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A Survey on Boundary Value Problems for Complex Partial Differential Equations

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Abstract

In this article, the recent results on basic boundary value problems of complex analysis are surveyed for complex model equations and linear elliptic complex partial differential equations of arbitrary order on simply connected bounded domains, particularly in the unit disc, on unbounded domains such as upper half plane and upper right quarter plane and on multiply connected domains containing circular rings.

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1 Introduction

The investigations on the boundary value problems for complex differential equations had a new starting point for complex differential equations in the middle of the nineteenth century. Riemann has stated the problem

"Find a function $w(z) = u + iv$ analytic in the domain Ω, which satisfies at every boundary point the relation

$$F(u, v) = 0 \ \ (on\ \partial\Omega)\text{"}$$

in his famous thesis [51]. Later, this statement is known as Riemann problem. However, he stated only some general considerations regarding with the solvability of the problem. The generalization of Riemann problem to a linear first-order differential equation together with the linear boundary condition

$$\alpha u + \beta v = \mathrm{Re}[\overline{\lambda(z)}w] = \gamma \text{ on } \partial\Omega$$

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was considered by Hilbert [45]. This new form of the problem is called as the generalized Riemann–Hilbert problem. Some extensions of this problem has been treated by many researchers [43, 46, 50, 53, 57, 59].

The first-order linear complex partial differential equation

$$w_{\bar{z}} = a(z)w(z) + b(z)\overline{w(z)}$$

has been considered by Vekua [58] and Bers [40] separately and simultaneously. Its solutions are called as generalized analytic (or pseudo-holomorphic) functions. The boundary value problems described above and their particular cases known as Schwarz, Dirichlet, Neumann and Robin problems are treated by many researchers.

Boundary value problems for higher-order linear complex partial differential equations gained attraction in the last twelve years. Dirichlet, Neumann, Robin, Schwarz and mixed boundary value problems for model equations, that is for the equations of the form

$$\partial_z^m \partial_{\bar{z}}^n w = f(z) ,$$

are introduced in the unit disc of the complex plane by Begehr [14]. In this article we want to give a directed survey of the relevant literature on the boundary value problems of complex analysis, and reveal some problems which are still open.

2 Dirichlet Problem for Complex PDEs

2.1 Dirichlet Problem for Complex Model PDEs

2.1.1 Simply Connected Bounded Domain Case

Many authors have investigated the Dirichlet problem in simply connected domains. To give the explicit representations for the solutions of the problems, we will consider the particular case of the unit disc $\mathbb{D}$ of the complex plane. Let us start by giving the related harmonic and polyharmonic Green functions.

In $\mathbb{D}$, the harmonic Green function is defined as

$$G_1(z,\zeta) = \log \left| \frac{1 - z\bar{\zeta}}{\zeta - z} \right|^2$$

and its properties are given in [16, 18]. A polyharmonic Green function G_n is defined iteratively by

$$G_n(z,\zeta) = -\frac{1}{\pi} \iint_{\mathbb{D}} G_1(z,\tilde{\zeta}) G_{n-1}(\tilde{\zeta},\zeta) d\tilde{\xi} d\tilde{\eta}$$

for $n \geq 2$, [33]. The explicit expressions of $G_n(z,\zeta)$ for $n = 2$ and for $n = 3$ are given in [16, 18] and in [41], respectively. $G_n(z,\zeta)$ are employed to solve the following n-Dirichlet problems for the n-Poisson equation, [33].

Theorem 2.1. *The Dirichlet problem*

$$(\partial_z\partial_{\bar z})^n w = f \ \text{ in } \ \mathbb{D},\ (\partial_z\partial_{\bar z})^\mu w = \gamma_\mu,\ 0 \le \mu \le n-1 \ \text{ on } \ \partial\mathbb{D}$$

$f \in L^1(\mathbb{D}), \gamma_\mu \in C(\partial\mathbb{D}),\ 0 \le \mu \le n-1$ *is uniquely solvable. The solution is*

$$w(z) = -\sum_{\mu=1}^{n} \frac{1}{4\pi i} \int\limits_{\partial\mathbb{D}} \partial_{\nu_\zeta} G_\mu(z,\zeta)\gamma_{\mu-1}(\zeta)\frac{d\zeta}{\zeta} - \frac{1}{\pi}\iint_{\mathbb{D}} G_n(z,\zeta) f(\zeta) d\xi d\eta. \tag{2.1}$$

The explicit forms of the solutions (2.1) are given in [15,16,18] in the case of $n = 2$. In [49], authors considered the problem given in Theorem 2.1 and they gave the explicit representation of the unique solution using the iterative sums for $n = 2$ and $n = 3$ are given. Begehr, Du and Wang [20] solved the Dirichlet problem for polyharmonic functions by using the decomposition of polyharmonic functions and transforming the problem into an equivalent Riemann boundary value problem for polyanalytic functions. In [39], authors solved the Dirichlet problem investigated in [20] by a new approach. The explicit expression of the unique solution for the Dirichlet problem of triharmonic functions in the unit disc is obtained by using the so-called weak decomposition of polyharmonic functions and converting the problem into an equivalent Dirichlet boundary value problem for analytic functions. In contrast to the boundary condition according to [20], the requirement of smoothness for the given functions is reduced.

Another polyharmonic kernel function is the so-called Green–Almansi function $\tilde{G}_n$ [9] which is given by

$$\begin{aligned}\tilde{G}_n(z,\zeta) &= \frac{|\zeta - z|^{2(n-1)}}{(n-1)!^2} \log\left|\frac{1 - z\bar\zeta}{\zeta - z}\right|^2 \\ &\quad - \sum_{\mu=1}^{n-1} \frac{1}{\mu(n-1)!^2}|\zeta - z|^{2(n-1-\mu)}(1-|z|^2)^\mu(1-|\zeta|^2)^\mu \,.\end{aligned} \tag{2.2}$$

Using $\tilde{G}_n$, the following theorem is proved in [33].

