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Reducibility and Stability Results for Dynamic Systems on Time Scales

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Abstract

We first establish a necessary and sufficient condition for the reducibility of linear dynamic systems on time scales. Next, by applying the Floquet theory, sufficient conditions are obtained for the stability.

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1 Introduction

In 1990, the theory of dynamic equations on time scales was introduced by Stefan Hilger [8] in order to unify continuous and discrete calculus. Since then, there have been many papers investigating analysis and dynamic equations on time scales, not only unifying the standard cases, that is ODEs and OΔEs, but also extending to other cases, for example q-difference equations.

A time scale is an arbitrary closed subset of reals, and the cases when this time scale is equal to the reals or to the integers represent the classical theories of differential and difference equations. Many other interesting time scales exist, and they give rise to plenty of applications. Since Stefan Hilger formed the definition of derivatives and integrals on time scales, several authors have expounded on various aspect of the new theory, see the paper by Agarwal et al [2] and the references cited therein. Books on the subject of time scales by Bohner and Peterson [3, 4] summarize and organize much of the time scales calculus. For the notions used below we refer to the next section about calculus on time scales and references given therein.

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In this paper we intent to give sufficient conditions for reducibility and stability of linear dynamic systems on time scales. The results of this paper include the results of [14, 15]. The paper is organized as follows: In the next section we present the basic definitions and the theory of calculus on time scales. Section 3 is devoted to the proof of the sufficient conditions for reducibility of dynamic systems on time scales. In Section 4, we give some new stability criteria by using some criteria which are given by C. Pötzsche et al [13] and by J. J. DaCunha in [9].

2 Some Preliminaries

In order to make the paper self contained, we first introduce some necessary definitions and results concerning time scales. For more detailed information see the books [3, 4] and the papers [2, 8].

Definition 2.1. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The mappings $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$ are called jump operators (forward and backward jump, respectively).

These jump operators enable us to classify the points t of a time scale as right-dense, right-scattered, left-dense, and left-scattered depending on whether $\sigma(t) = t$, $\sigma(t) > t$, $\rho(t) = t$, $\rho(t) < t$, respectively, for any $t \in \mathbb{T}$. The graininess μ of the time scale is defined by $\mu(t) := \sigma(t) - t$. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous provided it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist (finite) at left-dense points in $\mathbb{T}$.

If $\sup \mathbb{T} < \infty$ and $\sup \mathbb{T}$ is left-scattered, we let $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\sup \mathbb{T}\}$. Otherwise, we let $\mathbb{T}^\kappa = \mathbb{T}$. A time scale $\mathbb{T}$ with $\sup \mathbb{T} = \infty$ is called *homogeneous* if the graininess is constant. If $\lim\limits_{t\to\infty} \mu(t)$ exists, then $\mathbb{T}$ is said to be *asymptotically homogeneous*.

Time scale calculus unifies continuous and discrete calculus and is much more general as $\mathbb{T}$ can be any nonempty closed subset of the reals $\mathbb{R}$. For example it includes quantum calculus [10] which is time scales calculus on

$$q^{\mathbb{Z}} \cup \{0\} := \left\{0, 1, q^{\pm 1}, q^{\pm 2}, q^{\pm 3}, \ldots\right\} \quad \text{and} \quad h\mathbb{Z} = \{0, \pm h, \pm 2h, \pm 3h, \ldots\}$$

with $q > 1$ and $h > 0$.

Definition 2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and $t \in \mathbb{T}^\kappa$ (the range $\mathbb{R}$ of f may actually be replaced by any Banach space). Then we define $f^\Delta(t)$ to be the number (provided it exist) with the property that for any given $\varepsilon > 0$, there is a neighborhood of t (i.e., $U = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$) such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)\,[\sigma(t) - s]\,| \le \varepsilon|\sigma(t) - s| \quad \text{for all} \quad s \in U.$$

We call $f^\Delta(t)$ the delta (or Hilger) derivative of f at t. Moreover, we say f is delta differentiable (or in short: differentiable) on $\mathbb{T}$ provided $f^\Delta(t)$ exists for all $t \in \mathbb{T}^\kappa$.

