary flux measurements is assumed to be a control. For the case of constant ("average") velocity, as a characterization of the optimal control problem two necessary conditions are obtained. These conditions permit to reduce the nonlinear coupled two-dimensional problem to the two-point boundary value problem for the second-order nonlinear ordinary differential equation, and then, to a nonlinear equation, with respect to sludge concentration. For the approximate solution of the problem, an inverse polynomial method is used. This method is based on the Lagrange–Hermite interpolation of the solution of the corresponding direct problem and use of the flux information as well as *singular points*. The point $x \in (-L, L)$ will be defined as a singular point for the inverse problem if the first and/or second derivative of the direct problem's solution vanishes at that point x.

The determination of the leading unknown coefficient in ordinary and partial differential equations is one of actual problems in inverse problem theory and practice (see [1–3, 7, 8, 13] and references therein). For ordinary differential equations, the most widely studied inverse problems were formulated as classical Sturm–Liouville problems [4, 12], where the unknown coefficient needs to be determined from spectral data. However such a data is not available in many physical problems. The class of inverse problems considered here is based only on boundary measurements [5–11].

2 Problem Formulation

We consider the following identification / optimal control problem:

Find a triple $\langle u, D, c\rangle$, which satisfies the nonlocal problem

$$u_t(x,t) = (D(x)u_x(x,t))_x - \nu(c(x)u(x,t))_x, \;\; \forall (x,t) \in \Omega := (-L, L) \times (0, \infty)\,, \tag{2.1}$$

$$u(x,0) = \varphi(x), \;\; \forall x \in (-L, L), \tag{2.2}$$

$$u(-L,t) = u(L,t) = 0, \;\; \forall t \in \mathbb{R}_+\,, \tag{2.3}$$

$$\int_0^\infty u(x,t)dt = c(x), \;\; \forall x \in (-L, L). \tag{2.4}$$

We assume boundary flux measurement(s) as additional conditions:

$$-D(-L)u_x(-L,t) = f(t), \;\; t > 0 \quad (\text{or}, -D(L)u_x(L,t) = g(t), \;\; t > 0). \tag{2.5}$$

Although the homogeneous boundary conditions (2.3) are not realistic from the physical point of view, we assume here that the density of sludge particles on the boundary is small enough at the boundary $x = \pm L$. We also assume that the initial data and coefficients are continuous functions, i.e., $\varphi(x)$, $D(x), c(x) \in C[-L, L]$, and

$$0 < D_* \le D(x) \le D^*, \quad 0 \le c(x) \le c^*, \;\; \forall x \in [-L, L]. \tag{2.6}$$

For given coefficients $c, D \in C[-L, L]$, denote by $u = u(x, t; D, c)$ the unique classical solution of the parabolic initial value problem (2.1)–(2.3). The coefficients $c(x), D(x)$ are assumed to be unknown and the nonlocal condition (2.4) and measured data (2.5) can be treated as observations for determination of the coefficients $c(x), D(x)$. For this reason, the problem (2.1) to (2.5) is defined as a nonlocal optimal control or identification problem, and the triple $\langle u, D, c\rangle$ is called a solution of the identification problem (2.1)–(2.5). In this case, the problem (2.1)–(2.3) is called a direct problem corresponding to inverse problem (2.1)–(2.5).

3 Necessary Conditions for Optimality

Assume that $\langle u, D, c\rangle$ is a solution of the problem (2.1)–(2.5). Then integrating equation (2.1) on $[0, \infty)$ with respect to the variable $t > 0$, we obtain

$$u(x,\infty) - u(x,0) = \int_0^\infty (D(x)u_x(x,t))_x dt - \int_0^\infty \nu(c(x)u(x,t))_x dt, \quad \forall x \in (-L, L). \tag{3.1}$$

By using the nonlocal condition (2.4) and assuming differentiability of the function $u(x, t)$ under the integrals, we may write

$$\begin{aligned} c^2(x) &= \int_0^\infty c(x)u(x,t)dt, \ \ \forall x \in (-L, L), \\ c'(x) &= \int_0^\infty u_x(x,t)dt, \ \ \forall x \in (-L, L), \\ (c^2(x))' &= \int_0^\infty (c(x)u(x,t))_x dt, \ \ \forall x \in (-L, L), \\ (D(x)c'(x))' &= \int_0^\infty (D(x)u_x(x,t))_x dt, \ \ \forall x \in (-L, L). \end{aligned}$$

Taking into account the initial condition (2.2), we can rewrite the integro-differential equation (3.1) in the form

$$\begin{cases} -(D(x)c'(x))' + \nu(c^2(x))' = \varphi(x), \ \ \forall x \in (-L, L), \\ c(-L) = c(L) = 0. \end{cases} \tag{3.2}$$

Hence we obtain the following.

