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Nonlinear Monotone Potential Operators: From Nonlinear ODE and PDE to Computational Material Sciences

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Abstract

This paper deals with boundary value problems for nonlinear monotone potential operators. An analysis of the nonlinear (monotone potential) Sturm–Liouville operator $Au := -\left(k((u')^2)u'(x)\right)' + q(x)u(x)$, $x \in (a, b)$ shows that the potential of this operator as well as the potential of related boundary value problems play an important role not only for solvability of these problems, but also for linearization and convergence of solutions of corresponding linearized problems. This approach is then applied to boundary value problems for nonlinear elliptic equations with nonlinear monotone potential operators. As an extension of obtained results in the second part of the paper some applications to computational material science (COMMAT) are proposed. In this context, boundary value problems related to elastoplastic torsion of a bar, and the bending problem for an incompressible plate are considered.

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1 Introduction

Second and fourth-order nonlinear ordinary and elliptic partial differential equations form basis of mathematical models of various steady-state phenomena and processes

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in mechanics, physics and many other areas of science (see, for example, [4, 15, 16, 24]). One important class of these equations is related to nonlinear monotone potential operators [20–22, 24]. In the presented paper we study solvability and linearization of boundary value problems related to nonlinear monotone potential operators. The one-dimensional model of these problems is the boundary value problem

$$Au \equiv -\left(k((u')^2)u'\right)' + g(u) = f(x), \quad a < x < b, \tag{1.1}$$
$$u(a) = 0, \quad \left(k((u')^2)u'(x)\right)_{x=b} = \varphi. \tag{1.2}$$

The weak solution $u \in \overset{\circ}{H}{}^1[a, b]$ of the nonlinear boundary value problem (1.1)–(1.2) (subsequently, the problem (NBVP)) satisfies the integral identity

$$\begin{aligned} a(u; u, v) \ &:= \ \int_a^b [k((u')^2)u'(x)v'(x) + g(u)v(x)]dx \\ &= \ \int_a^b f(x)v(x)dx + \varphi v(b) := l(v), \end{aligned} \tag{1.3}$$

for all $v \in \overset{\circ}{H}{}^1[a, b]$, where $\overset{\circ}{H}{}^1[a, b] = \{u(x) \in H^1[a, b] : u(a) = 0\}$ and $H^1[a, b]$ is the Sobolev space [1]. Here and below $a(u; u, v) := \langle Au, v\rangle$.

For the linear operator $A_0 u \equiv -(k(x)u')' + q(x)u$ the left-hand side of the integral identity (1.3) corresponds to the symmetric bilinear form (functional)

$$a(u, v) := \int_a^b \left[k(x)u'(x)v'(x) + q(x)u(x)v(x)\right] dx,$$

and the well-known conditions

$$c_1 \geq k(x) \geq c_0 > 0, \quad c_2 \geq q(x) \geq 0, \tag{1.4}$$

guarantee the existence of the unique solution $u \in \overset{\circ}{H}{}^1[a, b]$, if $f \in H^0[a, b]$. The case $g(u) = q(x)u + p(x)$ corresponds to the quasilinear equation

$$Au \equiv -\left(k((u')^2)u'\right)' + q(x)u = f(x) - p(x), \quad a < x < b, \tag{1.5}$$

which can be considered as one-dimensional analogue of the well-known Plateau [5] and Kachanov equations [15]. Specifically, when $k(\xi) = (1 + \xi)^{-1/2}$, $\xi = (u')^2$, and $q(x) = 0$, the operator $Au \equiv -\left(k((u')^2)u'\right)'$ is a one-dimensional Plateau operator. Further, the case $k(\xi) = k_0\xi^{0.5(\kappa-1)}$, $\kappa \in (0, 1]$, $k_0 > 0$ and $q(x) = 0$, corresponds to the one-dimensional analogue of Kachanov's equation for engineering materials [23]. The differential operator $Au \equiv -\left(k((u')^2)u'\right)' + q(x)u$ sometimes is defined to be the nonlinear Sturm–Liouville operator.

Comparing the linear equation $-(k(x)u')' + q(x)u = f(x)$ with the nonlinear equation $-(k(x)u')' + g(u) = f(x)$, and taking account the second condition (1.4) for the linear equation, we conclude that extension of this condition for the function $g(u) = q(\eta)u$,

$\eta = u^2$, is the condition $c_2 \geq q(\eta) \geq 0$. Otherwise, if this condition does not hold, then the problem

$$\begin{cases} Au \equiv -(k(x)u')' + q(u^2)u = f(x), \quad a < x < b, \\ u(a) = 0, \quad \left(k\left((u')^2\right)u'(x)\right)_{x=b} = \varphi \end{cases}$$

may have an infinite number of solutions. For example, if

$$\lim_{u \to \pm\infty} \frac{g(u)}{u} \to \infty,$$

then, as it was shown in [2], the problem

$$\begin{cases} -u'' + g(u) = f(x), \quad x \in (a,b), \\ u(a) = 0, \quad u'(b) = \varphi \end{cases}$$

has an infinite number of solutions $u \in C^2[a,b]$, for every $f \in C^0[a,b]$.

The problems related to solvability of boundary value problems for the quasilinear and nonlinear equations of type (1.1) have been considered by various authors (see [3–5, 16, 18] and references therein). Some applications to evolution problems are presented in [12, 13]. Note that the iteration scheme for the quasilinear equations arising from elastoplasticity has first been given in [15] and then developed in [3, 6, 7]. An abstract iteration scheme and convergence criteria for these type of nonlinear problems were proposed in [8].