The following theorem concerning (delta) differentiation is due to Hilger [8]. See also [3, Theorem 1.16].

Theorem 2.3. *Let $f : \mathbb{T} \to \mathbb{R}$ be a function and let $t \in \mathbb{T}^\kappa$. Then we have the following:*

(i) *If f is differentiable at t, then f is continuous at t.*

(ii) *If f is continuous at t and t is right-scattered, then f differentiable at t with $f^\Delta(t) = \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}$.*

(iii) *If t is right-dense, then f is differentiable at t iff the limit $\lim_{s\to t} \dfrac{f(t) - f(s)}{t - s}$ exists as a finite number. In this case $f^\Delta(t) = \lim_{s\to t} \dfrac{f(t) - f(s)}{t - s}$.*

(iv) *If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^\Delta(t)$.*

We will use the product rule and the quotient rule for the derivative of the product fg and the quotient f/g (if $gg^\sigma \neq 0$ where g^σ denotes the composite function $g \circ \sigma$) of two differentiable functions $f, g : \mathbb{T} \to \mathbb{R}$:

$$(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma \text{ and } \left(\frac{f}{g}\right)^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}.$$

Remark 2.4. Note that in the case $\mathbb{T} = \mathbb{R}$ we have $\sigma(t) = t$, $\mu(t) = 0$, $f^\Delta(t) = f'(t)$ (ordinary derivative of f) and in the case $\mathbb{T} = \mathbb{Z}$ we have $\sigma(t) = t + 1$, $\mu(t) = 1$, $f^\Delta(t) = \Delta f(t) = f(t + 1) - f(t)$ (where Δ is the usual forward difference operator). Another important time scale is $\mathbb{T} = q^{\mathbb{N}_0} := \{q^m : m \in \mathbb{N}_0\}$ with $q > 1$, for which $\sigma(t) = qt$, $\mu(t) = (q - 1)t$, and then one gets the so-called q-derivative (quantum derivative) [10] $f^\Delta(t) = D_q f(t) = \dfrac{f(qt) - f(t)}{(q - 1)t}$.

Definition 2.5. Suppose $f : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$. Then the equation

$$y^\Delta = f(t, y, y^\sigma)$$

is called a first-order dynamic equation, sometimes also a differential equation. If $f(t, y, y^\sigma) = f_1(t)y + f_2(t)$ or $f(t, y, y^\sigma) = f_1(t)y^\sigma + f_2(t)$ for functions f_1 and f_2, then the first-order dynamic equations is called a linear dynamic equation.

Definition 2.6. Let A be an $m \times n$ matrix-valued function on $\mathbb{T}$. We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$ and similarly we say that A is differentiable on $\mathbb{T}$ provided each entry of A is differentiable on $\mathbb{T}$, and in this case we put

$$A^\Delta = \left(a_{ij}^\Delta\right)_{1\le i\le m, 1\le j\le\le n}, \quad \text{where} \quad A = (a_{ij})_{1\le i\le m, 1\le j\le\le n}.$$

An $m \times m$ matrix $A(t)$ defined on time scales $\mathbb{T}$ is called regressive if for all $t \in \mathbb{T}^\kappa$, $\det[I + \mu(t)A(t)] \neq 0$, where I is the $m \times m$ unit matrix. The set of all rd-continuous and regressive functions defined on $\mathbb{T}$ is denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T})$.

Theorem 2.7. *If A and B are differentiable $m \times m$-matrix-valued functions, then*

(i) $(A+B)^\Delta = A^\Delta + B^\Delta$*;*

(ii) $(\alpha A)^\Delta = \alpha A^\Delta$ *if α is constant;*

(iii) $(AB)^\Delta = A^\Delta B + A^\sigma B^\Delta = A^\Delta B^\sigma + AB^\Delta$*;*

(iv) $\left(A^{-1}\right)^\Delta = -\left(A^\sigma\right)^{-1} A^\Delta A^{-1} = -A^{-1}A^\Delta \left(A^\sigma\right)^{-1}$ *if AA^σ is invertible;*