Proposition 3.1. *If $\langle u, D, c\rangle$ is a solution of the identification problem* (2.1)–(2.5)*, then the couple of the functions $< c(x), D(x) >$ satisfy the two-point boundary value problem* (3.2) *for the second-order nonlinear ordinary differential equation.*

Proposition 3.2. *The identification problem* (2.1)–(2.5) *is equivalent to the following inverse problem: Find a pair* $\langle D, c\rangle$*, which satisfies the two-point boundary value problem* (3.2) *and the additional condition(s)*

$$D(-L)c'(-L;D) = \psi_0 \quad (\textit{or } D(L)c'(L;D) = \psi_1). \tag{3.3}$$

The reduced measured data

$$\psi_0 := -\int_0^\infty f(t)dt \quad \text{or} \quad \psi_1 := -\int_0^\infty g(t)dt$$

is obtained from nonlocal condition (2.4) and measured data (2.5). The function $c = c(x;D) \in C^2(-L,L) \cap C^1[-L,L]$ is the solution to the nonlinear boundary value problem (3.2) for a given coefficient $D = D(x)$. The problem (3.2) is called forward problem corresponding to the inverse problem (3.2)–(3.3).

According to Proposition 3.2, we can now consider the *reduced inverse problem* (3.2)–(3.3). Rewriting of equation in (3.2) with respect to the unknown function $D = D(x)$ and using the conditions in (3.3) as initial condition, we have the Cauchy problem

$$\begin{cases} D'(x) + \dfrac{c''(x)}{c'(x)}D(x) = \dfrac{\nu(c^2(x))' - \varphi(x)}{c'(x)}, \quad x \in (-L,L) \\ D(-L) = \dfrac{\psi_0}{c'(-L)}, \quad (\text{or } D(L) = \dfrac{\psi_1}{c'(L)}). \end{cases} \tag{3.4}$$

When the function $c = c(x)$ is known, the solution of the Cauchy problem (3.4) has the integral representation

$$D(x) = \frac{1}{c'(x)}\left\{\psi_0 + \nu c^2(x) - \int_{-L}^{x} \varphi(\xi)d\xi\right\}, \quad x \in [-L,L], \quad c'(x) \neq 0. \tag{3.5}$$

It is seen from (3.4) that ill-posedness of the inverse problem is a result of vanishing of the first or/and the second-order derivatives of $c = c(x)$ at one or more points of $(-L,L)$. This suggests to classify the inverse problem (3.2)–(3.3) as follows (see [13], [5]):

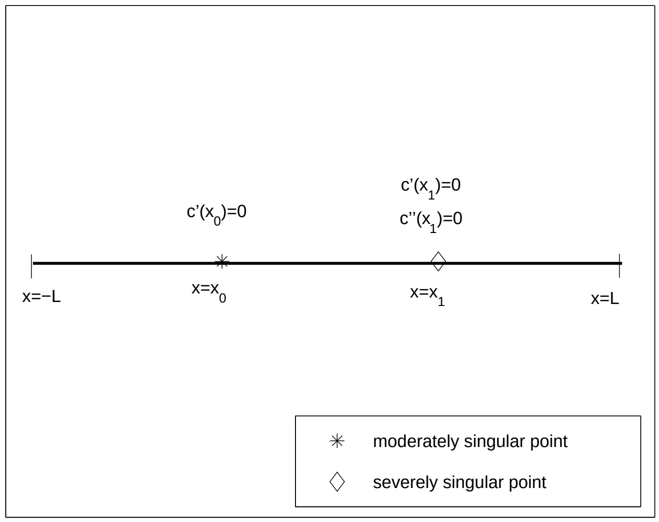
(i1) $c'(x) \neq 0$, $c''(x) \neq 0$, $\forall x \in [-L,L]$;

(i2) $c'(x_0) = 0$, $x_0 \in (-L,L)$; $c'(x) \neq 0$, $\forall x \neq x_0$; $c''(x) \neq 0$, $\forall x \in [-L,L]$;

(i3) $c'(x_1) = c''(x_1) = 0$, $x_1 \in (-L,L)$; $c'(x) \neq 0,\ c''(x) \neq 0$, $\forall x \neq x_1$.

In the case (i1) the inverse problem is *well-conditioned*. The cases (i2), (i3) correspond to the situations, when the problem is *moderately ill-conditioned* and *severely ill-conditioned*, respectively. The points $x_0,\ x_1 \in (-L,L)$ will be defined as *singular points* for the inverse problem. The geometry of singular points is given in Figure 3.1.

Figure 3.1: Simulation of the moderately and severely singular points



Corollary 3.3. *The inverse problem* (3.2)–(3.3) *is ill-conditioned in the above sense* (i2) *or* (i3).

We apply the inverse polynomial method (IPM), given in [5], for moderately and severely ill-conditioned inverse problems. The distinguished feature of this method is that it permits to determine the unknown coefficient in the neighborhood of the singular point also.

