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Existence of Hamiltonian Structure in 3D

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Abstract

In three dimensions, the construction of bi-Hamiltonian structure can be reduced to the solutions of a Riccati equation with the arclength coordinate of a Frenet–Serret frame being the independent variable. Explicit integration of conserved quantities are connected with the coefficients of Riccati equation which are elements of the third cohomology class. All explicitly constructed examples of bi-Hamiltonian systems are exhausted when this class along with the first one vanishes. The latter condition provides integrating factors for explicit integration of Hamiltonian functions. For the Darboux–Halphen system, the Godbillon–Vey invariant is shown to arise as obstruction to integrability of integrating factor.

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1 Introduction

A Poisson structure on a manifold is defined by a skew symmetric contravariant bilinear form subjected to the Jacobi identity expressed as the vanishing of the Schouten bracket of Poisson tensor with itself [32, 35, 42, 49]. This structure having no nondegeneracy requirement becomes the basic underlying geometry to study noncanonical Hamilton's equations on odd dimensional manifolds as well as the Hamiltonian structures of nonlinear evolution equations [35, 36, 42].

The first interesting case of a completely degenerate finite dimensional Hamiltonian structure occurs in three dimensions. Many works have been devoted to the study of three dimensional dynamical systems with primary concern on quantization, construction of conserved quantities, Hamiltonian structures, integrability problems and

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their numerical integration using techniques from various areas such as Poisson geometry, differential equations, Frobenius integrability theorem and theory of foliations [7, 10, 15, 17, 19, 20, 23–31, 38–41, 43, 46].

In [1], we reduced the problem of constructing Hamiltonian structures in three dimensions to the solutions of a Riccati equation in moving coordinates of Frenet–Serret frame. All known examples of dynamical systems having two compatible and explicit Hamiltonian structures are exhausted by constant solution. We concluded that in three dimensions vector fields which are not eigenvectors of the curl operator are at least locally bi-Hamiltonian.

In this work, we shall extend the discussion to analysis of obstructions to the construction of global Hamiltonian structures. We shall present the Darboux–Halphen system as an example for which the Godbillon–Vey three-form can be explicitly worked out as an obstruction to the existence of global structure.

2 Hamiltonian Systems in Three Dimensions

We shall summarize the necessary ingredients of the bi-Hamiltonian formalism in three dimensions. See [7, 10, 15, 17, 19, 20, 23–31, 38–41, 43, 46] for details and examples. For $\mathbf{x} = \left\{x^i\right\} = (x, y, z) \in \mathbb{R}^3$, $t \in \mathbb{R}$ and overdot denoting the derivative with respect to t, we consider the system of autonomous differential equations

$$\dot{\mathbf{x}} = \mathbf{v}\left(\mathbf{x}\right) \tag{2.1}$$

associated with a three-dimensional smooth vector field $\mathbf{v}$. Eq. (2.1) is said to be Hamiltonian if the right-hand side can be written as $\mathbf{v}\left(\mathbf{x}\right) = \Omega\left(\mathbf{x}\right)\left(dH\left(\mathbf{x}\right)\right)$ where $H\left(\mathbf{x}\right)$ is the Hamiltonian function and $\Omega\left(\mathbf{x}\right)$ is the Poisson bi-vector (i.e. a skew-symmetric, contravariant two-tensor) subjected to the Jacobi identity $\left[\Omega\left(\mathbf{x}\right), \Omega\left(\mathbf{x}\right)\right] = 0$ defined by the Schouten bracket [32]. In coordinates, if $\partial_i = \partial/\partial x^i$, the Poisson bi-vector is $\Omega\left(\mathbf{x}\right) = \Omega^{jk}\left(\mathbf{x}\right)\partial_j \wedge \partial_k$, with summation over repeated indices, and the Jacobi identity reads $\Omega^{i[j}\partial_i\Omega^{kl]} = 0$ where $[jkl]$ denotes the antisymmetrization over three indices. It follows that in three dimensions the Jacobi identity is a single scalar equation. One can exploit the vector calculus and the differential forms in three dimensions to have a more transparent understanding of Hamilton's equations as well as the Jacobi identity. Using the isomorphism