(v) $\left(AB^{-1}\right)^\Delta = \left(A^\Delta - AB^{-1}B^\Delta\right)\left(B^\sigma\right)^{-1} = \left(A^\Delta - \left(AB^{-1}\right)^\sigma B^\Delta\right)B^{-1}$ *if BB^σ is invertible.*

Definition 2.8. Let $p \in [0, \infty)$. Then the time scale $\mathbb{T}$ is called p-periodic if

(i) $t \in \mathbb{T}$ implies that $t + p \in \mathbb{T}$,

(ii) $\mu(t) = \mu(t+p)$

for all $t \in \mathbb{T}$. An $m \times m$-matrix-valued function $A : \mathbb{T} \to \mathbb{R}^{m\times m}$ is p-periodic if $A(t) = A(t+p)$ for all $t \in \mathbb{T}$.

3 Reducibility of Dynamic Systems

In [14] Tiryaki investigated necessary and sufficient conditions to reduce homogeneous linear system of differential equations

$$\dot{x} = A(t)x$$

with variable coefficients. Recently, Tiryaki and Misir [15] investigated necessary and sufficient conditions for the homogeneous linear system of difference equations

$$x(n+1) = A(n)x(n),$$

where $A(n) = (a_{ij})$ is an $m \times m$ nonsingular matrix with real entries and the vector $x(n) = (x_1(n), x_2(n), \ldots, x_m(n))^T \in \mathbb{R}^m$. In this section our aim is to give necessary and sufficient conditions for the reducibility of linear dynamic systems on time scales, which include the results of [14, 15].

Let $\mathbb{T}$ be a given time scale and consider the first-order linear system of dynamic equations

$$x^\Delta(t) = A(t)x(t) + b(t), \quad t \in \mathbb{T}, \tag{3.1}$$

where $A(t)$ and $b(t)$ are given $m \times m$ and $m \times 1$-matrix-valued functions, respectively. If for some $t_0 \in \mathbb{T}$,

$$x(t_0) = x_0 \tag{3.2}$$

is specified, then (3.1) is called an initial value problem (IVP). The equation (3.1) is reducible to the equation

$$y^{\Delta}(t) = B(t)y(t) + d(t) \tag{3.3}$$

if there exists a nonsingular matrix $S(t)$ with real entries such that

$$x(t) = H(t)y(t), \quad \text{where} \quad H^{\Delta}(t) = S(t)H(t) \quad \text{and} \quad H(t_0) = I. \tag{3.4}$$

Let $S(t)$ be an $m \times m$-matrix function whose entries are real-valued functions defined for $t \in \mathbb{T}$. Consider the system

$$z^{\Delta}(t) = S(t)z(t), \quad t \in \mathbb{T}. \tag{3.5}$$

Let $H(t)$ be a fundamental matrix of (3.5) (i.e., $H(t) = e_S(t, t_0)$) satisfying $H(t_0) = I$ and $H^{\Delta}(t) = S(t)H(t)$. $H(t)$ can be used to transform (3.1) into (3.3).

Therefore, we can give the following theorem concerning reducibility of (3.1) into the form of (3.3).

Theorem 3.1. *The inhomogeneous linear dynamic system* (3.1) *is reducible to* (3.3) *using the transformation* (3.4), *if and only if there exists an* $m \times m$ *regular real matrix* $S(t)$ *such that*

$$A(t_0) = S_1(t_0)B(t_0) + S(t_0) \tag{3.6}$$

and

$$A^{\Delta}(t) = S^{\Delta}(t) - (A - S)^{\sigma}(t)S(t) + D(t) + E(t) \tag{3.7}$$

holds for all $t \in \mathbb{T}$, *where* $D(t) = \left(S_1^{\Delta}(t)S_1^{-1}(t) + S_1^{\sigma}(t)S(t)S_1^{-1}(t)\right)(A - S)(t)$, $E(t) = S_1^{\sigma}(t)S_1(t)H(t)B^{\Delta}(t)H^{-1}(t)$ *and* $S_1(t) = I + \mu(t)S(t)$.