4 Inverse Polynomial Method (IPM)

By the inverse polynomial method as introduced in [5], we adopt the view that good approximation of the forward problem solution yields good solution of the corresponding inverse problem. The method consists of two steps: polynomial approximation of the direct problem solution $c = c(x)$ and solution of the Cauchy problem for the unknown coefficient $D = D(x)$.

For each considered ill-conditioned situation, the polynomial $c_p = c_p(x)$ will be defined from the boundary conditions and information about singular points. As a result, we need to construct the polynomial function

$$c_p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_m x^m, \quad x \in [-L, L], \tag{4.1}$$

where the parameter $m > 1$ will be defined from the number of additional conditions and the unknown coefficients α_i, $i = \overline{0, m}$ will be determined from the corresponding linear algebraic equations system:

$$\mathbf{A}\mathbf{x} = \mathbf{B}.$$

The coefficient matrix $\mathbf{A} = \{(a_{ij})\}$ and the column vector $\mathbf{B} = \{(b_j)\}$ will be determined separately for the cases (i2) and (i3).

In the second part of the method the following Cauchy problem, obtained from (3.4), will be solved for the first-order differential equation with respect to the unknown coefficient $D_{\text{appr.}} = D_{\text{appr.}}(x)$:

$$D'_{\text{appr.}}(x) + \frac{c''_p(x)}{c'_p(x)} D_{\text{appr.}}(x) = \frac{\nu (c_p^2(x))' - \varphi(x)}{c'_p(x)}, \quad x \in (-L, L), \tag{4.2}$$

$$D_{\text{appr.}}(-L) = \frac{\psi_0}{c'_p(-L)} \quad \left(\text{or} \quad D_{\text{appr.}}(L) = \frac{\psi_1}{c'_p(L)}\right). \tag{4.3}$$

It is seen from equation (4.2), if $c'_p(x) \neq 0$, $c''_p(x) \neq 0$, $x \in [-L, L]$, then for a given function $c_p(x)$, the Cauchy problem (4.2)–(4.3) has the unique solution

$$\left\{ \begin{array}{l} D_{\text{appr.}}(x) = \dfrac{1}{c'_p(x)} \left[\psi_0 + \nu c_p^2(x) - \displaystyle\int_{-L}^{x} \varphi(\xi) d\xi \right] \\ \text{or} \quad D_{\text{appr.}}(x) = \dfrac{1}{c'_p(x)} \left[\psi_1 + \nu c_p^2(x) + \displaystyle\int_{x}^{L} \varphi(\xi) d\xi \right] \end{array} \right. , \; x \in [-L, L], \;\; c'_p(x) \neq 0. \tag{4.4}$$

In the presented modification of the inverse method, we use the additional information to increase the order of the polynomial approximation for the function $c = c(x)$.

Lemma 4.1 (See [6]). *Suppose* $\langle D(x), c(x) \rangle$ *satisfies Eq.* (3.2) *and conditions* (3.3). *Then the following compatibility condition holds:*

$$\int_{-L}^{L} \varphi(x) dx = \psi_0 - \psi_1 \quad \textit{(conservation law)}.$$

Proof. This is obtained by integrating equation in (3.2) on the interval $[-L, L]$ and applying homogeneous boundary conditions in (3.2). □

Lemma 4.2 (See [6]). *Suppose* $\langle D_1(x), c_1(x)\rangle$ *and* $\langle D_2(x), c_2(x)\rangle$ *satisfy Eq.* (3.2) *and conditions* (3.3)*. If* $x_0 \in (-L, L)$ *is a singular point for the functions* $c_1(x)$ *and* $c_2(x)$*, i.e., if* $c_1'(x_0) = c_2'(x_0) = 0$*, then the following transmission conditions hold:*

$$c_1(x_a) = c_2(x_a), \quad c_1(x_b) = c_2(x_b), \quad x_a \in (-L, x_0), \quad x_b \in (x_0, L).$$

Proof. This is obtained by integrating equation in (3.2) on the interval $[x, L]$ and applying the hypothesis of the lemma. □

In this case the points x_a and x_b are called *transmission points* of the inverse problem (2.1)–(2.5).

5 Ill-Conditioned Situations and Numerical Examples

5.1 The Moderately Ill-Conditioned Situation

If we assume that the situation (i2) holds,

$$c'(x_0) = 0,\ x_0 \in (-L, L); \quad c'(x) \neq 0,\ \forall x \neq x_0; \quad c''(x) \neq 0,\ \forall x \in [-L, L],$$

then the polynomial function $c_p = c_p(x)$, given by (4.1), will be defined from the four conditions

$$c_p(-L) = 0, \quad c_p(L) = 0, \quad c_p'(x_0) = 0, \quad \int_{-L}^{x_0} \varphi(x)dx - \psi_0 - \nu c_p^2(x_0) = 0.$$