Theorem 2.2. *The Dirichlet problem*

$$(\partial_z\partial_{\bar z})^n w = f \ \text{ in } \ \mathbb{D},\ \ f \in L^1(\mathbb{D}) \tag{2.3}$$

$$(\partial_z\partial_{\bar z})^\mu w = \gamma_\mu,\ \gamma_\mu \in C^{n-2\mu}(\partial\mathbb{D}) \ \ 0 \le 2\mu \le n-1 \text{ on } \ \partial\mathbb{D} \tag{2.4}$$

$$\partial_{\nu_z}(\partial_z\partial_{\bar z})^\mu w = \hat\gamma_\mu,\ \hat\gamma_\mu \in C^{n-2\mu}(\partial\mathbb{D}) \ \ 0 \le 2\mu \le n-2 \ \text{ on } \ \partial\mathbb{D} \tag{2.5}$$

is uniquely solvable. The solution is

$$w(z) = -\sum_{\mu=0}^{[\frac{n}{2}]-1} \frac{1}{4\pi i}\int\limits_{\partial\mathbb{D}} \partial_{\nu_\zeta}(\partial_\zeta\partial_{\bar\zeta})^{n-\mu-1}\tilde{G}_n(z,\zeta)\gamma_\mu(\zeta)\frac{d\zeta}{\zeta}$$

$$+\sum_{\mu=0}^{[\frac{n-1}{2}]} \frac{1}{4\pi i} \int_{\partial\mathbb{D}} (\partial_\zeta \partial_{\bar{\zeta}})^{n-\mu-1} \tilde{G}_n(z,\zeta)\hat{\gamma}_\mu(\zeta)\frac{d\zeta}{\zeta}$$
$$-\frac{1}{\pi}\iint_{\mathbb{D}} \tilde{G}_n(z,\zeta) f(\zeta) d\xi d\eta.$$

A variant of the problem (2.3), (2.4), (2.5) is discussed in the case of γ_μ and $\hat{\gamma}_\mu$ are continuous on $\partial\mathbb{D}$, [17, 38]. The solution is attained by modifying the related Cauchy–Pompeiu representation with the help of the polyharmonic Green function.

The problem defined in Theorem 2.1 has also been solved using the Green–Almansi function $\tilde{G_n}$ by Kumar and Prakash [47, 49].

The Green-m-Green Almansi-n function $G_{m,n}(z,\zeta)$ for $m, n \in \mathbb{N}$ (which is also called as a polyharmonic hybrid Green function) is defined [7, 34] by the convolution of G_m and $\tilde{G}_n$ as

$$G_{m,n}(z,\zeta) = -\frac{1}{\pi}\iint_{\mathbb{D}} G_m(z,\tilde{\zeta})\tilde{G}_n(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}\ . \tag{2.6}$$

Note that, $G_{m,1}(z,\zeta) = G_{m+1}(z,\zeta)$ for $m \in \mathbb{N}$ and we take $G_{0,n}(z,\zeta) = \tilde{G}_n(z,\zeta)$ for $n \in \mathbb{N}$. Thus, $G_{0,1}(z,\zeta) = G_1(z,\zeta)$. Also, $G_{2,2}(z,\zeta)$ is defined in [33]. Eq. (2.6) is employed in the following (m,n)-type Dirichlet problem, [7].

Theorem 2.3. *The (m,n)-Dirichlet problem*

$$(\partial_z\partial_{\bar{z}})^n w = f \ \text{ in } \ \mathbb{D},$$

$$(\partial_z\partial_{\bar{z}})^\mu w = 0, \ 0 \le \mu \le m-1 \ \text{ on } \ \partial\mathbb{D}$$
$$(\partial_z\partial_{\bar{z}})^{\mu+m} w = 0, \ 0 \le 2\mu \le n-m-1 \ \text{ on } \ \partial\mathbb{D}$$
$$\partial_{\nu_z}(\partial_z\partial_{\bar{z}})^{\mu+m} w = 0, \ 0 \le 2\mu \le n-m-2 \ \text{ on } \ \partial\mathbb{D}$$

$for\ f \in L^1(\mathbb{D}) \cap C(\mathbb{D})$, *is uniquely solvable. The solution is*

$$w(z) = -\frac{1}{\pi}\iint_{\mathbb{D}} G_{m,n-m}(z,\zeta)f(\zeta)d\xi d\eta.$$

Begehr and Vaitekhovich [34] have considered the following similar problem with the inhomogeneous boundary conditions,

$$(\partial_z\partial_{\bar{z}})^{m+n} w = f \ \text{ in } \mathbb{D}$$

$$(\partial_z\partial_{\bar{z}})^\mu w = \gamma_\mu \ \text{ on } \partial\mathbb{D} \text{ for } 0 \le 2\mu \le m-1, m \le \mu \le m+n-1$$
$$\partial_\nu(\partial_z\partial_{\bar{z}})^\mu w = \tilde{\gamma}_\mu \ \text{ on } \partial\mathbb{D} \text{ for } 0 \le 2\mu \le m-2.$$

They have given the solution by an iterative technique.

The inhomogeneous polyanalytic equation is studied by Begehr and Kumar [28] in $\mathbb{D}$ with Dirichlet conditions and the following result is obtained.

Theorem 2.4. *The Dirichlet problem for the inhomogeneous polyanalytic equation in* $\mathbb{D}$

$$\partial_{\bar z}^n w = f \ \textit{in}\ \mathbb{D} \tag{2.7}$$

$$\partial_{\bar z}^\nu w = \gamma_\nu \ \ 0 \le \nu \le n-1 \ \ \textit{on} \ \ \partial\mathbb{D}$$

is uniquely solvable for $f \in L^1(\mathbb{D};\mathbb{C})$, $\gamma_\nu \in C(\mathbb{D};\mathbb{C}), 0 \le \nu \le n-1$, *if and only if for* $0 \le \nu \le n-1$

$$\sum_{\lambda=\nu}^{n-1} \left[\frac{\bar z}{2\pi i} \int_{\partial\mathbb{D}} (-1)^{\lambda-\nu} \frac{\gamma_\lambda(\zeta)}{(\lambda-\nu)!} \frac{\overline{(\zeta - z)}^{\lambda-\nu}}{1 - \bar z \zeta} d\zeta \right.$$
$$\left. + \frac{(-1)^\nu \bar z}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{(n-1-\nu)!} \frac{\overline{(\zeta - z)}^{n-1-\nu}}{1 - \bar z \zeta} d\xi d\eta \right] = 0\ .$$

If the problem is solvable, the solution is given by

$$w(z) = \sum_{\nu=0}^{n-1} \frac{(-1)^\nu}{2\pi i} \int_{\partial\mathbb{D}} \frac{\gamma_\nu(\zeta)}{\nu!} \frac{\overline{(\zeta - z)}^{\nu}}{\zeta - z} d\zeta + \frac{(-1)^n}{\pi} \iint_{\mathbb{D}} \frac{f(\zeta)}{(n-1)!} \frac{\overline{(\zeta - z)}^{n-1}}{\zeta - z} d\xi d\eta\ .$$

The Dirichlet problem for the equation

$$\partial_{\bar z}^{m+n} f + \alpha \overline{\partial_z^n \partial_{\bar z}^m} f = 0 \tag{2.8}$$

is investigated in [29] for $1 \le m, n \in \mathbb{N}$. (2.8) is known as bi-polyanalytic equation. In the case $m = 1$, the solutions are called bi-polyanalytic functions.