In this paper we present some review related to solvability of nonlinear boundary value problems for the second and fourth-orders nonlinear monotone potential differential operators. We also analyze questions related to linearization of these problems and convergence of approximate solutions in appropriate Sobolev spaces. The main subject of the analysis is to derive explicit, from the point of view practice, sufficient conditions for the leading coefficient $k = k(\xi)$, $\xi := |\nabla u|^2$. In Section 2 we discuss solvability of the problem (NBVP) in $\mathring{H}^1[a,b] \cap H^2[a,b]$ for the Sturm–Liouville operator $Au := -\left(k((u')^2)u'(x)\right)' + q(x)u(x)$. In Section 3 we extend the obtained results to the case of the nonlinear elliptic operator $Au \equiv -\nabla\left(k(|\nabla u|^2)\nabla u\right) + q(x)u$, and derive sufficient conditions for linearization and H^1-convergence of the approximate solution. Linearization of nonlinear problems, monotonicity on iterations and convergence issues for an abstract, as well as for concrete variational problems are discussed in Section 4. As a first application, in Section 5 the mathematical model of an elastoplastic torsion of a strain hardening bar is considered within the range of J_2-deformation theory. In the final Section 6 an elastoplastic bending problem for an incompressible thin plate is considered. Both applications show that the sufficient conditions obtained for abstract monotone potential elliptic operators are almost same with the basic conditions of J_2-deformation theory.

2 The Problem (NBVP) in $\mathbb{R}^1$

As a sample model consider first the boundary value problem (1.1)–(1.2) with $g(u) \equiv q(x)u$. The weak solution $u \in \mathring{H}^1[a,b] \cap H^2[a,b]$ of this problem is defined as a solution of the integral identity (or variational problem)

$$\begin{aligned} a(u,u;v) \ &:= \ \int_a^b [k((u')^2)u'v' + q(x)uv]dx \\ &= \ \int_a^b f(x)v(x)dx + \varphi v(b) := l(v), \quad \forall v \in \mathring{H}^1[a,b]. \end{aligned} \tag{2.1}$$

The multiple coefficient $q(x)$ and the source function $f(x)$ are assumed to be in

$$H^0[a,b] \equiv L_2[a,b].$$

To study solvability of the nonlinear problem (2.1), we shall use the variational approach and monotone operator theory (see [4, 19–22]). For this aim let us introduce the functional

$$J(u) = \frac{1}{2}\int_a^b \left\{ \int_0^{(u')^2} k(\xi)d\xi + q(x)u^2 \right\} dx \tag{2.2}$$

and calculate the first and the second Gateaux derivatives of this functional. We have

$$J'(u;v) = \int_a^b [k((u')^2)u'v' + q(x)uv]dx, \quad v \in \mathring{H}^1[a,b], \tag{2.3}$$

$$J''(u;v,h) = \int_a^b \{[k((u')^2)v'h' + 2k'((u')^2)u'h'u'v'] + q(x)vh\}dx, \tag{2.4}$$

for $v, h \in \mathring{H}^1[a,b]$. It is seen from the left-hand side of (1.5) and from (2.3) that $a(u,u;v) = J'(u,v)$ for all $v \in \mathring{H}^1[a,b]$ and hence the nonlinear operator A defined by (1.5) is a potential operator, with potential $J(u)$ defined by (2.2). In this context the functional $\mathcal{P}(u) = J(u) - l(u)$ is defined to be the potential of the variational problem (2.1).

Theorem 2.1. *Let us assume that in addition to conditions* (1.4)*, the coefficient* $k = k(\xi)$ *is piecewise differentiable and satisfies the condition*

$$k(\xi) + 2k'(\xi)\xi \geq \gamma_0 > 0, \quad \xi \in [\xi_*, \xi^*], \tag{2.5}$$

where $\xi_* = \inf_{[a,b]} (u'(x))^2 > 0$ *and* $\xi^* = \sup_{[a,b]} (u'(x))^2 < +\infty$*. If* $q(x), F(x) \in H^0[\xi_*, \xi^*]$*, then the problem* (*NBVP*) *has a unique solution* $u \in \mathring{H}^1[a,b] \cap H^2[a,b]$.

Proof. Due to the Browder–Minty theorem (see [4, 19]) the operator equation $Au = F$, the nonlinear operator A defined by (1.5), has a unique solution, if a potential operator A is bounded, radially continuous, coercive and uniform monotone. Hence, to prove the theorem we only need to show that the operator A defined by (1.5) has the above properties. Conditions (1.5) imply the boundedness of the operator A:

$$|\langle Au, v\rangle| \le c_1 \left|\int_a^b u'(x)v'(x)dx\right| + c_2 \left|\int_a^b u(x)v(x)dx\right| \le \max\{c_1; c_2\}\|u\|_1\|v\|_1.$$

Radial continuity of the operator A follows from continuity of the function

$$t \mapsto \langle A(u + tv), v\rangle,$$

for all $t \in \mathbb{R}$, and for fixed $u, v \in \mathring{H}^1[a, b] \cap H^2[a, b]$. Substituting $h = v$ in (2.4) and using condition (2.5), we obtain

$$J''(u; v, v) = \int_a^b \{[k((u')^2) + 2k'((u')^2)(u')^2](v')^2 + q(x)v^2\}dx \ge \gamma_0\|v'\|_0^2,$$

for all $h \in \mathring{H}^1[a, b]$. Applying the Poincaré inequality $\|v'\|_0^2 \ge c_\Omega^2\|v\|_0^2$ ($c_\Omega^2 = 2/(b-a)$), we obtain

$$J''(u; v, v) \ge \frac{\gamma_0 c_\Omega^2}{1 + c_\Omega^2}\|u\|_1^2, \quad \forall v \in \mathring{H}^1[a, b], \quad \gamma_0 > 0,$$

which means the uniform monotonicity

$$\langle Au - Av, u - v\rangle \ge \gamma_1\|u - v\|_1^2, \;\; \gamma_1 = \frac{\gamma_0 c_\Omega^2}{1 + c_\Omega^2} > 0, \quad \forall u, v \in \mathring{H}^1[a, b] \tag{2.6}$$

of the nonlinear operator. Since $A\theta = \theta$, where $\theta \in \mathring{H}^1[a, b]$ is the zero element, inequality (2.6) also implies the coercivity of the operator A, with the same coercivity constant $\gamma_1 > 0$. Due to the Browder–Minty theorem the problem (NBVP) has a unique solution in $\mathring{H}^1[a, b]$. □

Remark 2.2. Condition (2.5) has first been used in the form $k'(s^2)s^2 + k(s^2) \ge d > 0$ for the function $k(\xi) : \mathbb{R}_+ \to \mathbb{R}_+$ in the classical Kachanov method for stationary conservation laws (see [24, page 544]). In the case of Dirichlet problem for the nonlinear operator $Au \equiv -\nabla\left(k(|\nabla u|^2)\nabla u\right)$, the boundedness $\|\nabla u\|_C \le \xi^*$ of the norm $\|\nabla u\|_C$ has been proved in [11], where $\xi^* > 0$ is a positive constant. Hence condition (2.5) can also be considered as an extension of the above condition for the case $\xi \in [\xi_*, \xi^*]$.