As a result, we obtain the Lagrange–Hermite polynomial function

$$c_p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3, \quad x \in [-L, L] \tag{5.1}$$

of degree 3, and the unknown coefficients a_i, $i = \overline{0, 3}$ are determined from the system

$$\begin{pmatrix} 1 & -L & L^2 & -L^3 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2x_0 & 3x_0^2 \\ 1 & x_0 & x_0^2 & x_0^3 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \gamma \end{pmatrix}, \tag{5.2}$$

where the number $\gamma = \sqrt{\dfrac{\int_{-L}^{x_0} \varphi(x)dx - \psi_0}{\nu}} > 0$ is obtained from the fourth condition above. The approximate solution $D_{\text{appr.}} = D_{\text{appr.}}(x)$ of the inverse problem (3.2)–(3.3) can be computed from the integral representations (4.4), simultaneously (except the point $x = x_0$). At the singular point, the numerical solution is computed from

$$D_{\text{appr.}}(x_0) = \frac{\nu c_p^2(x_0) - \varphi(x_0)}{c_p''(x_0)}, \quad c_p''(x_0) \neq 0. \tag{5.3}$$

The error estimation of the inverse polynomial method in the considered situation is obtained as follows.

Lemma 5.1. *Let* $c(x) \in C^4[-L, L]$ *and* $c_p = c_p(x)$ *be the Lagrange–Hermite polynomial given by* (5.1)*. If* x_a *and* x_b *are transmission points given in Lemma 4.2, then the following estimate holds:* $\forall x \in [-L, L]$*,*

$$c(x) - c_p(x) = \frac{1}{4!}\, c^{(4)}(\xi)(x + L)(x - x_a)(x - x_b)(x - L), \quad \xi = \xi(x) \in (-L, L),$$

Proof. The points $-L, L, x_a$ and x_b are zeros of the function $F(x) = c(x) - c_p(x) + \kappa\delta(x)$, $\delta(x) = (x + L)(x - x_a)(x - x_b)(x - L)$. On the other hand, the constant κ can be determined for all $x \in (-L, L)$ such that $F(x) = 0$. Thus we get $\kappa = (c_p(x) - c(x))/\delta(x)$. Then by the Rolle theorem, there exists $\xi \in (-L, L)$ such that $F^{(4)}(\xi) = 0$. The proof is complete. □

Corollary 5.2. *If the conditions of Lemma 5.1 hold, then the following error formula holds:*

$$\|c - c_p\|_{C[-L,L]} \le \frac{1}{4!} M_4\, \alpha_1, \quad M_4 = \max_{[-L,L]} |c^{(4)}(x)|,$$
$$\alpha_1 = \max_{[-L,L]} |(x + L)(x - x_a)(x - x_b)(x - L)|.$$

Lemma 5.3. *Let* $c(x) \in C^5[0, 1]$ *and* $c_p = c_p(x)$ *be the polynomial given by* (5.1)*. If* x_0 *is a moderately singular point of the inverse problem* (3.2)–(3.3)*, then the following estimate holds:*

$$c'(x) - c_p'(x) = \frac{1}{4!}\, c^{(5)}(\xi)(x - \gamma_1)(x - \gamma_2)(x - \gamma_3)(x - x_0),$$
$$\gamma_1, \gamma_2, \gamma_3, \xi = \xi(x) \in (-L, L).$$

Proof. The points $-L, L, x_a$ and x_b are zeros of the function $W(x) := c(x) - c_p(x)$, so by the Rolle theorem there exists $\gamma_i \in (-L, L)$ such that $W'(\gamma_i) = 0$, $i = \overline{1, 3}$. On the other hand, $W'(x_0) = 0$. Introduce the function $G(x) = W'(x) + \kappa\delta(x)$, $\delta(x) = (x - \gamma_1)(x - \gamma_2)(x - \gamma_3)(x - x_0)$, and κ can be determined for all $x \in (-L, L)$ such that $G(x) = 0$. Thus we get $\kappa = -W'(x)/\delta(x)$. By applying the Rolle theorem again, we find that there exists $\xi \in (-L, L)$ such that $G^{(4)}(\xi) = 0$. Thus the proof is complete. □

Corollary 5.4. *If the conditions of Lemma 5.3 hold, then the following error formula holds:*

$$\|c' - c_p'\|_{C[-L,L]} \le \frac{1}{4!}\, M_5\, \alpha_2, \quad M_5 = \max_{[-L,L]} |c^{(5)}(x)|,$$
$$\alpha_2 = \max_{[-L,L]} |x - \gamma_1)(x - \gamma_2)(x - \gamma_3)(x - x_0)|.$$

In a first series of computational experiments, we consider the moderately ill-conditioned situation.