Proof. Let $S(t)$ and $H(t)$ be defined as above. Because of

$$e_s^{\sigma}(t, t_0) = [I + \mu(t)S(t)]\, e_s(t, t_0),$$

we get

$$H^{\sigma}(t) = S_1(t)H(t). \tag{3.8}$$

By using the transformation (3.4) and the fact (3.8) in (3.1), we get

$$[H(t)y(t)]^{\Delta} = A(t)H(t)y(t) + b(t),$$

and after reorganization of the above equation we get

$$y^{\Delta}(t) = H^{-1}(t)S_1^{-1}(t)\,(A - S)\,(t)H(t)y(t) + d(t).$$

Thus (3.1) is reducible to (3.3) with

$$B(t) = H^{-1}(t)S_1^{-1}(t)\,(A - S)\,(t)H(t) \quad \text{and} \quad d(t) = H^{-1}(t)S_1^{-1}(t)b(t).$$

If we differentiate both sides of the above first equation and if we use the facts in Theorem 2.3 and Theorem 2.7, we get

$$\begin{aligned}
B^{\Delta} &= \left\{\left[H^{-1}S_1^{-1}\right]\left[(A-S)\,H\right]\right\}^{\Delta} \\
&= \left\{[S_1H]^{-1}\left[(A-S)\,H\right]\right\}^{\Delta} \\
&= \left([S_1H]^{-1}\right)^{\Delta}(A-S)\,H + \left([S_1H]^{-1}\right)^{\sigma}\left[(A-S)\,H\right]^{\Delta} \\
&= -\left(S_1H\right)^{\sigma^{-1}}\Big\{\left(S_1^{\Delta}H + S_1^{\sigma}H^{\Delta}\right)H^{-1}S_1^{-1}(A-S)\,H \\
&\qquad\qquad - (A-S)^{\Delta}H - (A-S)^{\sigma}H^{\Delta}\Big\} \\
&= -\left(S_1H\right)^{\sigma^{-1}}\Big\{\left(S_1^{\Delta}S_1^{-1} + S_1^{\sigma}SS_1^{-1}\right)(A-S)\,H \\
&\qquad\qquad - (A-S)^{\Delta}H - (A-S)^{\sigma}SH\Big\}.
\end{aligned}$$

Thus we have

$$-S_1^{\sigma}S_1HB^{\Delta}H^{-1} = \left(S_1^{\Delta}S_1^{-1} + S_1^{\sigma}SS_1^{-1}\right)(A-S) - (A-S)^{\sigma}S - (A-S)^{\Delta}.$$

Finally we obtain

$$A^{\Delta} = S^{\Delta} - (A-S)^{\sigma}S + \left(S_1^{\Delta}S_1^{-1} + S_1^{\sigma}SS_1^{-1}\right)(A-S) + S_1^{\sigma}S_1HB^{\Delta}H^{-1}.$$

Clearly, $B(t)$ is the unique solution of the IVP

$$B^{\Delta}(t) = F(t) \tag{3.9}$$

$$B(t_0) = S^{-1}(t_0)\left[A(t_0) - S(t_0)\right], \tag{3.10}$$

where $F(t) := \left[\left(H^{-1}(t)S_1^{-1}(t)\left[A(t) - S(t)\right]H(t)\right)\right]^{\Delta}$. Hence, the solutions of (3.7)–(3.6) and (3.9)–(3.10) are equivalent. □

Note that if $b(t) = 0$ for all $t \in \mathbb{T}$, then $d(t) = 0$. Thus (3.1) becomes

$$x^{\Delta}(t) = A(t)x(t) \tag{3.11}$$

and (3.3) becomes

$$y^{\Delta}(t) = By(t). \tag{3.12}$$

Corollary 3.2. *The inhomogeneous linear dynamic system* (3.1) *is reducible to*

$$y^{\Delta}(t) = By(t) + d(t) \tag{3.13}$$

with a constant matrix B *by the use of transformation* (3.4) *if and only if there exists an* $m \times m$ *regular real matrix* $S(t)$ *defined on* $\mathbb{T}$ *such that*