3 Solvability of the Problem (NBVP) in $\mathbb{R}^n$ ($n > 1$)

Consider now the problem (NBVP) in $\mathbb{R}^n$ ($n > 1$) for the nonlinear elliptic operator

$$Au \equiv -\nabla\left(k(|\nabla u|^2)\nabla u\right) + q(x)u, \quad x \in \Omega \subset \mathbb{R}^n. \tag{3.1}$$

Specifically, we consider the mixed boundary value problem

$$\begin{cases} Au = F, \quad \text{in} \quad \Omega, \\ u = 0, \quad \text{on} \quad \Gamma_1 \subset \partial\Omega, \\ k(|\nabla u|^2)\dfrac{\partial u}{\partial n} = \varphi, \quad \text{on} \quad \Gamma_2 \subset \partial\Omega, \end{cases} \tag{3.2}$$

where $\overline{\Gamma}_1 \cup \Gamma_2 = \partial\Omega$, $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain with a piecewise smooth boundary $\partial\Omega$. The weak solution $u \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$ of the nonlinear problem (3.2) is defined as a solution of the variational problem

$$a(u, u; v) = l(v), \quad \forall v \in \mathring{H}^1(\Omega) \cap H^2(\Omega), \tag{3.3}$$

where

$$a(u; v, h) = \int_\Omega \{k(|\nabla u|^2)\nabla v \nabla h + q(x)vh\}dx, \quad l(v) = \int_\Omega F v dx + \int_{\Gamma_2} \varphi v ds,$$

and $\mathring{H}^1(\Omega) = \{u \in H^1(\Omega) : u(s) = 0, s \in \Gamma_1\}$. It is easy to verify that the functional

$$J(u) = \frac{1}{2}\int_\Omega \left\{ \int_0^{|\nabla u|^2} k(\xi)d\xi + q(x)u^2 \right\} dx$$

represents the potential of the operator A, defined by (2.1), since

$$J'(u; v) = \int_\Omega \{k(|\nabla u|^2)\nabla u \nabla v + q(x)uv\}dx \equiv a(u; v, h).$$

Calculating the second Gateaux derivative of this functional we obtain

$$\begin{aligned} J''(u; v, h) &\equiv \frac{d}{dt}\langle J'((u + th); v)\rangle_{t=0} \\ &= \frac{d}{dt}\left(\int_\Omega \{k(|\nabla(u + th)|^2)\nabla(u + th)\nabla v + q(x)(u + th)v\}dx \right)_{t=0} \\ &= \int_\Omega \{k(|\nabla u|^2)\nabla h \nabla v + 2k'(|\nabla u|^2)\nabla u \nabla h \nabla u \nabla v + q(x)vh\}dx. \end{aligned}$$

Hence for $h = v$ we have

$$J''(u; v, v) = \int_\Omega \{k(|\nabla u|^2)|\nabla v|^2 + 2k'(|\nabla u|^2)\nabla u \nabla v \nabla u \nabla v + q(x)v^2\}dx. \tag{3.4}$$

By the inequality $\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2$, for all $a_i, b_i \in \mathbb{R}^1$, we conclude

$$|\nabla u \nabla v|^2 \equiv \left(\sum_{i=1}^n \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_i}\right)^2 \leq \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2 \sum_{i=1}^n \left(\frac{\partial v}{\partial x_i}\right)^2 \equiv |\nabla u|^2 \, |\nabla v|^2.$$

Assuming the condition

$$k'(\xi) \leq 0, \quad \xi \in [\xi_*, \xi^*], \tag{3.5}$$

we obtain

$$2k'(|\nabla u|^2)|\nabla u \nabla v|^2 \geq 2k'(|\nabla u|^2)|\nabla u|^2 |\nabla v|^2.$$

Taking into account this inequality in the right-hand of (3.4) we obtain the following upper estimate

$$J''(u; v, v) \geq \int_\Omega \left\{ [\, k(|\nabla u|^2) + 2k'(|\nabla u|^2)|\nabla u|^2 \,] \, |\nabla v|^2 + q(x)v^2 \right\} dx.$$

This estimate with the condition (2.5) and the Poincaré inequality $\|\nabla v\|_0^2 \geq c_\Omega^2 \|v\|_0^2$, $c_\Omega^2 > 0$, implies the positivity of the second Gateaux derivative

$$J''(u; v, v) \geq \frac{\gamma_0 c_\Omega^2}{1 + c_\Omega^2} \|u\|_1^2, \;\; \gamma_0 > 0.$$

This means the strong convexity of the functional, which implies the uniform monotonicity of the nonlinear elliptic operator, defined by (3.1). Hence we get the following result.

Theorem 3.1. *Let us assume that in addition to the conditions of Theorem 2.1 and the conditions $F \in H^0(\Omega)$, $\varphi \in H^0(\Gamma)$, the coefficient $k(\xi)$ is piecewise differentiable and satisfies condition* (3.5)*. Then the problem* (*NBVP*) *has a unique solution $u \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$.*

The above theorems show that different from the one-dimensional case, for solvability of the multi-dimensional problem (NBVP) ($n > 1$), one needs to impose the additional conditions (3.5).