Example 5.5. Let

$$c(x) = \exp(-0.5x^2), \quad D(x) = 1 + \exp(-x^2)$$

be exact solutions of the inverse problem (3.2) for input data

$$\nu = 0.5,\ \varphi(x) = -D'(x)c'(x) - D(x)c''(x) + 2\nu c(x)c'(x), \quad x \in (-L, L).$$

Here $L = 1.5$ and the value ψ_0 is assumed to be synthetic noise free data given on the left boundary of the interval $(-L, L)$, and $x_0 = 0$ is a moderately singular point. Figure 5.1 illustrates the exact solutions $c(x),\ D(x)$ and their polynomial approximations $c_{\text{appr.}}(x),\ D_{\text{appr.}}(x)$ on the the interval $[-L, L]$. The relative supnorm-errors (in absence of noise in data)

$$\varepsilon_c = \left\| \frac{c - c_{\text{appr.}}}{c} \right\|_\infty, \quad \varepsilon_D = \left\| \frac{D - D_{\text{appr.}}}{D} \right\|_\infty$$

are 0.87 and 0.61, respectively. Here we used $n = 45$ mesh points in the interval $[L, L]$. Figure 5.2 shows the exact and approximate solutions (on top $c(x)$, on bottom $D(x)$) for the above test functions with noisy data. The curves $-o-$ in both figures correspond to synthetic noisy Neumann data $D(-L)c(L) = \psi_\gamma, \psi_\gamma = \psi \pm \gamma\psi$ with $\gamma = 0.05$. Here we used the same input parameters as before. The relative supnorm-error, defined above, for the noise factor $\gamma = 0.05$ are 0.87 and 0.62, respectively. Here we used $n = 45$ mesh points in the interval $[L, L]$.

5.2 The Severely Ill-Conditioned Situation

Let us assume situation (i3) holds and x_1 is a singular point of the inverse problem (3.2)–(3.3)

$$c'(x_1) = c''(x_1) = 0, \quad x_1 \in (-L, L);\ \ c'(x) \neq 0,\ \ c''(x) \neq 0,\ \ \forall x \neq x_1.$$

Since x_1 is a severely singular point of the problem, we get

$$\varphi(x_1) = 0.$$

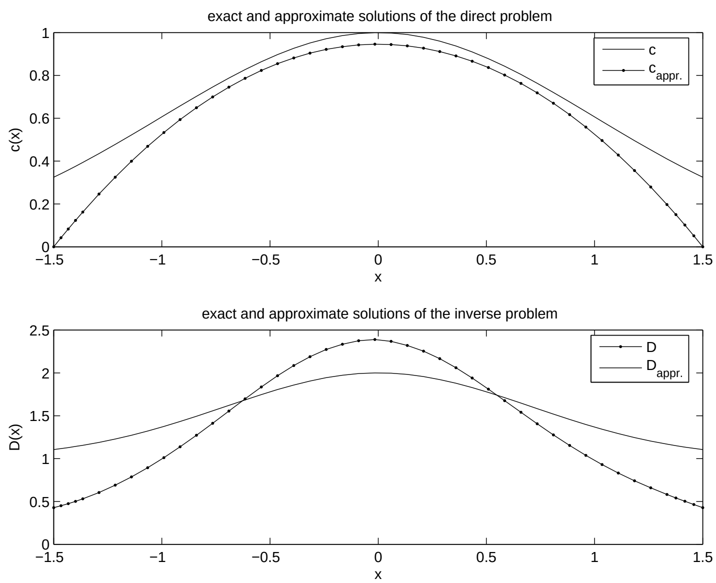
We can split Cauchy problem (3.4) into two subproblems

$$\begin{cases} D'_{\text{appr.}}(x) + \dfrac{c''_p(x)}{c'_p(x)} D_{\text{appr.}}(x) = \dfrac{\nu(c_p^2(x))' - \varphi(x)}{c'_p(x)}, \quad x \in (-L, x_1 - \delta) \\ D_{\text{appr.}}(-L) = \dfrac{\psi_0}{c'_p(-L)} \end{cases} \tag{5.4}$$

and

$$\begin{cases} D'_{\text{appr.}}(x) + \dfrac{c''_p(x)}{c'_p(x)} D_{\text{appr.}}(x) = \dfrac{\nu(c_p^2(x)_p)' - \varphi(x)}{c'_p(x)}, \quad x \in (x_1 + \delta, L) \\ D_{\text{appr.}}(L) = \dfrac{\psi_1}{c'_p(L)}, \end{cases} \tag{5.5}$$

Figure 5.1: Exact and approximate solutions for Example 5.5



where $\delta > 0$ is a tolerance parameter. Due to the conditions above, we can determine the polynomial function $c_p = c_p(x)$, given in (4.1), from the five conditions

$$c_p(-L) = 0, \quad c_p(L) = 0, \quad c_p'(x_1) = 0, \quad c_p''(x_1) = 0,$$
$$\int_{-L}^{x_1} \varphi(x)dx - \psi_0 - \nu c_p^2(x_1) = 0.$$