$$J_i = \varepsilon_{ijk}\Omega^{jk}, \qquad i, j, k = 1, 2, 3 \tag{2.2}$$

between skew-symmetric matrices and (pseudo)-vectors defined by the completely anti-symmetric Levi–Civita tensor ε_{ijk}, we can write the Hamilton equations and the Jacobi identity as

$$\mathbf{v} = \mathbf{J} \times \nabla H, \qquad \mathbf{J} \cdot \left(\nabla \times \mathbf{J}\right) = 0, \tag{2.3}$$

respectively. In this form the Jacobi identity is equivalent to the Frobenius integrability condition $J \wedge dJ = 0$ for the one form $J = J_i dx^i$. It is the condition for J to define a foliation of codimension one in three dimensional space [2, 19, 44, 47]. A distinguished property of Poisson structures in three dimensions is the invariance of the Jacobi identity under the multiplication of Poisson vector $\mathbf{J}(\mathbf{x})$ by an arbitrary but nonzero factor. The identities

$$\mathbf{J} \cdot \mathbf{v} = 0, \quad \nabla H \cdot \mathbf{v} = 0 \tag{2.4}$$

follow directly from the Hamilton equations in (2.3). The second equation in (2.4) is the expression for the conservation of the Hamiltonian function. A three dimensional vector $\mathbf{v}(\mathbf{x})$ is said to be bi-Hamiltonian if there exist two different compatible Hamiltonian structures [34, 42]. In the notation of equation (2.3), this implies

$$\mathbf{v} = \mathbf{J}_1 \times \nabla H_2 = \mathbf{J}_2 \times \nabla H_1 \tag{2.5}$$

for the dynamical equations. The compatibility condition for $\mathbf{J}_1$ and $\mathbf{J}_2$ is defined by the Jacobi identity for the Poisson pencil $\mathbf{J}_1 + c\mathbf{J}_2$ for arbitrary constant c.

3 Frenet–Serret Frame

Let $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ denote the Frenet–Serret frame associated with a differentiable curve $t \to \mathbf{x}(t)$ in some domain of the three dimensional space $\mathbb{R}^3$. Throughout, $\nabla = (\partial_x, \partial_y, \partial_z)$ will denote the usual gradient operator in local Cartesian coordinates. Given a vector field $\mathbf{v}$, the unit tangent vector $\mathbf{t}$, the unit normal $\mathbf{n}$, and the unit bi-normal $\mathbf{b}$ can be constructed as

$$\mathbf{t}(\mathbf{x}) = \frac{\mathbf{v}(\mathbf{x})}{\|\mathbf{v}(\mathbf{x})\|}, \qquad \mathbf{n}(\mathbf{x}) = \frac{-\mathbf{t} \times (\nabla \times \mathbf{t})}{\|\mathbf{t} \times (\nabla \times \mathbf{t})\|}, \qquad \mathbf{b}(\mathbf{x}) = \mathbf{t}(\mathbf{x}) \times \mathbf{n}(\mathbf{x}) \tag{3.1}$$

and they form a right-handed orthonormal frame except at points $\mathbf{x}$ corresponding to equilibrium solutions and, those vector fields $\mathbf{v}$ satisfying the condition imposed by $\mathbf{t} \times (\nabla \times \mathbf{t}) = 0$. This condition excludes essentially the flows with constant unit tangent and the points $\mathbf{x}$ at which the unit normal $\mathbf{n}$ (hence the bi-normal $\mathbf{b}$) have zeros. That is, the cases the Frenet–Serret frame is not well-defined. To avoid this we may assume that

$$(\nabla \times \mathbf{t}) \neq \lambda(\mathbf{x})\,\mathbf{t} \tag{3.2}$$

for arbitrary nonzero function $\lambda(\mathbf{x})$. That is, we exclude the dynamical systems whose unit tangent vectors are the eigenvectors of the curl operator [11, 37].