4 Linearization of Nonlinear Problems

Consider first the abstract equation

$$Au = F, \quad u \in H, \quad F \in H^*, \tag{4.1}$$

for the nonlinear strong monotone potential operator, acting from the Hilbert space H to its dual H^*. Assume that $a(u; \cdot, \cdot)$ is a bounded symmetric bilinear form generated by the operator A, i.e., $a(u; u, v) = \langle Au, v \rangle$, $\forall u, v \in H$. Suppose that $A\Theta = \Theta$, i.e., A transforms zero element of H to zero element of H^*. These mean that the operator satisfies the conditions

$$\begin{cases} \langle Au - Av, u - v \rangle \geq \gamma_1 \|u - v\|_H^2, \;\; \gamma_1 > 0, \quad \forall u, v \in H, \\ a(u; u, u) \geq \gamma_1 \|u\|_H^2, \quad \forall u \in H, \\ |a(u; u, v)| \leq M_1 \|u\|_H \|v\|_H, \quad \forall u, v \in H. \end{cases} \tag{4.2}$$

We denote again by $J(u)$ the potential of the operator A, and by $\mathcal{P}(u) = J(u) - l(u)$, $l(u) = \langle F, u \rangle$, the potential of the corresponding variational problem

$$a(u; u, v) = l(v), \quad \forall v \in H. \tag{4.3}$$

A monotone iteration scheme for the abstract variational problem (4.3) corresponding to the nonlinear problem (4.1) has been proposed in [8]. For this aim, the inequality

$$0.5a(u; v, v) - 0.5a(u; u, u) - J(v) + J(u) \geq 0, \quad \forall u, v \in H \tag{4.4}$$

has been introduced in [8], as a convexity argument for nonlinear monotone potential operators. To analyze this inequality from the point of view the leading coefficient $k = k(\xi)$, let us consider the one-dimensional variational problem (2.1), assuming without loss of generality, that $q(x) \equiv 0$. For this nonlinear problem we may rewrite this inequality as

$$\frac{1}{2} \int_a^b \left\{ k((u')^2)[(v')^2 - (u')^2] - \int_{(v')^2}^{(u')^2} k(\xi) d\xi \right\} dx \geq 0, \tag{4.5}$$

for all $u, v \in \overset{\circ}{H}{}^1[a, b] \cap H^2[a, b]$, using (2.2) and (2.3). Letting $\xi_1 = (v')^2$ and $\xi_2 = (u')^2$ in (4.5), we conclude that the inequality

$$k(\xi_1)[\xi_2 - \xi_1] - \int_{\xi_1}^{\xi_2} k(z) dz \geq 0, \quad \forall \xi_1, \xi_2 \in [\xi_*, \xi^*] \tag{4.6}$$

is a sufficient condition for fulfilment of the convexity argument, i.e., inequality (4.4), for the variational problem (2.1). Introducing the new function

$$\mathcal{K}(\xi) = \int_{\xi_*}^{\xi} k(z) dz, \quad \forall \xi \in [\xi_*, \xi^*],$$

we conclude that inequality (4.6) means concavity of the function $\mathcal{K}(\xi)$. Since $\mathcal{K}'(\xi) = k(\xi)$, the condition $k'(\xi) < 0$, $\forall \xi \in [\xi_*, \xi^*]$, is evidently a sufficient condition for fulfilment of the convexity argument for the variational problem (2.1).

Theorem 4.1. *Let us assume that in addition to conditions of Theorem 2.1, the coefficient $k(\xi)$ is piecewise differentiable and satisfies condition* (4.6)*. Then the convexity argument* (4.4) *holds in the one-dimensional variational problem* (2.1).

The same result remains true for the n-dimensional problem (3.1)–(3.2). Detailed proofs of these result are given in [8].

The first application of the convexity argument for nonlinear monotone potential operators, is the monotone iteration scheme

$$a(u^{(n-1)}; u^{(n)}, v) = l(v), \quad \forall v \in H, \quad n = 1, 2, 3, \ldots, \tag{4.7}$$

for the abstract variational problem (4.3). Here $u^{(0)} \in H$ is an initial iteration. This results asserts that the sequence of potentials $\{\mathcal{P}(u^{(n)})\}$ of the linearized problem (4.7) is a monotone decreasing one, i.e., $\mathcal{P}(u^{(n+1)}) \leq \mathcal{P}(u^{(n)})$, $\forall n = 1, 2, 3, \ldots$ (see [8, Lemma 1]). Since

$$\mathcal{P}(u^{(n)}) = \frac{1}{2} a(u^{(n-1)}; u^{(n)}, u^{(n)}) - l(u^{(n)}), \quad n = 1, 2, 3, \ldots, \tag{4.8}$$

substituting in (4.7) $v = u^{(n)}$ we get $a(u^{(n-1)}; u^{(n)}, u^{(n)}) = l(u^{(n)})$. This, with (4.8), implies

$$\mathcal{P}(u^{(n)}) = -\frac{1}{2} a(u^{(n-1)}; u^{(n)}, u^{(n)}) < 0, \quad n = 1, 2, 3, \ldots. \tag{4.9}$$

Thus the sequence $\{\mathcal{P}(u^{(n)})\}$ is bounded below, and hence it converges. Using this lemma, it is proved that (see [8, Theorem 1]), the difference $\|u - u^{(n)}\|_H$ between the solution $u \in H$ of the variational problem (4.3) and its approximation $u^{(n)} \in H$ obtained by the iteration scheme (4.7), can be estimated as

$$\|u - u^{(n)}\|_H \leq \frac{\sqrt{2} M_1}{\gamma_1^{3/2}} [\mathcal{P}(u^{(n-1)}) - \mathcal{P}(u^{(n)})]^{1/2}, \quad n = 1, 2, 3, \ldots. \tag{4.10}$$

This estimate shows that the sequence of approximate solutions $\{u^{(n)}\} \subset H$, obtained by the iteration scheme (4.7) converges to the solution $u \in H$ of the variational problem (4.3) in H-norm.