In a half disc and a half ring of the complex plane, the Green function is given and the Dirichlet problem for the Poisson equation is explicitly solved by Begehr and Vaitekhovich, [35].

2.1.2 Simply Connected Unbounded Domain Case

In this subsection we give an overview of the problems defined in the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : 0 < \operatorname{Im} z\}$ and in right upper quarter plane $\mathbb{Q}_1 = \{z \in \mathbb{C} : 0 < \operatorname{Re}\ z, 0 < \operatorname{Im}\ z\}$.

The polyharmonic Green function for $\mathbb{H}$ is given using the Almansi expansion [22]:

$$\tilde G_n(z,\zeta) = \frac{|\zeta - z|^{2(n-1)}}{(n-1)!^2} \log \left| \frac{\bar\zeta - z}{\zeta - z} \right|^2$$
$$- \sum_{\mu=1}^{n-1} \frac{1}{\mu (n-1)!^2} |\zeta - z|^{2(n-1-\mu)} (\zeta - \bar\zeta)^\mu (z - \bar z)^\mu.$$

Begehr and Gaertner [22] have proved the following theorem.

Theorem 2.5. *For given f satisfying $|z|^{2(n-1)}f(z) \in L_1(\mathbb{H};\mathbb{C}), \gamma_\nu \in C^{n-2\nu}(\mathbb{R};\mathbb{C})$ for $0 \leq 2\nu \leq n-1$, $\hat{\gamma}_\nu \in C^{n-1-2\nu}(\mathbb{R};\mathbb{C})$ for $0 \leq 2\nu \leq n-2$ with the respective derivatives bounded, the Dirichlet problem*

$$(\partial_{\bar{z}}\partial_z)^n w = f \text{ in } \mathbb{H}$$

$$(\partial_z\partial_{\bar{z}})^\nu w = \gamma_\nu \text{ for } 0 \leq 2\nu \leq n-1,$$
$$\partial_z^\nu \partial_{\bar{z}}^{\nu+1} w = \hat{\gamma}_\nu \text{ for } 0 \leq 2\nu \leq n-2 \text{ on } \mathbb{R}$$

is uniquely solvable in a weak sense by

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\Bigg[\left(\frac{z-\bar{z}}{t-\bar{z}}\right)^n \frac{\gamma_0(t)}{t-z} + \sum_{\mu=1}^{n-1}(-1)^\mu \frac{(z-\bar{z})^\mu}{\mu} g_\mu(z,t)\hat{\gamma}_0(t) \\ &+ \sum_{\nu=1}^{[\frac{n-1}{2}]}\Bigg[\sum_{\mu=2\nu}^{n-1}(-1)^{\mu-\nu}\frac{(\mu-\nu-1)!}{\mu!(\nu-1)!}\frac{(z-\bar{z})^\mu}{(t-z)^{\mu-2\nu+1}} \\ &+ \sum_{\mu=2\nu}^{n-1}(-1)^{\nu-1}\frac{(\mu-\nu)!}{\mu!\nu!}\frac{(z-\bar{z})^\mu}{(t-\bar{z})^{\mu-2\nu+1}}\Bigg]\gamma_\nu(t) \\ &+ \sum_{\nu=1}^{[\frac{n}{2}]-1}\sum_{\mu=2\nu+1}^{n-1}(-1)^{\mu-\nu}\frac{(\mu-\nu-1)!}{\mu!\nu!}(z-\bar{z})^\mu g_{\mu-2\nu}(z,t)\hat{\gamma}_\nu(t)\Bigg]dt \\ &- \frac{1}{\pi}\iint_{\mathbb{H}} G_n(z,\zeta)f(\zeta)d\xi d\eta, \end{aligned}$$

where for $1 \leq \alpha$,

$$g_\alpha(z,\zeta) = \frac{1}{(\bar{\zeta}-z)^\alpha} + \frac{(-1)^\alpha}{(\zeta-\bar{z})^\alpha}\ .$$

In $\mathbb{Q}_1 = \{z \in \mathbb{C} : 0 < \text{Re } z, 0 < \text{Im } z\}$, the following result for the Dirichlet boundary value problem is given for the inhomogeneous Cauchy–Riemann equation in [23].

Theorem 2.6. *The Dirichlet problem*

$$w_{\bar{z}} = f \text{ in } \mathbb{Q}_1,$$

$$w = \gamma_1 \text{ for } x \geq 0, y = 0, w = \gamma_2 \text{ for } x = 0, y \geq 0$$

for $f \in L_{p,2}(\mathbb{Q}_1;\mathbb{C}), 2 < p, \gamma_1, \gamma_2 \in C(\mathbb{R};\mathbb{C})$ such that $(1+t)^\delta\gamma_1(t), (1+t)^\delta\gamma_2(t)$ are bounded for some $\delta > 0$ and satisfying the compatibility condition $\gamma_1(0) = \gamma_2(0) = 0$ is uniquely weakly solvable in the class $C^1(\mathbb{Q}_1;\mathbb{C}) \cap C(\overline{\mathbb{Q}}_1;\mathbb{C})$ if and only if

$$\frac{1}{2\pi i}\int_0^\infty \gamma_1(t)\frac{dt}{t-\bar{z}} - \frac{1}{2\pi i}\int_0^\infty \gamma_2(t)\frac{dt}{t+i\bar{z}} - \frac{1}{\pi}\iint_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta-\bar{z}} = 0\ ,$$

$$\frac{1}{2\pi i}\int_0^\infty \gamma_1(t)\frac{dt}{t+\bar{z}} - \frac{1}{2\pi i}\int_0^\infty \gamma_2(t)\frac{dt}{t-i\bar{z}} - \frac{1}{\pi}\iint_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta+\bar{z}} = 0 \,,$$
$$\frac{1}{2\pi i}\int_0^\infty \gamma_1(t)\frac{dt}{t+z} - \frac{1}{2\pi i}\int_0^\infty \gamma_2(t)\frac{dt}{t-iz} - \frac{1}{\pi}\iint_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta+z} = 0 \,.$$

The solution is

$$w(z) = \frac{1}{2\pi i}\int_0^\infty \gamma_1(t)\frac{dt}{t-z} - \frac{1}{2\pi i}\int_0^\infty \gamma_2(t)\frac{dt}{t+iz} - \frac{1}{\pi}\iint_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta-z} \,.$$

The equation $\partial_{\bar{z}}^{n+1} f + \alpha\overline{\partial_z^n \partial_{\bar{z}} f} = 0$ which represents the bi-polyanalytic functions, is investigated in the upper half plane and different forms of boundary conditions leading to the well-known Schwarz, Dirichlet and Neumann problems in complex analysis are solved in the upper half plane in [19].