$$A^{\Delta}(t) = S^{\Delta}(t) + \left(S_1^{\Delta} S_1^{-1} + S_1^{\sigma} S S_1^{-1}\right)(t)\,(A - S)\,(t) - (A - S)^{\sigma}\,(t) S(t) \quad (3.14)$$

and

$$A(t_0) = S_1(t_0) B + S(t_0). \quad (3.15)$$

Additionally if the matrix $b(t)$ *satisfies the IVP*

$$b^{\Delta}(t) = -\left(S_1^{\Delta} + S_1^{\sigma} S\right)(t) S_1^{-1}(t) b(t), \quad b(t_0) = S_1^{-1}(t_0) d(t_0),$$

then (3.13) *becomes*

$$y^{\Delta}(t) = By(t) + d$$

with a constant matrix d*.*

Remark 3.3. If $\sigma(t) = t$, then we have $\mu(t) = 0$, $A^{\sigma} = A$ and $S_1 = S_1^{-1} = S_1^{\sigma} = I$. Thus the inhomogeneous linear dynamic system (3.1) is reducible to (3.3) by the help of the transformation of (3.4) if and only if there exists an $m \times m$ regular real matrix $S(t)$ such that

$$A'(t) = S'(t) + S(t)A(t) - A(t)S(t) + H(t)B'(t)H^{-1}(t)$$

and

$$A(t_0) = B(t_0) + S(t_0).$$

The inhomogeneous linear dynamic system (3.1) is reducible to (3.13) by the help of the transformation of (3.4) if and only if there exists an $m \times m$ regular real matrix $S(t)$ such that

$$A'(t) = S'(t) + S(t)A(t) - A(t)S(t) \quad (3.16)$$

and

$$A(t_0) = B + S(t_0). \quad (3.17)$$

Naturally, Theorem 3.1 and Corollary 3.2 and Remark 3.3 coincide with [14, Theorem 1 and Corollary 1] when $\mathbb{T} = \mathbb{R}$ and $b(t) = 0$ for all $t \in \mathbb{T}$.

Remark 3.4. If $\sigma(t) > t$, then the inhomogeneous linear dynamic system (3.1) is reducible to (3.3) by the help of the transformation of (3.4) if and only if there exists an $m \times m$ regular real matrix $S(t)$ such that

$$[A(t) - S(t)]^{\sigma} S_1(t) = S_1^{\sigma}(t)\,[A(t) - S(t)] + S_1^{\sigma}(t) S_1(t) H(t) \mu(t) B^{\Delta}(t) H^{-1}(t)$$

and

$$A(t_0) = S_1(t_0) B(t_0) + S(t_0).$$

The inhomogeneous linear dynamic system (3.1) is reducible to (3.13) by the help of the transformation of (3.4) if and only if there exists an $m \times m$ regular real matrix $S(t)$ such that

$$[A(t) - S(t)]^{\sigma} S_1(t) = S_1^{\sigma}(t)\,[A(t) - S(t)] \tag{3.18}$$

and

$$A(t_0) = S_1(t_0)B + S(t_0). \tag{3.19}$$

Naturally, Theorem 3.1 and Corollary 3.2 and Remark 3.4 coincide with [15, Theorem 2.1 and Corollary 2.2.] when $\mathbb{T} = \mathbb{N}$ and $b(t) = 0$ for all $t \in \mathbb{N}$.

4 Stability of Dynamic Systems

It is well known that exponential decay of the solution of a linear autonomous ordinary differential equation (ODE) $x'(t) = Ax(t)$, $t \in \mathbb{R}$ or of an autonomous difference equation (OΔE) $x(n+1) = Ax(n)$, $n \in \mathbb{Z}$, can be characterized by spectral properties of A. Namely, the solutions tend to 0 exponentially as $t \to \infty$, if and only if all the eigenvalues of $A \in \mathbb{C}^{m\times m}$ have negative real parts or a modulus smaller than 1, respectively (cf. Hahn [7, p. 14], Agarwal [1, p. 227]). C. Pötzsche et al in [13] generalized this classical result to linear time-invariant dynamic equations $x^{\Delta} = Ax$ on arbitrary time scales.