Let us apply the abstract monotone iteration scheme (4.7) to the problem (NBVP) given by (4.1)–(4.2). The sequence of approximate solutions $\{u^{(n)}\} \subset \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$ is defined from the linearized mixed boundary value problem

$$\left\{ \begin{array}{l} -\nabla\left(k(|\nabla u^{(n-1)}|^2)\nabla u^{(n)}\right) + q(x)u^{(n)} = F, \quad \text{in} \quad \Omega, \\ u^{(n)} = 0, \quad \text{on} \quad \Gamma_1 \subset \partial\Omega, \\ k(|\nabla u^{(n-1)}|^2)\dfrac{\partial u^{(n)}}{\partial n} = \varphi, \quad \text{on} \quad \Gamma_2 \subset \partial\Omega. \end{array} \right. \tag{4.11}$$

The potential of this linearized problem is defined as

$$\begin{aligned} \mathcal{P}(u^{(n)}) &:= J(u^{(n)}) - l(u^{(n)}) \\ &= \frac{1}{2} \int_\Omega \left\{ k(|\nabla u^{(n-1)}|^2)|\nabla u^{(n)}|^2 + q(x)(u^{(n)})^2 \right\} dx - \int_\Omega F(x) u^{(n)} dx. \end{aligned} \tag{4.12}$$

Hence

$$a(u^{(n-1)}; u^{(n)}, v) := \int_\Omega \left\{ k(|\nabla u^{(n-1)}|^2)\nabla u^{(n)} \nabla v + q(x)u^{(n)} v \right\} dx.$$

Evidently, if the coefficient $k(\xi)$ is piecewise differentiable and satisfies conditions (1.4), (2.5) and (3.5), the coefficient $q(x)$ satisfies conditions (1.4), and $q(x), F(x) \in H^0[\xi_*, \xi^*]$, then all conditions in (4.2) hold. Thus the above results obtained for the abstract monotone iteration scheme (4.7) remain true also for the problem (NBVP) given by (4.1)–(4.2).

Theorem 4.2. *Let us assume that the conditions of Theorem 3.1 hold. Then*

(i) *the sequence of potentials* $\{\mathcal{P}(u^{(n)})\}$ *defined by* (4.12) *is a monotone decreasing and convergent one;*

(ii) *the sequence of approximate solutions* $\{u^{(n)}\} \subset \mathring{H}^1(\Omega) \cap H^2(\Omega)$*, defined by* (4.11)*, converges to the weak solution* $u \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$ *of the problem* (*NBVP*) (4.1)–(4.2) *in* H^1*-norm;*

(iii) *for the rate of convergence the following estimate holds:*

$$\|u - u^{(n)}\|_H \leq \frac{\sqrt{2}M_1}{\gamma_1^{3/2}}[\mathcal{P}(u^{(n-1)}) - \mathcal{P}(u^{(n)})]^{1/2}, \quad n = 1, 2, 3, \ldots, \tag{4.13}$$

where $M_1 = \max\{c_1, c_2\} > 0$*, and* $\gamma_1 > 0$ *is defined in* (2.6).

The main distinguished feature of this theorem is that it requires the same conditions as Theorem 3.1. In other words, for the convergence of the monotone iteration (4.12) scheme, no additional conditions are required.

5 An Elastoplastic Torsion of a Strain Hardening Bar

The concept of torsional rigidity is well known in structural mechanics as one of main characteristics of a beam of uniform cross section during elastoplastic torsion. Torsional rigidity is defined as the torque required for per unit angle of twist $\varphi > 0$ per unit length, when the elastic modulus of the material is set equal to one [11, 17]. Specifically, if $u = u(x);\ \ x = (x_1, x_2) \in \Omega \subset \mathbb{R}^2$ denotes the deflection function, then the torque (or torsional rigidity) is defined as to be

$$\mathcal{T}[u; g, \varphi] = 2 \int_\Omega u(x; g; \varphi) dx, \tag{5.1}$$

where $\Omega := (0, l_1) \times (0, l_2)$, $l_1, l_2 > 0$, denotes the cross section of a bar, and is assumed to be in $\mathbb{R}^2$, with piecewise smooth boundary. For given $g = g(\xi^2)$ and $\varphi > 0$, the function $u(x) := u(x; g, \varphi)$ is the solution of the nonlinear boundary value problem

$$\begin{cases} -\nabla(g(|\nabla u|^2)\nabla u) = 2\varphi, \quad x \in \Omega \subset \mathbb{R}^2, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{5.2}$$

corresponding to a given function $g = g(\xi^2)$. The boundary value problem (5.2) represents an elastoplastic torsion of a strain hardening bar, which lower end is fixed, i.e., rigid clamped. The function $u(x)$ is the Prandtl's stress function and $\xi(u) = [(\partial u/\partial x_1)^2 + (\partial u/\partial x_2)^2]^{1/2}$ is the stress intensity. In view of J_2-deformation theory, the function $g = g(\xi^2)$, $\xi^2 = |\nabla u|^2$, defined to be the *plasticity function*, describes elastoplastic properties of a homogeneous isotropic material, and satisfies the conditions (see [6, 9, 11, 15, 17, 23] and references therein)

$$\begin{cases} 0 \le c_0 \le g(\xi^2) \le c_1; \\ g'(\xi^2) \le 0; \\ g(\xi^2) + 2\xi^2 g'(\xi^2) \ge \gamma_0 > 0, \quad \xi \in [\xi_*, \xi^*], \ \ \xi_* > 0; \\ g(\xi^2) = G, \quad \xi \in [\xi_*, \xi_0], \quad \xi_0 \in (\xi_*, \xi^*). \end{cases} \tag{5.3}$$

Here $G > 0$ is the shift modulus and $\xi_0 > 0$ is assumed to be the elasticity limit of a material. Note that $G = E/(1 + \nu)$, where $E > 0$ is the elasticity modulus and $\nu \in (0, 0.5)$ is the Poisson coefficient.