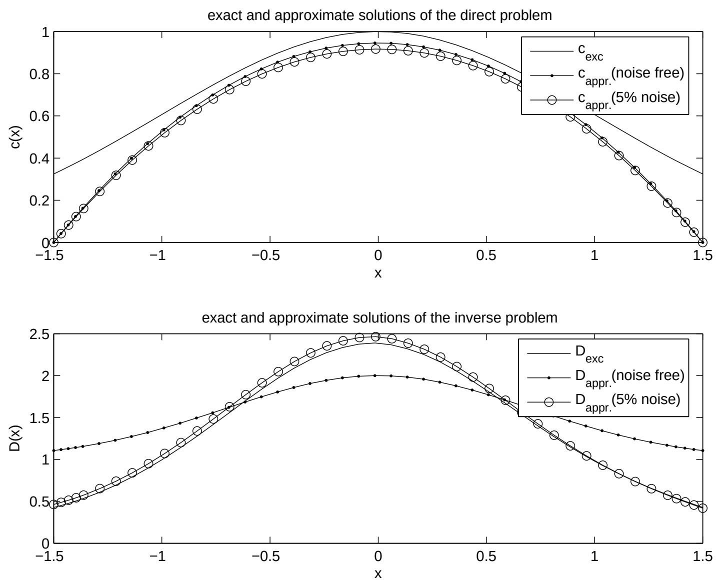
As a result we obtain the Lagrange–Hermite polynomial function

$$c_p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4, \quad x \in [-L, L] \tag{5.6}$$

of degree 4, and the unknown coefficients a_i are determined from the system

$$\begin{pmatrix} 1 & -L & L^2 & -L^3 & L^4 \\ 1 & L & L^2 & L^3 & L^4 \\ 0 & 1 & 2x_1 & 3x_1^2 & 4x_1^3 \\ 0 & 0 & 2 & 6x_1 & 12x_1^2 \\ 1 & x_1 & x_1^2 & x_1^3 & x_1^4 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \gamma \end{pmatrix}, \tag{5.7}$$

Figure 5.2: Exact and noise free and noisy approximate solutions for Example 5.5



where the number $\gamma = \sqrt{\dfrac{\int_{-L}^{x_0} \varphi(x)dx - \psi_0}{\nu}}$ is obtained from the fifth condition above. Contrary to previous situation, the approximate solution $c_p = c_p(x)$ is not defined at the singular point $x = x_1$, but this approximate solution can be computed in a sufficient $\delta-$neighborhood of the point x_1 from the Cauchy problems (5.4) and (5.5).

The error estimation of the inverse polynomial method in the considered situation is obtained as follows.

Lemma 5.6. *Let $c(x) \in C^4[-L, L]$ and $c_p = u_p(x)$ be the Lagrange–Hermite polynomial of degree 4 given by* (5.6)*. Then for all $x \in (-L, L)$,*

$$c(x) - c_p(x) = \frac{c^{(5)}(\xi) - 24a_4}{5!}(x+L)(x-x_a)(x-x_b)(x-L),$$
$$x_a, x_b, \xi = \xi(x) \in (-L, L),$$

where x_a and x_b are transmission points which are defined in Lemma 4.2.

Proof. The proof is similar to the proof of Lemma 5.3 and hence is omitted. □

Corollary 5.7. *If the conditions of Lemma 5.6 hold, then the following error formula holds:*

$$\|c - c_p\|_{C[-L,L]} \le \frac{1}{4!} M_4\,\alpha, \quad M_4 = \max_{[-L,L]} |c^{(4)}(x) - 24a_4|,$$
$$\alpha = \max_{[-L,L]} |(x+L)(x-x_a)(x-x_b)(x-L)|.$$

Lemma 5.8. *Let* $c(x) \in C^5[-L,L]$ *and* $c_p = c_p(x)$ *be the Lagrange–Hermite polynomial of degree 4 given by* (5.6). *If* x_1 *is a severely singular point of the inverse problem* (3.2)–(3.3), *then the following estimate holds:*

$$c'(x) - c_p'(x) = \frac{1}{5!} c^{(5)}(\xi)(x-\gamma_1)(x-\gamma_2)(x-\gamma_3)(x-x_1),$$
$$\gamma_1, \gamma_2, \gamma_3, \xi = \xi(x) \in (-L, L),$$

Proof. The proof is similar to the proof of Lemma 5.3 and hence is omitted. □

Corollary 5.9. *If the conditions of Lemma 5.8 hold, then the following error formula holds:*

$$\|c' - c_p'\|_{C[-L,L]} \le \frac{1}{5!} M_5\,\alpha, \quad M_5 = \max_{[-L,L]} |c^{(5)}(x)|,$$
$$\alpha = \max_{[-L,L]} |(x-\gamma_1)(x-\gamma_2)(x-\gamma_3)(x-x_1)|.$$

Now we consider an example for the severely ill-conditioned situation.