First, however, we fix some notation. In the following, $\mathbb{K}$ denotes the real ($\mathbb{K} = \mathbb{R}$) or the complex ($\mathbb{K} = \mathbb{C}$) field. For a complex number $z \in \mathbb{C}$, we denote by $\operatorname{Re} z$ and $\operatorname{Im} z$ the real and imaginary part, respectively, and $B_{\varepsilon}(z)$ is the open ball with center z and radius $\varepsilon > 0$ in the complex plane. As usual, $\mathbb{K}^{m\times m}$ is the space of square matrices with m rows, and $\sigma(A) \subset \mathbb{C}$ denotes the set of eigenvalues of a matrix $A \in \mathbb{K}^{m\times m}$.

Let $A : \mathbb{T}^{\kappa} \to \mathbb{K}^{m\times m}$ be rd-continuous and consider the m-dimensional linear system of dynamic equations

$$x^{\Delta} = A(t)x. \tag{4.1}$$

Let $\Phi_A : \{(t,\tau) \in \mathbb{T}^{\kappa} \times \mathbb{T}^{\kappa} : t \geq \tau\} \to \mathbb{K}^{m\times m}$ denote the *transition matrix* corresponding to (4.1), that is, $\varphi(t,\tau,\xi) = \Phi_A(t,\tau)\xi$ solves the initial value problem (4.1) with initial condition $x(\tau) = \xi$ for $\xi \in \mathbb{K}^m$ and $t, \tau \in \mathbb{T}$ with $t \geq \tau$.

We are interested in the stability of the equilibrium position $x^* = 0$ of system (4.1) and introduce the following definitions.

Definition 4.1 (Exponential Stability [13]). Let $\mathbb{T}$ be a time scale which is unbounded above. We call system (4.1) exponentially stable if there exists a constant $\alpha > 0$ such that for every $t_0 \in \mathbb{T}$ there exists a $M = M(t_0) \geq 1$ with

$$\|\Phi_A(t,t_0)\| \leq Me^{-\alpha(t-t_0)} \text{ for } t \geq t_0,$$

and uniformly exponentially stable if M can be chosen independently of t_0 in the definition of exponential stability.

We need the following well-known theorems to obtain a stability result [9, 13].

Proposition 4.2 (See [13]). *Let $\mathbb{T}$ be a time scale which is unbounded above and let $\lambda \in \mathbb{C}$. The scalar equation*

$$x^{\Delta} = \lambda x, \quad x \in \mathbb{C} \tag{4.2}$$

is exponentially stable if and only if one of the following conditions is satisfied for arbitrary $t_0 \in \mathbb{T}$:

(i) $\gamma(\lambda) := \limsup_{T \to \infty} \dfrac{1}{T - t_0} \displaystyle\int_{t_0}^{T} \lim_{s \searrow \mu(t)} \dfrac{\log|1 + s\lambda|}{s} \Delta t < 0,$

(ii) $\forall T \in \mathbb{T} : \exists t \in \mathbb{T}$ *with* $t > T$ *such that* $1 + \mu(t)\lambda = 0$,

where we use the convention $\log 0 = -\infty$ *in* (i).

Theorem 4.3 (Characterization of Exponential Stability [13]). *Let $\mathbb{T}$ be a time scale which is unbounded above and $A \in \mathbb{K}^{m \times m}$ be regressive. If for all eigenvalues of A there exists $\gamma > 0$ such that*

$$\gamma^{-1} \leq |1 + \mu(t)\lambda(t)| \ \textit{for} \ \ t \in \mathbb{T} \tag{4.3}$$

and $\sigma(A) \subset S_{\mathbb{C}}(\mathbb{T})$, then the time-invariant system

$$x^{\Delta} = Ax \tag{4.4}$$

is exponentially stable, where

$$S_{\mathbb{C}}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C} : \ \limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu(t)} \frac{\log|1 + s\lambda|}{s} \Delta t < 0, \ t_0 \in \mathbb{T} \right\}.$$

Theorem 4.4 (See [13]). *Let $\mathbb{T}$ be a time scale which is unbounded above and $A \in \mathbb{K}^{m \times m}$ and consider the linear system* (4.4)*. Then the following assertions hold:*