Evidently the variational problem (4.7) corresponds here to the linearized problem

$$\begin{cases} -\nabla(g(|\nabla u^{(n-1)}|^2)\nabla u^{(n)}) = 2\varphi, \quad x \in \Omega; \\ u^{(n)}(s) = 0, \quad s \in \partial\Omega, \end{cases} \tag{5.4}$$

where $n = 1, 2, 3, \ldots$ and $u^{(0)} \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$ is the initial iteration. The potential of the linearized problem (5.4) is defined to be

$$\mathcal{P}(u^{(n)}) = \frac{1}{2}\int_\Omega \left\{g(|\nabla u^{(n-1)}|^2)|\nabla u^{(n)}|^2\right\} dx - 2\varphi \int_\Omega u^{(n)} dx,$$
$$u^{(n)} \in \mathring{H}^1(\Omega) \cap H^2(\Omega), \ \ n = 1, 2, 3, \ldots.$$

With definition (5.1) of the torque, this potential has the form

$$\mathcal{P}(u^{(n)}) = \frac{1}{2}\int_\Omega \left\{g(|\nabla u^{(n-1)}|^2)|\nabla u^{(n)}|^2\right\} dx - 2\varphi\mathcal{T}[u^{(n)}; g, \varphi], \ \ n = 1, 2, 3, \ldots. \tag{5.5}$$

On the other hand, the weak solution $u^{(n)} \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$ of the linearized problem (5.4) is defined by the integral identity

$$\int_\Omega \left\{g(|\nabla u^{(n-1)}|^2)\nabla u^{(n)}\nabla v\right\} dx = 2\varphi \int_\Omega v dx, \quad \forall v \in \mathring{H}^1(\Omega) \cap H^2(\Omega),$$

for all $n = 1, 2, \ldots$. Substituting here $v = u^{(n)}$ and using the definition of the torque, we obtain the following energy identity for the linearized problem (5.4):

$$\int_\Omega \left\{g(|\nabla u^{(n-1)}|^2)|\nabla u^{(n)}|^2\right\} dx = 2\varphi\mathcal{T}[u^{(n)}; g, \varphi].$$

This, with (5.5), permits one to define the potential of the linearized problem (5.4) via the torque by

$$\mathcal{P}(u^{(n)}) = -\varphi\mathcal{T}[u^{(n)}; g, \varphi], \ \ n = 1, 2, 3, \ldots.$$

This result agrees with (4.9), since the torque is positive.

Comparing conditions (3.3), (2.5) and (3.5), with the assumptions of J_2-deformation theory, we conclude that all conditions (5.3) hold. Therefore based on Theorem 3.1 and Theorem 4.2, we can derive the following results for the nonlinear boundary value problem (5.2) related to the elastoplastic torsion of a strain hardening bar.

Theorem 5.1. *Let assumption* (5.3) *of J_2-deformation theory hold. Then*

(i) *the weak solution $u \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$ of the nonlinear boundary value problem* (5.2) *exists and unique;*

(ii) *the sequence of potentials $\{\mathcal{P}(u^{(n)})\}$ defined by* (5.5) *is a monotone decreasing and convergent one;*

(iii) *the sequence of approximate solutions $\{u^{(n)}\} \subset \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$, defined by* (5.4), *converges to the weak solution $u \in \overset{\circ}{H}{}^1(\Omega) \cap H^2(\Omega)$ of problem* (5.2) *in H^1-norm;*

(iv) *the rate of convergence can be estimated via the torque as follows:*

$$\|u - u^{(n)}\|_H \leq \frac{\sqrt{2}M_1\varphi}{\gamma_1^{3/2}} \left\{\mathcal{T}[u^{(n)}; g, \varphi] - \mathcal{T}[u^{(n-1)}; g, \varphi\}\ \right]^{1/2}, \quad n = 1, 2, 3, \ldots.$$

6 Elastoplastic Bending of Incompressible Plates

The mathematical model of inelastic bending of an isotropic homogeneous incompressible plate under the loads normal to the middle surface of the plate, within the range of J_2-deformation theory, is described by the following nonlinear boundary value problem (see [9, 10, 14]):

$$\begin{cases} Au \equiv \dfrac{\partial^2}{\partial x_1^2}\left[g(\xi^2(u))\left(\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x_2^2}\right)\right] + \dfrac{\partial^2}{\partial x_1 \partial x_2}\left[g(\xi^2(u))\dfrac{\partial^2 u}{\partial x_1 \partial x_2}\right] \\ +\dfrac{\partial^2}{\partial x_2^2}\left[g(\xi^2(u))\left(\dfrac{\partial^2 u}{\partial x_2^2} + \dfrac{1}{2}\dfrac{\partial^2 u}{\partial x_1^2}\right)\right] = F(x)\ , \quad x \in \Omega, \\ \\ u(x) = \dfrac{\partial u}{\partial n}(x) = 0, \quad x \in \partial\Omega. \end{cases} \tag{6.1}$$

The function $u = u(x)$ represents deflection of a point $x \in \Omega$ on the middle surface of a plate, occupying the square domain $\Omega \subset \mathbb{R}^2$, being in equilibrium under the action of normal loads. The coordinate plane Ox_1x_2 is assumed to be the middle surface of an isotropic homogeneous incompressible plate with the thickness $h > 0$. $F(x) =$

$3q(x)/h^3$, and $q = q(x)$ is the intensity (per unit area) of the loads normal to the middle surface of a plate, and n is a unit outward normal to the boundary $\partial\Omega$. It is assumed that the load $q = q(x_1, x_2)$ acts on the upper surface only in the x_3-axis direction, and the lower surface of the plate is free.

The coefficient $g = g(\xi^2(u))$ is defined to be the plasticity function, and satisfies assumptions (5.3) of J_2-deformation theory. The dependent variable $\xi = \xi(u)$, being referred to as effective value of the plate curvature [10], satisfies

$$\xi^2(u) = \left(\frac{\partial^2 u}{\partial x_1^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_2^2}\right)^2 + \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 + \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 u}{\partial x_2^2}. \tag{6.2}$$

Besides the above clamped boundary conditions, simply supported and other natural boundary conditions can also be considered.