We introduce the directional derivatives along the triad $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ as

$$\partial_s = \mathbf{t} \cdot \nabla, \qquad \partial_n = \mathbf{n} \cdot \nabla, \qquad \partial_b = \mathbf{b} \cdot \nabla \tag{3.3}$$

so that the variables (s, n, b) are the coordinates associated with the Frenet–Serret frame based at the point $\mathbf{x}$. Assuming the Cartesian coordinates are functions $\mathbf{x} = \mathbf{x}(s, n.b)$ of the Frenet–Serret coordinates one concludes easily that the Jacobian determinant is nonzero and hence the inverse transformation $s = s(\mathbf{x})$, $n = n(\mathbf{x})$, $b = b(\mathbf{x})$ exists locally, that is, in a sufficiently small neighborhood of a given point $\mathbf{x}_0 \in \mathbb{R}^3$. These functions may be obtained by integrating the one-forms

$$\tau = \mathbf{t} \cdot d\mathbf{x}, \;\; \eta = \mathbf{n} \cdot d\mathbf{x}, \;\; \beta = \mathbf{b} \cdot d\mathbf{x} \tag{3.4}$$

the last two of which implies n =constant and b =constant when restricted to the curve $\mathbf{x}(t)$. By inverting equations (3.3) we get the expression $\nabla = \mathbf{t}\partial_s + \mathbf{n}\partial_n + \mathbf{b}\partial_b$ for the Cartesian gradient in the Frenet–Serret frame. We finally note that Eq. (3.1) is not the only way to construct Frenet–Serret frame, it rather shows the existence under the condition in Eq. (3.2). For the example of the Darboux–Halphen system, we will use its symmetries to construct the moving frame.

4 Jacobi Identity in Frenet–Serret Frame

It follows from the identity $\mathbf{J} \cdot \mathbf{v} = 0$ that the Poisson vector $\mathbf{J}$ has no component along the unit tangent vector $\mathbf{t}$. Hence, we set

$$\mathbf{J} = A\mathbf{n} + B\mathbf{b} \tag{4.1}$$

for unknown functions $A(\mathbf{x})$ and $B(\mathbf{x})$ satisfying $A^2 + B^2 \neq 0$. Assuming $A \neq 0$ and defining the function $\mu = B/A$ the Jacobi identity for $\mathbf{J} = A(\mathbf{n} + \mu\mathbf{b})$ reduces to the Riccati equation

$$\partial_s \mu = \mathcal{H}_{\mathbf{n}} + \mu \mathcal{H}_{\mathbf{nb}} + \mu^2 \mathcal{H}_{\mathbf{b}} \tag{4.2}$$

in the arclength variable s. Here, we define the helicity densities

$$\mathcal{H}_{\mathbf{n}} = \mathbf{n} \cdot (\nabla \times \mathbf{n}), \;\; \mathcal{H}_{\mathbf{nb}} = \mathbf{n} \cdot \nabla \times \mathbf{b} + \mathbf{b} \cdot \nabla \times \mathbf{n}, \;\; \mathcal{H}_{\mathbf{b}} = \mathbf{b} \cdot (\nabla \times \mathbf{b})$$

associated with the Frenet–Serret triad [1]. These functions are the coefficients of the Cartesian volume $dx \wedge dy \wedge dz$ in the three-forms

$$\Omega_{\mathbf{n}} = \eta \wedge d\eta, \;\; \Omega_{\mathbf{nb}} = \eta \wedge d\beta + \beta \wedge d\eta, \;\; \Omega_{\mathbf{b}} = \beta \wedge d\beta. \tag{4.3}$$

We refer to [3, 4, 9, 45] for physical and geometric interpretations of these quantities in the context of hydrodynamics. The Riccati equation (4.2) is equivalent to a linear second order equation and hence possesses two linearly independent solutions leading to two Poisson vectors for the dynamical systems under consideration. The Hamiltonian form of dynamical equations implies that the Poisson vectors obtained from solutions of Riccati equation are always compatible. Thus, we conclude that the following results holds.