(i) *If $\sigma(A) \subset S_{\mathbb{C}}(\mathbb{T})$, then the time scale $\mathbb{T}$ has bounded graininess and if for all defective $\lambda \in \sigma(A)$ the scalar equation* (4.2) *is uniformly exponentially stable, then system* (4.4) *is exponentially stable.*

(ii) *If A is diagonalizable, then system* (4.4) *is exponentially stable if and only if $\sigma(A) \subset S_{\mathbb{C}}(\mathbb{T})$.*

Recall that an eigenvalue is called *defective* if it is not semi-simple, i.e., if geometric and algebraic multiplicities do not coincide.

Definition 4.5 (See [9]). Let $x_0 \in \mathbb{R}^m$ be a nonzero vector and $\Psi(t)$ be any fundamental matrix for the p-periodic system (3.11). The vector solution of the system with initial condition $x(t_0) = x_0$ is given by $x(t) = \Psi(t)\Psi^{-1}(t_0)x_0$. The operator $M : \mathbb{R}^m \to \mathbb{R}^m$ given by

$$M(x_0) = \Phi_A(t_0 + p, t_0)x_0 = \Psi(t_0 + p)\Psi^{-1}(t_0)x_0$$

is called a monodromy operator. The eigenvalues of the monodromy operator are called the Floquet (or characteristic) multipliers of the system (3.11).

Theorem 4.6 (See [9]). *Suppose that $\lambda_1, \ldots, \lambda_m$ are the Floquet multipliers for the p-periodic system*

$$x^{\Delta}(t) = A(t)x(t), \quad x(t_0) = x_0, \tag{4.5}$$

where $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{K}^{m \times m})$ and p-periodic for all $t \in \mathbb{T}$. Then the following assertions hold:

(i) *If all the Floquet multipliers have modulus less one, then the system* (4.5) *is exponentially stable.*

(ii) *If all the Floquet multipliers have modulus less than or equal to one, then the system* (4.5) *is stable.*

Definition 4.7 (See [6]). Let A be an $m \times n$-matrix-valued function on $\mathbb{T}$. Then its spectral radius $\rho(A)$ is defined as

$$\rho(A) := \max \{|\lambda| : \lambda \text{ is an eigenvalue of } A\}.$$

Since stability of (3.1) and (3.11) are equivalent, in view of Proposition 4.2, Theorem 4.3, Theorem 4.4 and Theorem 4.6, we obtain from Corollary 3.2 the following new stability criteria for (3.1).

Theorem 4.8. *Let $\mathbb{T}$ be a time scale which is unbounded above. Assume there exists a regressive and periodic matrix $S(t)$ which satisfies* (3.14)–(3.15) *and the following conditions:*

(i) $S_1^{-1}(t_0)\,[A(t_0) - S(t_0)] \in \mathcal{R}$*;*

(ii) *all the Floquet multipliers of $S(t)$ have modulus less one;*

(iii) $\rho(S_1^{-1}(t_0)\,[A(t_0) - S(t_0)]) \subset S_{\mathbb{C}}(\mathbb{T})$*;*

(iv) *for all $\lambda \in \rho(S_1^{-1}(t_0)\,[A(t_0) - S(t_0)])$, the inequality* (4.3) *holds.*

Then (3.1) *is exponentially stable.*

Definition 4.9 (See [13]). Given a time scale $\mathbb{T}$ which is unbounded above, we define for arbitrary $t_0 \in \mathbb{T}$

$$S_{\mathbb{C}}(\mathbb{T}) = \left\{ \lambda \in \mathbb{C} : \limsup_{T \to \infty} \frac{1}{T - t_0} \int_{t_0}^{T} \lim_{s \searrow \mu(t)} \frac{\log|1 + s\lambda|}{s} \Delta t < 0, \ t_0 \in \mathbb{T} \right\}$$

and

$$S_{\mathbb{R}}(\mathbb{T}) = \{\lambda \in \mathbb{R} : \ \forall T \in \mathbb{T} : \exists t \in \mathbb{T} \text{ with } t > T \text{ such that } 1 + \mu(t)\lambda = 0\}.$$