Open Problem 2.7. The results given for half disc, half ring and unbounded domains have not been extended to linear higher-order differential equations yet.

2.1.3 Multiply Connected Domain Case

The main contributions for boundary value problems are given by Begehr and Vaitekhovich [35, 54, 56]. They have considered the boundary value problems for inhomogeneous Cauchy–Riemann equation and Poisson equation in concentric ring domains.

Open Problem 2.8. The Dirichlet problems for higher-order linear differential equations in multiply connected domains have not been solved yet.

2.2 Dirichlet Problem for Complex Linear Elliptic PDEs

Now we take the linear differential equations which have the form

$$(\partial_z\partial_{\bar{z}})^n w + \sum_{\substack{k+l=2n\\(k,l)\neq(n,n)}} (q_{kl}^{(1)}(z)\partial_z^k\partial_{\bar{z}}^l w + q_{kl}^{(2)}(z)\partial_z^l\partial_{\bar{z}}^k\overline{w})$$
$$+ \sum_{0\le k+l<2n} (a_{kl}(z)\partial_z^k\partial_{\bar{z}}^l w + b_{kl}(z)\partial_z^l\partial_{\bar{z}}^k\overline{w}) = f(z) \quad \text{in } \mathbb{D} \tag{2.9}$$

where

$$a_{kl}, b_{kl}, f \in L^p(\mathbb{D}), \tag{2.10}$$

and $q_{kl}^{(1)}$ and $q_{kl}^{(2)}$, are measurable bounded functions subject to

$$\sum_{\substack{k+l=2r\\(k,l)\neq(r,r)}} (|q_{kl}^{(1)}(z)| + |q_{kl}^{(2)}(z)|) \le q_0 < 1 \,. \tag{2.11}$$

The equation (2.9) is called a generalized higher-order Poisson equation. We pose the following problem [7] in the unit disc.

Dirichlet-(m,n) Problem. Find $w \in W^{2n,p}(\mathbb{D})$ as a solution to equation (2.9) satisfying the Dirichlet condition

$$(\partial_z\partial_{\bar z})^{\mu} w = 0,\ 0 \le \mu \le m-1 \text{ on } \partial\mathbb{D} \tag{2.12}$$

$$(\partial_z\partial_{\bar z})^{\mu+m} w = 0,\ 0 \le 2\mu \le n-m-1 \text{ on } \partial\mathbb{D} \tag{2.13}$$

$$\partial_{\nu_z}(\partial_z\partial_{\bar z})^{\mu+m} w = 0,\ 0 \le 2\mu \le n-m-2 \text{ on } \partial\mathbb{D}\,. \tag{2.14}$$

We need some preparations to find the solutions.

2.2.1 A Class of Integral Operators Related to Dirichlet Problems

In this section, using $G_{m,n}(z,\zeta)$ and its derivatives with respect to z and $\bar z$ as the kernels, we define a class of integral operators related to (m,n)-type Dirichlet problems.

Definition 2.9. For $m,k,l \in \mathbb{N}_0$, $n \in \mathbb{N}$ with $(k,l) \neq (n,n)$ and $k+l \le 2n$, we define

$$G^{k,l}_{m,n} f(z) := -\frac{1}{\pi} \iint_{\mathbb{D}} \partial_z^k \partial_{\bar z}^l G_{m,n-m}(z,\zeta) f(\zeta) d\xi d\eta$$

for a suitable complex valued function f given in $\mathbb{D}$.

It is easy to observe that the operators $G^{k,l}_{m,n}$ are weakly singular for $k+l<2n$ and strongly singular for $k+l=2n$.

Using Definition 2.9, we can obtain the following operators by some particular choices of n,k,l:

$$\begin{aligned}
G^{0,0}_{0,1} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} G_1(z,\zeta) f(\zeta) d\xi d\eta = -\frac{1}{\pi} \iint_{\mathbb{D}} \log \left| \frac{1-z\bar\zeta}{\zeta - z} \right|^2 f(\zeta) d\xi d\eta, \\
G^{1,0}_{0,1} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} \partial_z G_1(z,\zeta) f(\zeta) d\xi d\eta \\
&= -\frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{1}{(\zeta - z)} - \frac{\bar\zeta}{(1 - z\bar\zeta)} \right) f(\zeta) d\xi d\eta, \\
G^{2,0}_{0,1} f(z) &= -\frac{1}{\pi} \iint_{\mathbb{D}} \partial_z^2 G_1(z,\zeta) f(\zeta) d\xi d\eta \\
&= -\frac{1}{\pi} \iint_{\mathbb{D}} \left(\frac{1}{(\zeta - z)^2} - \frac{\bar\zeta^2}{(1 - z\bar\zeta)^2} \right) f(\zeta) d\xi d\eta.
\end{aligned}$$

One should observe that, $G^{0,0}_{0,1}$, $G^{1,0}_{0,1}$ and $G^{2,0}_{0,1}$ are the operators Π_0, Π_1 and Π_2 respectively, which were investigated by Vekua [58]. It is easy to show that these operators satisfy

$$\partial_z G^{0,0}_{0,1} f = G^{1,0}_{0,1} f \ \text{ and } \ \partial_z^2 G^{0,0}_{0,1} f = G^{2,0}_{0,1} f \tag{2.15}$$

for $f \in L^p(\mathbb{D})$, $p > 2$, in Sobolev's sense. The other properties of these operators can be found in [7]. We will mention just one of them:

For $k \in \mathbb{N}$, if $f \in W^{k,p}(\mathbb{D})$, then

$$\partial_z^{k-1} G_{0,1}^{2,0} f(z) = G_{0,1}^{1,0}((D - D_*)^k f(z)) \tag{2.16}$$

where $Df(z) = \partial_z f(z)$, $D_* f(z) = \partial_{\bar{z}}(\bar{z}^2 f(z))$.