Proposition 4.1. *All dynamical systems in three dimensions subjected to the condition* (3.2) *possess two compatible Poisson vectors.*

5 Local Form of Hamiltonian Structures

By constant solution of Eq.(4.2) we mean a function of normal coordinates n and b. In particular, such solutions of the Riccati equation include the case

$$\mathcal{H}_{\mathbf{n}}=0,\ \ \mathcal{H}_{\mathbf{b}}=0,\ \mathcal{H}_{\mathbf{nb}}=0$$

the first two of which are the conditions for the normal vectors $\mathbf{n}$ and $\mathbf{b}$ to satisfy the Jacobi identity or Frobenius integrability criterion. The last equation turns out to be the compatibility condition for the normal vectors to form a Poisson pencil and thereby to define bi-Hamiltonian structure. Integrability criterion implies the existence of one-forms ξ and ς such that $d\eta = \xi \wedge \eta$ and $d\beta = \varsigma \wedge \beta$. From the compatibility condition we see that $\xi = \varsigma$ and the common integrating factor ξ satisfies $\eta \wedge d\xi = 0$, $\beta \wedge d\xi = 0$. Thus, we have $d\xi = g\eta \wedge \beta$ for some arbitrary function g and this closes the algebra of one-forms. The integrating factor ξ is itself integrable, that is, $\xi \wedge d\xi = 0$ when $g = 0$ or equivalently $d\xi = 0$. By the Poincaré lemma, the integrating factor is the differential of a function and this makes possible to integrate η and β explicitly in the given coordinate representation. These results can be drawn from the requirement that the one-form

$$\Gamma_{\text{global}} = \eta + \beta\varepsilon + f d\varepsilon$$

with ε being a parameter, is integrable $\Gamma_{\text{global}} \wedge d\Gamma_{\text{global}} = 0$ to all orders in ε [19]. Eventually, all the known examples of bi-Hamiltonian systems in three dimensions are included in this case. The normal coordinates n and b obtained from explicit integration of the one-forms η and β represent the global conserved quantities. They appear arbitrarily in the Riccati equation and hence in the Poisson structures. Conversely, if we are given a bi-Hamiltonian dynamical system of the form $\mathbf{v} = \psi \nabla H_1 \times \nabla H_2$ as, for example, in [7, 10, 15, 17, 19, 20, 23–31, 38–41, 43, 46], the representation in the Frenet–Serret frame requires $\mathbf{t} = \psi \parallel \mathbf{v} \parallel^{-1} \nabla H_1 \times \nabla H_2$ which implies the orthogonality of the unit tangent vector to the gradients of Hamiltonian functions. The normal vectors $\mathbf{n}$ and $\mathbf{b}$ can then be identified via orthonormalization procedure applied to the linearly independent vectors ∇H_1 and ∇H_2 taking also the constraint $\parallel \nabla H_1 \times \nabla H_2 \parallel = \parallel \mathbf{v} \parallel / \psi$ into account. Thus, the normal vectors $\mathbf{n}$ and $\mathbf{b}$ defining the bi-Hamiltonian structure corresponds, in the globally integrable case, to the gradients of Hamiltonian functions defining Poisson vectors. Local structure arises when we are not able to integrate the conserved functions explicitly. Before discussing this case, we remark that, as far as the Frenet–Serret frame is constructable, one can still use the normal vectors to cast the dynamical equations into formal bi-Hamiltonian form. To summarize, we have the following result.

Proposition 5.1. *The manifestly bi-Hamiltonian equation* $\mathbf{t} = \mathbf{n} \times \mathbf{b}$ *is the local form of the bi-Hamiltonian structure in three dimensions. Using an appropriate volume three-form* ν*, this equation can be written as*

$$i_{\partial_s}\nu = \eta \wedge \beta, \tag{5.1}$$

where the one-forms η and β are defined by Eq. (3.4).