We will show here that conditions (5.3) of J_2-deformation theory are sufficient not only for the existence and uniqueness of the weak solution $u \in \mathring{H}^2(\Omega)$ of the nonlinear problem (6.1), but also for the convergence of the linearized problem solution in the norm of the space $\mathring{H}^2(\Omega)$.

Let $H^2(\Omega)$ be the Sobolev space of functions [1] defined on the domain Ω with piecewise smooth boundary $\partial\Omega$ and

$$\mathring{H}^2(\Omega) = \{v \in H^2(\Omega) : \ u(x) = \partial u(x)/\partial n = 0, \ x \in \partial\Omega\}.$$

Multiplying both sides of equation (6.1) by $v \in \mathring{H}^2(\Omega)$, integrating on Ω and using the boundary conditions (6.1), we obtain the integral identity

$$\int_\Omega g(\xi^2(u))H(u,v)dx = \int_\Omega F(x)v(x)dx, \ \ \forall v \in \mathring{H}^2(\Omega), \tag{6.3}$$

where

$$H(u,v) = \frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_1 \partial x_2}\frac{\partial^2 v}{\partial x_1 \partial x_2} + \frac{1}{2}\left(\frac{\partial^2 u}{\partial x_1^2}\frac{\partial^2 v}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_2^2}\frac{\partial^2 v}{\partial x_1^2}\right). \tag{6.4}$$

The function $u \in \mathring{H}^2(\Omega)$ satisfying the integral identity (6.3) for all $v \in \mathring{H}^2(\Omega)$ is said to be a weak solution of the nonlinear problem (6.1). Recalling the definition $a(u; u, v) = \langle Au, v\rangle$, we may write

$$a(u; u, v) = \int_\Omega g(\xi^2(u))H(u,v)dx, \ \ \forall v \in \mathring{H}^2(\Omega). \tag{6.5}$$

Let us introduce now the functional $\mathcal{P}(u) = J(u) - l(u)$, where

$$J(u) = \frac{1}{2}\int_\Omega \left\{\int_0^{\xi^2(u)} g(\tau)d\tau\right\} dx, \ \ l(u) = \int_\Omega F(x)u(x)dx, \ \ u \in \mathring{H}^2(\Omega). \tag{6.6}$$

It easy to prove that the above defined functionals $J(u)$ and $\mathcal{P}(u)$ are the potentials of the nonlinear operator A and the nonlinear problem (6.1), respectively. Indeed, calculating the Gateaux derivative of the functional $J(u)$, we find

$$\langle J'(u), v\rangle = \int_\Omega g(\xi^2(u))H(u,v)dx.$$

Hence the nonlinear bending operator A, defined by (6.1), is a potential operator with the potential $J(u)$, defined by (6.6). To analyze monotonicity of this operator A, we will use the equivalence (see [10])

$$\exists\alpha_1,\alpha_2>0,\ \ \|v\|_2^2 \le \|v\|_E^2 \le 2\|v\|_2^2,\ \ \forall v\in \overset{\circ}{H}{}^2(\Omega)$$

of the norm $\|\cdot\|_2$ of the Sobolev space $H^2(\Omega)$ and the energy norm

$$\|v\|_E = \left\{\int_\Omega H(v,v)dx\right\}^{1/2}.$$

Lemma 6.1. *If the plasticity function* $g = g(\xi^2)$ *satisfies conditions* (5.3)*, then the nonlinear bending operator* A*, defined by* (6.1)*, is strong monotone in* $\overset{\circ}{H}{}^2(\Omega)$*, i.e.,*

$$\forall u,v\in \overset{\circ}{H}{}^2(\Omega),\ \ \langle Au-Av,u-v\rangle \ge \gamma_1\|u-v\|_2^2,\ \ \gamma_1>0. \tag{6.7}$$

Proof. Calculating the second Gateaux derivative of the functional $J(u)$, defined by (6.6), we have

$$\begin{aligned}
\langle J''(u),v,w\rangle &= \frac{d}{dt}\langle J'(u+tw),v\rangle|_{t=0} = \frac{d}{dt}\left\{\int_\Omega g(\xi^2(u+tw))H(u+tw,v)dx\right\}_{t=0}\\
&= \left\{\int_\Omega \left[2g'(\xi^2(u+tw))H(u,w)H(u+tw,v)+g(\xi^2(u+tw))H(w,v)\right]dx\right\}_{t=0}\\
&= \int_\Omega \left[2g'(\xi^2(u))H(u,w)H(u,v)+g(\xi^2(u))H(w,v)\right]dx.
\end{aligned}$$

For $w=v$, we have ($H(v,v)=\xi^2(v)$)

$$\langle J''(u),v,v\rangle = \int_\Omega \left[g(\xi^2(u))\xi^2(v)+2g'(\xi^2(u))H^2(u,v)\right]dx.$$

The second condition (5.3), with the inequality $(H(u,v))^2 \le H(u,u)H(v,v)$ and the formula $\xi^2(v)=H(v,v)$ (by (6.2) and (6.4)), implies

$$\begin{aligned}
\langle J''(u),v,v\rangle &\ge \int_\Omega \left[g(\xi^2(u))\xi^2(v)+2g'(\xi^2(u))\xi^2(u)\xi^2(v)\right]dx\\
&= \int_\Omega [g(\xi^2(u))+2g'(\xi^2(u))\xi^2(u)]\xi^2(v)dx.
\end{aligned}$$

Using now the third condition in (5.3) on the right-hand side and equivalence of norms, we obtain

$$\langle J''(u), v, v\rangle \geq c_2 \int_\Omega \xi^2(v)dx = c_2 \int_\Omega H(v,v)dx \geq \alpha_1 c_2 \|v\|_2^2.$$

The positivity of the second Gateaux derivative of the functional $J(u)$ means that the operator A is strongly monotone. $\square$

Since $A\Theta = \Theta$, where $\Theta \in \overset{\circ}{H}{}^2(\Omega)$ is zero element, monotonicity condition (6.7) for the nonlinear operator A also means its coercivity, i.e., $\langle Av, v\rangle \geq \gamma_1 \|v\|_2^2$, $\gamma_1 > 0$. Further the operator A is radially continuous (hemicontinuous), i.e., the real-valued function $t \to \langle A(u+tv), v\rangle$, for fixed $u, v \in \overset{\circ}{H}{}^2(\Omega)$, is continuous, since both functions $t \to g(\xi^2(u+tv))$, $t \to H(u+tv, v)$ are continuous, the proof of this assertion follows immediately from (6.5).