(2.16) is very important for the solution of the boundary value problem for linear partial differential equations. Our main result is given by the following theorem.

Theorem 2.10. *The equation* (2.9) *with the conditions* (2.12), (2.13) *and* (2.14) *is solvable if*

$$q_0 \max_{k+l=2n} \|G_{m,n}^{k,l}\|_{L^p(\mathbb{D})} \leq 1 \tag{2.17}$$

and a solution is of the form $w(z) = G_{m,n-m}^{0,0} g(z)$ where $g \in L^p(\mathbb{D})$, $p > 2$, is a solution of the singular integral equation

$$(I + D + K)g = f \tag{2.18}$$

where

$$Dg = \sum_{\substack{k+l=2n \\ (k,l)\neq(n,n)}} (q_{kl}^{(1)}(z) G_{m,n}^{k,l} + q_{kl}^{(2)}(z) \overline{G_{m,n}^{k,l}}) \, ,$$

$$Kg = \sum_{0\leq k+l<2n} (a_{kl}(z) G_{m,n}^{k,l} + b_{kl}(z) \overline{G_{m,n}^{k,l}}) \, .$$

3 Neumann Problem for Complex PDEs

3.1 Neumann Problem for Complex Model PDEs

3.1.1 Simply Connected Bounded Domain Case

The harmonic Neumann function for the domain $\mathbb{D}$ is given by

$$N_1(z, \zeta) = \log |(\zeta - z)(1 - z\bar{\zeta})|^2 \tag{3.1}$$

for $z, \zeta \in \mathbb{D}$, [37]. (3.1) satisfies

$$\partial_{\nu_z} N_1(z, \zeta) = (z\partial_z + \bar{z}\partial_{\bar{z}}) N_1(z, \zeta) = 2 \tag{3.2}$$

for $z \in \partial\mathbb{D}$, $\zeta \in \mathbb{D}$. But the higher-order Neumann functions are not easy to find in their explicit forms. They may be defined iteratively for $n \in \mathbb{N}$ where $n \geq 2$, as

$$N_n(z, \zeta) = \frac{1}{\pi} \iint_{\mathbb{D}} N_1(z, \tilde{\zeta}) N_{n-1}(\tilde{\zeta}, \zeta) d\tilde{\xi} d\tilde{\eta} \, . \tag{3.3}$$

For the explicit form in the case of $n = 2$ and $n = 3$, see [18, 37, 41]. By the aid of (3.3), the higher-order Poisson equation is investigated under the Neumann conditions and the following result is obtained [37].

Theorem 3.1. *The Neumann-n problem*

$$\begin{aligned}(\partial_z\partial_{\bar z})^n w &= f \text{ in } \mathbb{D}\,,\ f\in L^p(\mathbb{D}) \text{ for } 1<p<+\infty,\\ \partial_\nu(\partial_z\partial_{\bar z})^\sigma w &= \gamma_\sigma \text{ on } \partial\mathbb{D}\,,\ \gamma_\sigma\in C(\partial\mathbb{D}) \text{ for } 0\le\sigma\le n-1,\end{aligned}$$

satisfying

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(\partial_\zeta\partial_{\bar\zeta})^\sigma w(\zeta)\frac{d\zeta}{\zeta}=c_\sigma\,,\ c_\sigma\in\mathbb{C} \text{ for } 0\le\sigma\le n-1,$$

is solvable if and only if

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}\gamma_\sigma(\zeta)\frac{d\zeta}{\zeta}=\sum_{\mu=\sigma+1}^{n-1}\alpha_{\mu-\sigma}c_\mu+\frac{1}{\pi}\iint_{\mathbb{D}}\partial_{\nu_z}N_{n-\sigma}(z,\zeta)f(\zeta)d\xi d\eta\,. \tag{3.4}$$

Here $\alpha_1=2$ *and for* $3\le k$

$$\alpha_{k-1}=-\sum_{\mu=[\frac{k}{2}]}^{k-2}\frac{\mu!^2}{(k-1)!(k-1-\mu)!^2(2\mu-k+1)!}\,\alpha_\mu. \tag{3.5}$$

The solution is unique and given by

$$\begin{aligned}w(z)=&\sum_{\mu=0}^{n-1}\left\{\frac{1}{2}c_\mu\partial_{\nu_\zeta}N_{\mu+1}(z,\zeta)-\frac{1}{4\pi i}\int_{\partial\mathbb{D}}N_{\mu+1}(z,\zeta)\gamma_\mu(\zeta)\frac{d\zeta}{\zeta}\right\}\\ &+\frac{1}{\pi}\iint_{\mathbb{D}}N_n(z,\zeta)f(\zeta)d\xi d\eta\,.\end{aligned}$$

Particularly, for the inhomogeneous biharmonic equation, analogous results are presented in [16,18]. We may consult with [13] for the solutions of Bitsadze equation under Neumann conditions.

The inhomogeneous polyanalytic equation (2.7) with the half-Neumann conditions

$$z\partial_{\bar z}^\nu\partial_z w=\gamma_\nu \text{ on } \partial\mathbb{D}, \partial_{\bar z}^\nu w(0)=c_\nu$$

is uniquely solved with some solvability conditions in [28].

3.1.2 Unbounded Domain Case

The Neumann boundary value problem is considered for the inhomogeneous Cauchy–Riemann equation in a quarter plane and the solvability conditions and solutions are given in explicit form in [23].