In the case that the solution μ of the Riccati equation is not a constant, at least one of the three-forms in Eq. (4.3) is nonzero. Thus, they arise as obstructions to extent globally the local bi-Hamiltonian structure. To our knowledge, there is only one example in the literature for which we can construct explicitly the obstruction to existence of global Hamiltonian structure. This is the Darboux–Halphen system which was obtained from the theory of surfaces more than a hundred years ago and resurrected in theoretical physics around eighties.

6 The Darboux–Halphen System

In 1878, Darboux [13] obtained, in his investigation of the family of second-degree surfaces in $\mathbb{R}^3$, the system of equations

$$\frac{d(x+y)}{dt} = xy \ , \quad \frac{d(y+z)}{dt} = yz \ , \quad \frac{d(x+z)}{dt} = xz. \tag{6.1}$$

Soon after, Halphen [21, 22] has given the time-dependent transformations

$$(t, x^i) \mapsto \left(\frac{\alpha t + \beta}{\gamma t + \delta}, 2\gamma \frac{\gamma t + \delta}{\alpha\delta - \gamma\beta} + \frac{(\gamma t + \delta)^2}{\alpha\delta - \gamma\beta} x^i \right) \tag{6.2}$$

with $(x^1, x^2, x^3) = (x, y, z)$, which leave Eq. (6.1) invariant. He has also obtained the solution of the system. Starting from 1979, growing attractions on the Darboux–Halphen equations (6.1) come from the observations that various important models of theoretical physics admit reductions to equations (6.1) or its generalizations. It appeared in the analysis of $SO(3)$ invariant anti-self-dual Einstein metrics in gravitation theory [16], in connection with the moduli space of two-monopole problem [5, 6], the complex Bianchi IX cosmological models and the reductions of self-dual Yang-Mills fields [12], the WDVV equations of topological field theories [14] and, renormalization group flow of WZW theory [8]. It manifests several interesting properties concerning integrability. Namely, the Darboux–Halphen system is an algebraically nonintegrable system [33, 48] whose general solution can be expressed in terms of elliptic integrals [21].

6.1 Halphen's Symmetries

We let the Darboux–Halphen system (6.1) be represented in a local coordinate system $\mathbf{x} \equiv (x, y, z)$ by the vector field

$$v = (yz - xy - xz)\frac{\partial}{\partial x} + (xz - xy - yz)\frac{\partial}{\partial y} + (xy - xz - yz)\frac{\partial}{\partial z} \tag{6.3}$$

for which the corresponding differential equations are equivalent to Eq. (6.1) provided we perform the replacement $t \mapsto -t/2$. For notational convenience, we also define the autonomous vector fields

$$u = 2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right), \quad w = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \tag{6.4}$$

and observe that they satisfy the Lie bracket relations

$$[u, v] = 2v, \ \ [u, w] = -2w, \ \ [v, w] = u$$

of the three-dimensional Lie algebra $\mathfrak{sl}(2)$. We call the set (v, u, w) of vector fields having weights $(-1, 0, 1)$, respectively, an $\mathfrak{sl}(2)$-triple. The generators of the Halphen symmetries are the vector fields

$$V_t = \frac{\partial}{\partial t}, \ U_t = -2t\frac{\partial}{\partial t} + u, \ W_t = -t^2\frac{\partial}{\partial t} - w + tu \tag{6.5}$$

on the time-extended space $I \times M \subset \mathbb{R} \times \mathbb{R}^3$. These vector fields satisfy the conditions

$$\left[\frac{\partial}{\partial t} + v, V_t\right] = 0, \quad \left[\frac{\partial}{\partial t} + v, U_t\right] = -2\left(\frac{\partial}{\partial t} + v\right), \quad \left[\frac{\partial}{\partial t} + v, W_t\right] = -2t\left(\frac{\partial}{\partial t} + v\right) \tag{6.6}$$

to be infinitesimal time-dependent geometric symmetries of the Darboux–Halphen vector field v. Eq. (6.6) guarantee that if a function H is a time-dependent conserved quantity for v, then $X(H)$ with X being any of the vectors in Eq. (6.5), is also a conserved quantity. Geometric consequences of the time-dependent symmetries are discussed in [18]. Below, we will use these symmetries to show the local existence of time-independent conserved quantity. This will arise as a result of a locally integrable one-form. However, we will not be able to integrate it explicitly because the integrating factor turns out to be not integrable in the sense of Frobenius.