Thus, the potential operator A is radially continuous, strongly monotone and coercive. Then, by Browder–Minty theorem, we get the following.

Theorem 6.2. *If conditions* (5.3) *hold, then the nonlinear problem* (6.1) *has a unique solution in* $\overset{\circ}{H}{}^2(\Omega)$, *defined by the integral identity* (6.3).

Now we apply the abstract iteration scheme (4.7) linearizing the variational problem (6.3) as follows:

$$\int_\Omega g(\xi^2(u^{(n-1)}))H(u^{(n)}), v)dx = \int_\Omega F(x)v(x)dx, \ \ \forall v \in \overset{\circ}{H}{}^2(\Omega), \ \ n = 1, 2, 3, \ldots, \tag{6.8}$$

where $u^{(0)}) \in \overset{\circ}{H}{}^2(\Omega)$ is an initial iteration. The solution $u^{(n)} \in \overset{\circ}{H}{}^2(\Omega)$ of the linearized problem (6.8) is defined to be an approximate solution of the variational problem (6.3).

To apply the abstract iteration scheme we need a sufficient condition for the fulfilment of the convexity argument (4.4) for the nonlinear bending problem.

Lemma 6.3. *Let the function* $g = g(\xi^2)$ *satisfy the condition* $g(\xi^2) \leq 0$. *Then the convexity argument* (4.4) *holds for the nonlinear bending operator* A, *defined by* (6.1).

Proof. Using definitions (6.5) and (6.6), on the left-hand side of inequality (4.4), we have

$$\begin{aligned}
&\frac{1}{2}a(u; v, v) - \frac{1}{2}a(u; u, u) - J(v) + J(u) \\
&\quad = \ \frac{1}{2}\int_\Omega g(\xi^2(u))H(v,v)dx - \frac{1}{2}\int_\Omega g(\xi^2(u))H(u,u)dx \\
&\qquad -\frac{1}{2}\int_\Omega \left\{ \int_0^{\xi^2(v)} g(\tau)d\tau \right\} dx + \frac{1}{2}\int_\Omega \left\{ \int_0^{\xi^2(u)} g(\tau)d\tau \right\} dx \\
&\quad = \ \frac{1}{2}\int_\Omega \left\{ g(\xi^2(u))[\xi^2(v) - \xi^2(u)] + \int_{\xi^2(v)}^{\xi^2(u)} g(\tau)d\tau \right\} dx.
\end{aligned}$$

Hence

$$\frac{1}{2}a(u;v,v) - \frac{1}{2}a(u;u,u) - J(v) + J(u)$$
$$= \frac{1}{2}\int_\Omega \left\{ g(\xi^2(u))[\xi^2(v) - \xi^2(u)] + \int_{\xi^2(v)}^{\xi^2(u)} g(\tau)d\tau \right\} dx. \quad (6.9)$$

As in the proof of Theorem 4.1, introducing the function

$$\mathcal{Q}(t) = \int_{\xi_*}^{\xi} g(z)dz,$$

we conclude $\mathcal{Q}''(t) = g'(t) \leq 0$, which means $\mathcal{Q}(t)$, is a concave function. Hence inequality (4.6) holds for this function. This, with (6.9) completes the proof. □

Lemma 6.3 implies that the sequence of potentials

$$\mathcal{P}(u^{(n)}) = \frac{1}{2}\int_\Omega g(\xi^2(u^{(n-1)}))\xi^2(u^{(n)})dx - \int_\Omega F(x)u^{(n)}(x)dx, \; n = 1, 2, 3, \ldots, \quad (6.10)$$

defined on approximate solutions $u^{(n)} \in \overset{\circ}{H}{}^2(\Omega)$, is a monotone decreasing one.

Next we need to show that the functional $a(u;v,w)$, defined by (6.5), is bounded. This follows from the condition $g(\xi^2(u)) \leq c_1$ and the equivalence of norms $\|\cdot\|_2$ and $\|\cdot\|_E$:

$$a(u;v,w) = \int_\Omega g(\xi^2(u))H(v,w)dx \leq c_1 \int_\Omega |H(v,w)|dx \leq c_1\alpha_2^2\|v\|_2\|w\|_2.$$

Therefore all conditions of [8, Theorem 2.1] hold, and we have the following result.

Theorem 6.4. *Let* $u \in \overset{\circ}{H}{}^2(\Omega)$ *and* $u^{(n)} \in \overset{\circ}{H}{}^2(\Omega)$ *be the solutions of the nonlinear problem* (2.3)*, and the linearized problem* (4.4)*, respectively. If conditions* (*i*)–(*iii*) *hold, then*

(a) *the iteration scheme defined by* (4.7) *is a monotone decreasing one:*

$$\Pi(u^{(n)}) \leq \Pi(u^{(n-1)}), \;\; \forall u^{(n-1)}, u^{(n)} \in \overset{\circ}{H}{}^2(\Omega);$$

(b) *the sequence of approximate solutions* $\{u^{(n)}\} \subset \overset{\circ}{H}{}^2(\Omega)$ *defined by the iteration scheme* (4.7) *converges to the solution* $u \in \overset{\circ}{H}{}^2(\Omega)$ *of the nonlinear problem* (4.1) *in the norm of the Sobolev space* $\overset{\circ}{H}{}^2(\Omega)$*;*

(c) *for the rate of convergence the following estimate holds:*

$$\|u - u^{(n)}\|_H \leq \frac{\sqrt{2}c_1\alpha_2^2}{\gamma_1^{3/2}} \{\mathcal{P}(u^{(n-1)}) - \mathcal{P}(u^{(n)})\}^{1/2}, \quad n = 1, 2, 3, \ldots.$$

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