Neumann Problem. Let $f\in L_{p,2}(\mathbb{Q}_1;\mathbb{C})\cap C^\alpha(\overline{\mathbb{Q}}_1;\mathbb{C})$ for $2<p, 0<\alpha<1, \gamma_1,\gamma_2\in C(\mathbb{R}^+;\mathbb{C})$ such that $(1+t)^\delta\gamma_1(t),(1+t)^\delta\gamma_2(t),(1+t)^\delta f(t),(1+t)^\delta f(it)$ are bounded for some $0<\delta, c\in\mathbb{C}$. Find $w\in C^1(\overline{\mathbb{Q}}_1;\mathbb{C})$ satisfying

$$w_{\bar z}=f \text{ in } \mathbb{Q}_1, \partial_y w=\gamma_1 \text{ for } 0<x, y=0,\ \partial_x w=\gamma_2 \text{ for } 0<y, x=0, w(0)=c.$$

Theorem 3.2. *The Neumann problem is uniquely solvable in the weak sense if and only if for any* $z \notin \overline{\mathbb{Q}_1}$

$$\frac{1}{2\pi}\int_0^{+\infty}[\gamma_1(t)+if(t)]\frac{dt}{t-z}+\frac{1}{2\pi i}\int_0^{+\infty}[\gamma_2(t)-f(it)]\frac{dt}{t+iz}+\frac{1}{\pi}\iint_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{(\zeta-z)^2}=0 \tag{3.6}$$

holds. The solution is

$$\begin{aligned} w(z) &= c+\frac{1}{2\pi}\int_0^{+\infty}[\gamma_1(t)+if(t)]\log\left|\frac{t^2-z^2}{t^2}\right|^2 dt \\ &\quad +\frac{1}{2\pi}\int_0^{+\infty}[\gamma_2(t)-f(it)]\log\left|\frac{t^2+z^2}{t^2}\right|^2 dt-\frac{z}{\pi}\iint_{\mathbb{Q}_1}\frac{f(\zeta)}{\zeta}\frac{d\xi d\eta}{\zeta-z}. \end{aligned} \tag{3.7}$$

Also, in the upper half plane the Neumann problem is considered for the inhomogeneous Cauchy–Riemann equation and Poisson equation, [42].

3.1.3 Multiply Connected Domain Case

The Neumann problem for analytic functions, more generally for the inhomogeneous Cauchy–Riemann equation and Poisson equation are investigated in a circular ring domain; the representations to the solutions and solvability conditions are given in an explicit form by Vaitekhovich [54–56].

3.2 Neumann Problem for Complex Linear Elliptic PDEs

For $n \in \mathbb{N}$, $k,l \in \mathbb{N}_0$ with $(k,l) \neq (n,n)$ and $k+l \leq 2n$, the operators given by

$$S_{n,k,l}f(z)=\frac{1}{\pi}\iint_{\mathbb{D}}\partial_z^k\partial_{\bar z}^l N_n(z,\zeta)f(\zeta)d\xi d\eta$$

for a suitable complex valued function f given in $\mathbb{D}$, are the operators related to Neumann problem for generalized n-Poisson equations. In [6], these operators are shown to be uniformly bounded and uniformly continuous for the case $n \leq 2$ and $f \in L^p(\mathbb{D})$ for $p > 2$ and bounded in $L^p(\mathbb{D})$ for $f \in L^p(\mathbb{D})$ and $n > 1$. Using these operators and a property similar to (2.16), the following problem is investigated.

Neumann Problem. Find $w \in W^{2n,p}(\mathbb{D})$ as a solution of the linear complex partial differential equation (generalized n-Poisson equation)

$$\frac{\partial^{2n}w}{\partial z^n\partial\bar z^n}+\sum_{\substack{k+l=2n\\ k\neq l}}\left(q_{kl}^{(1)}(z)\frac{\partial^{2n}w}{\partial z^k\partial\bar z^l}+q_{kl}^{(2)}(z)\frac{\partial^{2n}\overline{w}}{\partial z^k\partial\bar z^l}\right)$$

$$+\sum_{0\le k+l<2n}\left[a_{kl}(z)\frac{\partial^{k+l}w}{\partial\bar{z}^k\partial z^l}+b_{kl}(z)\frac{\partial^{k+l}\overline{w}}{\partial z^k\partial\bar{z}^l}\right]=f(z)\text{ in }\mathbb{D},\tag{3.8}$$

where

$$a_{kl},b_{kl},f\in L^p(\mathbb{D}),\tag{3.9}$$

and $q_{kl}^{(1)}$ and $q_{kl}^{(2)}$, are measurable bounded functions satisfying

$$\sum_{\substack{k+l=2n\\k\neq l}}(|q_{kl}^{(1)}(z)|+|q_{kl}^{(2)}(z)|)\le q_0<1\tag{3.10}$$

with Neumann conditions

$$\partial_\nu(\partial_z\partial_{\bar z})^\sigma w \;=\; \gamma_\sigma \text{ on } \partial\mathbb{D}\ ,\ \gamma_\sigma\in C(\partial\mathbb{D};\mathbb{C}) \text{ for } 0\le\sigma\le n-1\tag{3.11}$$

satisfying

$$\frac{1}{2\pi i}\int_{\partial\mathbb{D}}(\partial_\zeta\partial_{\bar\zeta})^\sigma w(\zeta)\frac{d\zeta}{\zeta}=c_\sigma\ ,\ c_\sigma\in\mathbb{C}\text{ for }0\le\sigma\le n-1.\tag{3.12}$$

The solvability of this problem is given in the following theorem.

Theorem 3.3. *If the inequality*

$$q_0\max_{\substack{k+l=2n\\|k-l|\neq 2}}\|S_{n,k,l}\|_{L^p(\mathbb{D})}\|(I+\hat{N}_1)^{-1}-K\|_{L^p(\mathbb{D})}<1\tag{3.13}$$

holds for some $K\in K(L^p(\mathbb{D}))$, $0<p-2<\epsilon$, then the equation (3.8) *with the boundary conditions* (3.11) *and normalization conditions* (3.12) *has a solution of the form $w(z)=S_{n,0,0}g(z)+\varphi(z)$, where $g\in L^p(\mathbb{D})$ is a solution of the singular integral equation*

$$(I+\hat{N}+\hat{K})g=\hat{f}\ ,\tag{3.14}$$

where

$$\varphi(z)=\sum_{\mu=0}^{n-1}\left\{\frac{1}{2}c_\mu\partial_{\nu_\zeta}N_{\mu+1}(z,\zeta)-\frac{1}{4\pi i}\int_{\partial\mathbb{D}}N_{\mu+1}(z,\zeta)\gamma_\mu(\zeta)\frac{d\zeta}{\zeta}\right\}$$

and

$$\hat{N}g=\sum_{\substack{k+l=2n\\|k-l|=2}}(q_{kl}^{(1)}S_{n,k,l}g+q_{kl}^{(2)}\overline{S_{n,k,l}g})+\sum_{\substack{k+l=2n\\|k-l|\neq 2}}(q_{kl}^{(1)}S_{n,k,l}g+q_{kl}^{(2)}\overline{S_{n,k,l}g})$$

$$:=\hat{N}_1g+\hat{N}_2g,$$

$$\hat{K}g=\sum_{k+l<2n}\left(a_{kl}S_{n,k,l}g+b_{kl}\overline{S_{n,k,l}g}\right),$$