The set of exponential stability for the time scale $\mathbb{T}$ is then defined by

$$S(\mathbb{T}) = S_{\mathbb{C}}(\mathbb{T}) \cup S_{\mathbb{R}}(\mathbb{T}).$$

Theorem 4.10. *Let $\mathbb{T}$ be a time scale which is unbounded above and has bounded graininess. Assume there exists a regressive and periodic matrix $S(t)$ which satisfies* (3.14)–(3.15) *and following conditions:*

(i) *All the Floquet multipliers of $S(t)$ have modulus less one;*

(ii) *for all defective $\lambda \in \rho(S_1^{-1}(t_0)\,[A(t_0) - S(t_0)])$, the equation* (4.2) *is uniformly exponentially stable;*

(iii) $\rho(S_1^{-1}(t_0)\,[A(t_0) - S(t_0)]) \subset S(\mathbb{T})$.

Then (3.1) *is exponentially stable.*

Theorem 4.11. *Let $\mathbb{T}$ be a time scale which is unbounded above. Then* (3.1) *is exponentially stable if there exists a regressive and periodic matrix $S(t)$ which satisfies* (3.14)–(3.15) *and the following conditions:*

(i) *All the Floquet multipliers of $S(t)$ have modulus less one;*

(ii) $\rho(S_1^{-1}(t_0)\,[A(t_0) - S(t_0)]) \subset S(\mathbb{T})$*;*

(iii) $S_1^{-1}(t_0)\,[A(t_0) - S(t_0)]$ *is diagonalizable.*

Corollary 4.12. *Let $\mathbb{T}$ be a time scale which is unbounded above with $\mu(t) > 0$. Then* (3.1) *is exponentially stable if one of the following conditions holds for a regressive and periodic matrix $S(t)$ which satisfies* (3.18)–(3.19)*:*

(i) $S_1^{-1}(t_0)\,[A(t_0) - S(t_0)] \in \mathcal{R}$ *and the conditions* (i) *and* (ii) *of Theorem 4.8 hold;*

(ii) *the conditions* (i) *and* (ii) *of Theorem 4.10 hold;*

(iii) $S_1^{-1}(t_0)\,[A(t_0) - S(t_0)]$ *is diagonalizable and the conditions* (i) *and* (ii) *of Theorem 4.11 hold.*

Example 4.13. Let $\mathbb{T} = \mathbb{N} = \{0, 1, 2, 3, \ldots\}$. Then we can consider (4.1) in the form of

$$x(n+1) = A_1(n)x(n), \tag{4.6}$$

and the conditions (3.18) and (3.19) become

$$A_1(n+1)S_1(n) = S_1(n+1)A_1(n) \tag{4.7}$$

and

$$A_1(n_0) = S_1(n_0)B_1, \tag{4.8}$$

where $A_1(n) = A(n) + I$, $S_1(n) = S(n) + I$ and $B_1 = B + I$. Consider the system

$$x(n+1) = \begin{pmatrix} -\frac{1}{4}(-1)^n & \frac{1}{8}\beta^{n+1} \\ 0 & -\frac{1}{8}(-1)^n \end{pmatrix} x(n), \quad 0 < \beta < 1. \tag{4.9}$$

If we define a 2-periodic matrix

$$S_1(n) = \begin{pmatrix} \frac{\beta}{2}(-1)^n & 0 \\ 0 & \frac{1}{2}(-1)^{n+1} \end{pmatrix},$$

then the matrix $S_1(n)$ satisfies the condition (4.7) and all the Floquet multipliers of $S_1(n)$, $\lambda_1 = -\frac{\beta^2}{4}$ and $\lambda_2 = -\frac{1}{4}$, have modulus less one. From (4.8) we get

$$B_1 = S_1^{-1}(0)A_1(0) = \begin{pmatrix} \frac{1}{2}\beta & \frac{1}{4} \\ 0 & -\frac{1}{4} \end{pmatrix}$$

and $\rho(B_1) < 1$. Applying Theorem 4.8 we see that the zero solution of (4.9) or equivalently (4.1) is exponentially stable.

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