Example 5.10. Let $x_1 = 0$ be a severely singular point and

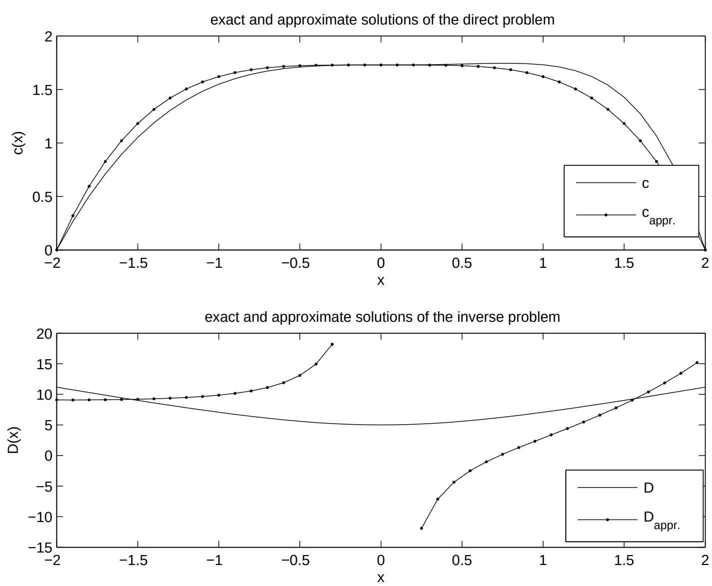
$$\begin{aligned} c(x) &= -\frac{-12L - 6e^{-L}(L+2) - 6e^{L}(L-2)}{12L^3} x^3 \\ &\quad -e^{x}(x-2) - x - \frac{6e^{-L}(L+2) - 6e^{L}(L-2)}{12} \\ D(x) &= 5\sqrt{x^2+1} \end{aligned}$$

be exact solutions of the inverse problem (3.2)–(3.3) in case (i3) holds for given input data

$$\nu = 0.5, \quad \varphi(x) = -D'(x)c'(x) - D(x)c''(x) + 2\nu c(x)c'(x), \quad x \in (-L, L).$$

Here $L = 1.5$ and the value $\psi_0 = D(-L)c'(L)$ is assumed to be synthetic noise free data given on the left boundary of the interval $(-L, L)$. Figure 5.3 illustrates the exact solution $c(x)$ and its polynomial approximations $c_{\text{appr.}}(x)$ of the direct problem and exact solution $D(x)$ and the approximate solution $D_{\text{appr.}}(x)$ of the inverse problem on the interval $[-L, x_1 - \delta) \cup (x_1 + \delta, L]$, $\delta = 0.25$. The relative supnorm-errors (in absence of noise in data) are 0.28 and 2.4 and 3.3, respectively. Here we used $n = 36$ mesh points in the interval $[-L, x_1 - \delta) \cup (x_1 + \delta, L]$. Consider now this example in the case of noisy data. The curve in Figure 5.4 corresponds to the synthetic noisy Neumann data $D(-L)c'(-L) = \psi_\gamma$, $\psi_\gamma = \psi_0 \pm \gamma\psi_0$. The relative supnorm-errors of the functions $\langle c(x); D^1(x), D^2(x)\rangle$ are 1.04, 1.03 and 2.34 for the noise factor $\gamma = 0.05$, respectively. Here the approximate solutions $D^1(x), D^2(x)$ are solutions of the Cauchy problems (5.4) and (5.5), respectively. We used $n = 36$ mesh points in the interval $[-L, x_1 - \delta) \cup (x_1 + \delta, L]$ for $\delta = 0.25$.

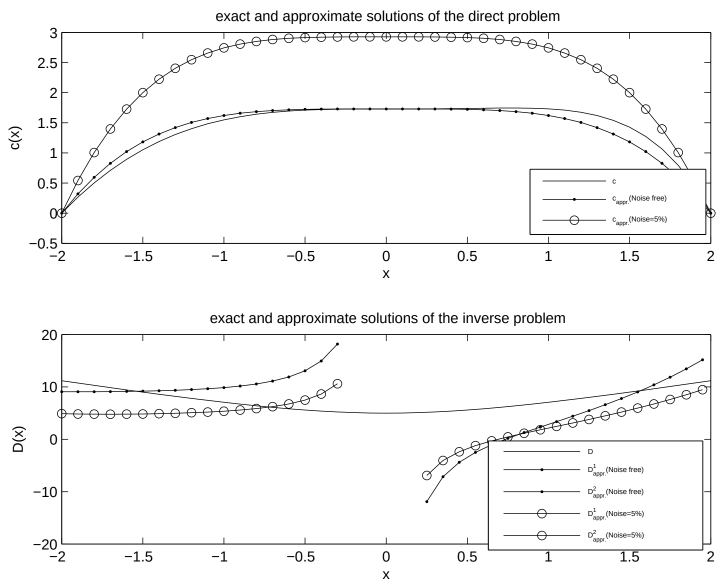
Figure 5.3: Exact and approximate solutions for Example 5.10



6 Conclusions

We have considered inverse coefficient problem (2.1)–(2.5) with nonlocal condition for parabolic equation and reduced it to an inverse problem in ordinary differential equations. We have applied inverse polynomial method for approximate solution of the inverse problem which is based on the Lagrange–Hermite interpolation of the solution of the corresponding direct problem and use of the information about singular points.