$$\hat{f}=f-L\varphi$$

in which

$$L\varphi := \sum_{\substack{k+l=2n\\k\neq l}} \left(q_{kl}^{(1)}(z)\frac{\partial^{2n}\varphi}{\partial z^k \partial \bar{z}^{(l)}} + q_{kl}^{(2)}(z)\frac{\partial^{2n}\overline{\varphi}}{\partial z^l \partial \bar{z}^k} \right)$$
$$+ \sum_{0\leq k+l<2n} \left[a_{kl}(z)\frac{\partial^{k+l}\varphi}{\partial \bar{z}^k \partial z^l} + b_{kl}(z)\frac{\partial^{k+l}\overline{\varphi}}{\partial z^k \partial \bar{z}^l} \right]$$

subject to the solvability conditions

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \gamma_\sigma(\zeta)\frac{d\zeta}{\zeta} = \sum_{\mu=\sigma+1}^{n-1} \alpha_{\mu-\sigma}c_\mu + \frac{1}{\pi}\iint\limits_{\mathbb{D}} \partial_{\nu_z} N_{n-\sigma}(z,\zeta)g(\zeta)d\xi d\eta, \tag{3.15}$$

where $0 \leq \sigma \leq n-1$, $\alpha_1 = 2$ *and for* $3 \leq k$

$$\alpha_{k-1} = -\sum_{\mu=[\frac{k}{2}]}^{k-2} \frac{\mu!^2}{(k-1)!(k-1-\mu)!^2(2\mu-k+1)!}\,\alpha_\mu. \tag{3.16}$$

Open Problem 3.4. The Neumann problems for higher-order linear complex partial differential equations are not considered in half plane, quarter plane and concentric rings. These problems can be handled after some studies of the corresponding integral operators.

4 Robin Problem for Complex PDEs

4.1 Robin Problem for Complex Model PDEs

4.1.1 Simply Connected Domain Case

Begehr and Harutyunyan [24] obtained the following result for the Robin problem in $\mathbb{D}$.

Theorem 4.1. *The Robin problem for the inhomogeneous Cauchy–Riemann equation in the unit disc*

$$w_{\bar{z}} = f \text{ in } \mathbb{D},\ w + \partial_\nu w = \gamma \text{ on } \mathbb{D}$$

is uniquely solvable for given $f \in L^1(\mathbb{D}) \cap C(\partial D)$, $\gamma \in C(\partial D)$ *if and only if for all z,* $|z| < 1$,

$$z\left[\frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\gamma(\zeta) - \bar{\zeta}f(\zeta))\frac{d\zeta}{1-\bar{z}\zeta} + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{\bar{z}\zeta}{(1-\bar{z}\zeta)^2}f(\zeta)d\xi d\eta\right] = 0,$$

and the solution is

$$w(z) = \frac{1}{2\pi i}\int_{\partial\mathbb{D}} (\bar{\zeta}f(\zeta) - \gamma(\zeta))\frac{\ln(1-z\bar{\zeta})}{z}d\zeta + \frac{1}{\pi}\iint_{\mathbb{D}} \frac{1}{z-\zeta}f(\zeta)d\xi d\eta.$$

In the same article, they have also investigated the problem

$$w^n_{\bar z} = f \text{ in } \mathbb{D},$$
$$\partial^{\nu-1}_{\bar z} w + z\partial^{\nu-1}_{\bar z} w + \bar z \partial^{\nu}_{\bar z} w = \gamma_\nu \text{ on } \mathbb{D},\ \nu = 1,\dots,n$$

and obtained the representation of the solution with the corresponding solvability conditions by converting the problem into an equivalent system of n Robin problems for the Cauchy–Riemann operator.

The Robin function for harmonic operator is

$$R_1(z,\zeta) = \log\left|\frac{1-z\bar\zeta}{\zeta - z}\right|^2 - 2\left[\frac{\log(1-z\bar\zeta)}{z\bar\zeta} + \frac{\log(1-\bar z\zeta)}{\bar z\zeta} + 1\right]$$

and for polyharmonic operator

$$R_n(z,\zeta) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}} R_1(z,\tilde\zeta)R_{n-1}(\tilde\zeta,\zeta)d\tilde\xi d\tilde\eta\ .$$

The Robin problem for inhomogeneous harmonic equation is treated in [16, 18, 36]. For the higher-order Poisson operators the problem is defined by

$$(\partial_z\partial_{\bar z})^n w = f \text{ in } \mathbb{D}$$
$$(\partial_z\partial_{\bar z})^{\nu-1} w + z\partial^\nu_z\partial^{\nu-1}_{\bar z} w + \bar z\partial^{\nu-1}_z\partial^\nu_{\bar z} w = \gamma_\nu,\ \ \nu = 1,\dots n \text{ on } \partial\mathbb{D}$$

This problem is studied by Begehr and Harutyunyan [25]. In the cases $n = 1$ and $n = 2$, the explicit solutions are given for the corresponding problems.

4.1.2 Unbounded Domain Case

The following Robin boundary value problem is investigated in $\mathbb{Q}_1$ for the inhomogeneous Cauchy–Riemann equation in [1].

Robin Problem. Let $f \in L_{p,2}(\mathbb{Q}_1;\mathbb{C}) \cap C^\alpha(\overline{\mathbb{Q}}_1;\mathbb{C})$ for $2 < p, 0 < \alpha < 1, \gamma_1, \gamma_2 \in C(\mathbb{R}^+;\mathbb{C})$ such that for some $0 < \delta$ the functions $(1+t)^\delta\gamma_1(t), (1+t)^\delta\gamma_2(t), (1+t)^\delta f(t), (1+t)^\delta f(it)$ are bounded on $\mathbb{R}^+, c \in \mathbb{C}$. Find $w \in C^1(\overline{\mathbb{Q}}_1;\mathbb{C})$ satisfying

$$\begin{aligned} &w_{\bar z} = f \text{ in } \mathbb{Q}_1, w(0) = c,\\ &w - i\partial_y w = \gamma_1 \text{ for } 0 < x, y = 0,\\ &w + \partial_x w = \gamma_2 \text{ for } 0 < y, x = 0.\end{aligned}$$