Having the $\mathfrak{sl}(2)$-triple (v, u, w) intrinsically associated with the Darboux–Halphen phase space, one can define a Frenet–Serret frame by orthonormalizing these vector fields and proceed as described in previous sections. Namely, one can introduce a Poisson vector on the plane of normal vectors, obtain a Riccati equation from the Jacobi identity and investigate its solutions. It is obvious from the explicit form of vector fields that the coefficients of the three-forms in Eq. (4.3) will be nonvanishing. In spite of the explicit expressions, this computational way is quite cumbersome and will not be much useful unless one guarantees the explicit integration of the one-forms defining the moving coordinates. That means, one presumes existence of global bi-Hamiltonian structure. Instead, we will first cast the Darboux–Halphen system into the form of a local Hamiltonian structure as described by Eq. (5.1) and see if we can integrate the one-forms involved, that is, if we can extend globally the local structure. For this purpose, we will proceed by exploiting the Halphen symmetries further.

A direct computation shows that Eq. (6.6) are equivalent to

$$\left[\frac{\partial}{\partial t}+v, V_M\right] = \left[\frac{\partial}{\partial t}+v, U_M\right] = \left[\frac{\partial}{\partial t}+v, W_M\right] = 0, \tag{6.7}$$

where the time-dependent vector fields

$$V_M = -v \;, \quad U_M = u + 2tv \;, \quad W_M = -w + tu + t^2 v \tag{6.8}$$

on $M \subset \mathbb{R}^3$ are the unique characteristic (or evolutionary) forms [42] of V_t, U_t, W_t along the Darboux–Halphen field v. One can check easily the following result.

Proposition 6.1. *Each of the sets* (v, u, w)*,* (V_t, U_t, W_t) *and* (V_M, U_M, W_M) *of vector fields form* $\mathfrak{sl}(2)$*-triples.*

It follows from Eqs. (6.5) or (6.8) that the function ρ defined by

$$\begin{aligned}
\rho^{-1} &= \det((1,\mathbf{v}), \mathbf{V}_t, \mathbf{U}_t, \mathbf{W}_t) \\
&= \det((1,\mathbf{v}), (1,0), (-2t,\mathbf{u}), (-t^2, -\mathbf{w}+t\mathbf{u})) \\
&= \det((1,\mathbf{v}), \mathbf{V}_M, \mathbf{U}_M, \mathbf{W}_M) \\
&= \det((1,\mathbf{v}), -\mathbf{v}, \mathbf{u}+2t\mathbf{v}, -\mathbf{w}+t\mathbf{u}+t^2\mathbf{v}) \\
&= -(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -4(x-y)(y-z)(z-x)
\end{aligned}$$

is the so-called last multiplier [50]. Here, the boldface letters $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are the components of u, v, w, respectively.

Proposition 6.2. $\nu(\mathbf{x}) = \rho dx \wedge dy \wedge dz$ *is the invariant volume element on the Darboux–Halphen phase space.*

In other words, with respect to the volume ν, all three vector fields u, v, w are divergence free. This will be seen easily when we complete the algebra of dual one-forms. Let (β, α, γ) be one-forms dual to the $\mathfrak{sl}(2)$-triple (v, u, w) so that they satisfy

$$i_v\beta = i_u\alpha = i_w\gamma = 1$$

with the rest of the pairings being zero. Using the components of the representative vector fields in Eqs. (6.3) and (6.4) of the chosen basis, we find the expressions

$$\beta = \rho\mathbf{u}\times\mathbf{w}\cdot d\mathbf{x}, \quad \alpha = \rho\mathbf{w}\times\mathbf{v}\cdot d\mathbf{x}, \quad \gamma = \rho\mathbf{v}\times\mathbf{u}\cdot d\mathbf{x}$$

from which the orthonormality relations follow easily. For these dual basis one-forms, corresponding to the Lie bracket relations of (v, u, w), the $\mathfrak{sl}(2)$-Maurer–Cartan equations

$$d\beta = -2\alpha\wedge\beta \;, \quad d\alpha = \gamma\wedge\beta \;, \quad d\gamma = 2\alpha\wedge\gamma \tag{6.9}$$

can be obtained using the invariant definition

$$2d\omega(u,v) = u(i_v\omega) - v(i_u\omega) - i_{[u,v]}\omega$$

of the exterior derivative on a one-form ω and the Lie algebra relations.