Figure 5.4: Exact and noise free and noisy approximate solutions for Example 5.10



An efficiency and applicability of the method is demonstrated on various numerical examples with noisy free and noisy data.

On the other hand, the Dirichlet type conditions (2.3) and Neumann type additional conditions (2.5) are interchangeable in the inverse problem (2.1)–(2.5), i.e., the conditions (2.5) can be taken as boundary conditions of the direct problem, as the condition (2.3) are taken as additional condition of the inverse problem. So we can get different new inverse problems as follows:

Problem 1 (NN-DD)

$$\begin{cases} u_t(x,t) = (D(x)u_x(x,t))_x - \nu(c(x)u(x,t))_x, \;\; \forall (x,t) \in (-L,L) \times \mathbb{R}_+ \,, \\ u(x,0) = \varphi(x), \;\; \forall x \in (-L,L), \\ -D(-L)u_x(-L,t) = f_0(t), \quad -D(L)u_x(L,t) = f_1(t), \;\; t > 0), \\ \displaystyle\int_0^\infty u(x,t)dt = c(x), \;\; \forall x \in (-L,L). \end{cases}$$

Additional conditions:

$$u(-L,t) = g_0(t), \;\; u(L,t) = g_1(t), \;\; \forall t \in \mathbb{R}_+.$$

Problem 2 (DN-DN)

$$\begin{cases} u_t(x,t) = (D(x)u_x(x,t))_x - \nu(c(x)u(x,t))_x, \ \ \forall(x,t) \in (-L,L) \times \mathbb{R}_+ \,, \\ u(x,0) = \varphi(x), \ \ \forall x \in (-L,L), \\ u(-L,t) = g_0(t), \quad - D(L)u_x(L,t) = f_1(t), \ \ t > 0), \\ \displaystyle\int_0^\infty u(x,t)dt = c(x), \ \ \forall x \in (-L,L). \end{cases}$$

Additional conditions:

$$-D(-L)u_x(-L,t) = f_0(t), \ \ u(L,t) = g_1(t), \ \ \forall t \in \mathbb{R}_+.$$

Acknowledgments

The author wishes to thank Alemdar Hasanov for the problem formulation and fruitful suggestions on the topic of polynomial approximation. The author also thanks A. Okay Çelebi for carefully reading the paper and the valuable suggestions that improved the paper.

References

[1] J. R. Cannon and P. C. DuChateau, Determining of unknown coefficients in a nonlinear heat conduction problem, *SIAM J. Appl. Math.* 24 (1973), 298–314.

[2] M. Choulli and P. Stefano, Reconstruction of the coefficients of the stationary transport equation from boundary measurements, *Inverse Problems* 12 (1996) L19–L23.

[3] P. C. DuChateau, Introduction to inverse problems in partial differential equations for engineers, physicists and mathematicians, in: *Parameter Identification and Inverse Problems in Hydrology, Geology and Ecology* (J. Gottlieb and P DuChateau eds.), Netherland, Kluver Academic Publishers (1996) 3–38.

[4] I. M. Gelfand and I. M. Levitan, On the determination of a differential equation from its spectrum, *Trans. Amer. Math. Soc.* 1 (1955) 253–304.

[5] A. Hasanov, An inverse polynomial method for the identification of the leading coefficient in the Sturm-Liouville operator from boundary measurements, *Appl. Math. Comput.* 140 (2003) 501–515.

[6] A. Hasanov, Error analysis of a multisingular inverse coefficient problem for the Sturm-Liouville operator based on boundary measurement, *Appl. Math. Comput.* 150 (2004) 493–524.

[7] A. Hasanov, P. C. Duchateau and B. Pektaş, *Inverse and ill-posed Problems* 14(5) (2006) 435–463.

[8] A. Hasanov and B. Pektaş, Simulation of ill-conditioned situations of inverse coefficient problem for the Sturm-Liouville operator based on boundary measurements, *Math. Comput. Simulation* 61 (2001) 47–52.

[9] A. Hasanov and Z. Seyidmamedov, Determination of leading coefficients in Sturm-Liouville operator from boundary measurements. I. A stripping algorithm, *Appl. Math. Comput.* 125 (2001) 1–21.

[10] A. Hasanov and Z. Seyidmamedov, Determination of leading coefficients in Sturm-Liouville operator from boundary measurements. II. Unicity and an engineering approach, *Appl. Math. Comput.* 125 (2001) 23–34.

[11] A. Hasanov and T. S. Shores, Solution of an inverse coefficient problem for an ordinary differential equation *Appl. Anal.* 67 (1997) 11–20.

[12] H. Hochstedt, The inverse Sturm-Liouville problem, *Comm. Pure Appl. Math.* 26 (1973) 715–729.

[13] G. R. Richter, An inverse problem for the steady state diffusion equation, *SIAM J. Appl. Math* 41 (1981) 210–221.