Proposition 6.3. *The Darboux–Halphen dynamical system can be written as*

$$i_v\mu = \alpha \wedge \gamma = d\gamma/2$$

which is the form of local Hamiltonian structure.

Similar relations for the other vector fields of the $\mathfrak{sl}(2)$-triple are

$$i_u\mu = \gamma \wedge \beta = d\alpha, \quad i_w\mu = \beta \wedge \alpha = d\beta/2,$$

and these relations verify also the invariance of the volume form under the flows of $\mathfrak{sl}(2)$-triple. It also follows that

$$\beta \wedge d\beta = 0, \quad \alpha \wedge d\alpha = \rho^{-2}\nu, \quad \gamma \wedge d\gamma = 0$$

so that the one-forms β and γ are integrable in the sense of Frobenius. To this end, we recall the integration of one-forms by the homotopy formula. Given a one-form $\omega = \omega_i(x)dx^i$, let $H(x) = \int_0^1 x^i\omega_i(sx)ds$. Then, $dH = \omega$ [42]. However, for each of the one-forms β and γ the integrands $\rho\mathbf{u} \times \mathbf{w} \cdot \mathbf{x}$ and $\rho\mathbf{v} \times \mathbf{u} \cdot \mathbf{x}$ vanish identically and one cannot apply the homotopy formula for their integration.

6.2 Nonintegrable Integrating Factors

The algebraic structure on the solution space of the Darboux–Halphen system can be obtained, as in the case of globally bi-Hamiltonian structures, from a locally integrable one-form involving an arbitrary parameter, this time to second order. More precisely, we consider the one form

$$\Gamma_{local} = \beta + (2\alpha m - dm)\varepsilon - m^2\gamma\varepsilon^2 + (m + n\varepsilon)d\varepsilon$$

and require this to satisfy the local integrability condition $\Gamma_{local} \wedge d\Gamma_{local} = 0$ to all orders in the parameter ε. The resulting equations, beyond immediate consequences of the Maurer–Cartan equations, include relations for integration of locally integrable one-forms. Referring to [19] for details of this computation, we find that the one-forms

$$e^{-2\int\alpha}\gamma, \quad e^{2\int\alpha}\beta \tag{6.10}$$

are closed. Provided these are not elements of the first cohomology class and, by the Poincaré lemma, they must be locally exact, that is, $e^{-2\int\alpha}\gamma = dH_1$ for some function H_1 and similarly for the other. Since, $i_v\gamma = 0$ by duality relations, this function is a conserved quantity of the Darboux–Halphen system and, as we pointed out earlier, the Halphen symmetries, when acted upon H_1 produces new conserved quantities two of which will be sufficient to establish the bi-Hamiltonian structure. For a given parametrized path the integral of the one-form α can be computed locally and hence the local conserved quantity along the given path can be obtained. This structure remains only local because the integrating factor α, for explicit integration of H_1, is nonintegrable

$$\alpha \wedge d\alpha \neq 0$$

and from the $\mathfrak{sl}(2)$-Maurer–Cartan equations, we see that this condition is equivalent to the nonvanishing of the Godbillon–Vey three-form [44, 47]

$$\alpha \wedge \beta \wedge \gamma \neq 0 \, .$$

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The existence of local bi-Hamiltonian structures was somehow known to Prof. Dr. Yavuz Nutku while we were working on the Darboux–Halphen system. I remember asking him, to be sure not misunderstood what we have discussed, “you mean all these systems are bi-Hamiltonian?” He said, “of course they are”.

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