Theorem 4.2. *This particular Robin problem is uniquely solvable in the weak sense if and only if for* $z \notin \overline{\mathbb{Q}}_1$

$$\frac{1}{2\pi i}\int_0^{+\infty}\int_0^t\left[\gamma_1(\tau) + \frac{3}{2}f(\tau)\right]e^{\tau - t}d\tau\frac{dt}{t-z}$$
$$-\frac{i}{2\pi i}\int_0^{+\infty}\int_0^t\left[\gamma_2(\tau) - \frac{1}{2}f(i\tau)\right]e^{i(\tau-t)}d\tau\frac{idt}{it-z} = 0. \qquad (4.1)$$

The solution is

$$\begin{aligned}
w(z) &= [c - Tf(0)]e^{-z} + \frac{1}{2\pi i}\int_0^{+\infty}\int_0^t \left[\gamma_1(\tau) + \frac{3}{2}f(\tau)\right]e^{\tau - t}d\tau\frac{dt}{t-z} \\
&\quad - \frac{1}{2\pi i}\int_0^{+\infty}\int_0^t \left[\gamma_2(\tau) - \frac{1}{2}f(i\tau)\right]e^{i(\tau-t)}id\tau\frac{idt}{it-z} \qquad (4.2) \\
&\quad - \int_0^z T(f+f_\zeta)(\zeta)e^{\zeta - z}d\zeta + \int_0^z \frac{1}{2\pi i}\int_{\partial\mathbb{Q}_1} f(\tilde{\zeta})\frac{d\tilde{\zeta}}{\tilde{\zeta}-\zeta}e^{\zeta - z}d\zeta + Tf(z).
\end{aligned}$$

In Theorem 4.2,

$$Tf(z) = -\frac{1}{\pi}\iint_{\mathbb{Q}_1} f(\zeta)\frac{d\xi d\eta}{\zeta - z}.$$

4.1.3 Multiply Connected Domain Case

Explicit Robin functions are given for Poisson equation for a circular ring in the complex plane by Begehr and Vaitekhovich [36]. Robin boundary value problem for analytic functions and for the inhomogeneous Cauchy–Riemann equation are investigated in ring domains [56].

4.2 Open Problems with Robin Conditions

For the higher-order linear differential equations, Robin boundary value problem is not considered particularly. Just in the last section of this survey, it will be considered as a part of a mixed problem in the unit disc. The Robin problem is not considered for higher-order model and linear differential equations in the case of unbounded domains and multiply connected domains. The corresponding Robin functions are not known yet.

5 Schwarz Problem for Complex PDEs

5.1 Schwarz Problem for Complex Model PDEs

5.1.1 Simply Connected Domain Case

The first article in Schwarz problem for analytic functions is given in [52]. The following theorem gives the unique solution of the Schwarz problem for inhomogeneous polyanalytic equation, [10, 14, 31].

Theorem 5.1. *The Schwarz problem for the homogeneous polyanalytic equation in the unit disc $\mathbb{D}$ defined by*

$$\partial_{\bar z}^k w = f \text{ in } \mathbb{D}\ , \ \ \mathrm{Re}\ \partial_{\bar z}^l w = 0 \text{ on } \partial\mathbb{D}\ , \ \ \mathrm{Im}\ \partial_{\bar z}^l w(0) = 0\ , \ 0 \le l \le n-1\ ,$$

is uniquely solvable for $f \in L^1(\mathbb{D})$. The solution is

$$w(z) = \frac{(-1)^k}{2\pi(k-1)!} \iint\limits_{\mathbb{D}} \Big(\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\Big)(\zeta - z + \overline{\zeta - z})^{k-1} d\xi d\eta\ .$$

Previously the cases of $k=1$ and $k=2$ have been studied [13].

5.1.2 Unbounded Domain Case

In the upper right quarter plane, the following problem is defined and solved by Abdymanapov et al [2].

Schwarz Problem. Let $f \in L_1(\mathbb{Q}_1;\mathbb{C}), \gamma_1, \gamma_2 \in C(\mathbb{R}^+;\mathbb{R})$ be bounded on $\mathbb{R}^+ = (0,+\infty)$. Find a solution of

$$w_{\bar z} = f \text{ in } \mathbb{Q}_1 \text{ satisfying}$$

$$\mathrm{Re}\ w = \gamma_1 \text{ on } 0 < x, y = 0,$$

$$\mathrm{Im}\, w = \gamma_2 \text{ on } 0 < y, x = 0.$$

Theorem 5.2. *The Schwarz problem is uniquely weakly solvable. The solution is*

$$\begin{aligned} w(z) &= \frac{2}{\pi i}\int_0^{+\infty} \gamma_1(t)\frac{z}{t^2 - z^2}dt - \frac{2}{\pi i}\int_0^{+\infty}\gamma_2(t)\frac{z}{t^2+z^2}dt \\ &\quad -\frac{2}{\pi}\iint\limits_{\mathbb{Q}_1}\left[\frac{zf(\zeta)}{\zeta^2 - z^2} - \frac{z\overline{f(\zeta)}}{\bar\zeta^2 - z^2}\right]d\xi d\eta. \end{aligned} \tag{5.1}$$

In the case of upper half plane $\mathbb{H}$, the following result is obtained in [42].

Theorem 5.3. *Let $f \in L_{p,2}(\mathbb{H};\mathbb{C}), 2 < p, \gamma \in C(\mathbb{R})$, $c \in \mathbb{R}$ such that γ is bounded on $\mathbb{R}$. Then the Schwarz problem*

$$w_{\bar z} = f \ \text{ in } \ \mathbb{H}$$

$$\mathrm{Re}\, w = \gamma \ \text{ on } \ \mathbb{H}, \quad \mathrm{Im}\, w(i) = c$$

is uniquely solvable in the weak sense. The solution is

$$\begin{aligned} w(z) = ic &+ \frac{1}{\pi i}\int_{-\infty}^{\infty}\gamma(t)\left(\frac{1}{t-z} - \frac{t}{t^2+1}\right)dt \\ &- \frac{1}{\pi i}\iint_{\mathbb{H}}\left[f(\zeta)\left(\frac{1}{\zeta - z} - \frac{\zeta}{\zeta^2+1}\right) - \overline{f(\zeta)}\left(\frac{1}{\bar\zeta - z} - \frac{\bar\zeta}{\bar\zeta^2+1}\right)\right]d\xi d\eta. \end{aligned}$$

5.1.3 Multiply Connected Domain Case

Schwarz problems are solved for the inhomogeneous Cauchy–Riemann equation and Poisson equation in a concentric ring domain (concentric annulus) by Vaitekhovich [54–56]

5.2 Schwarz Problem for Complex Linear Elliptic PDEs

Begehr [10, 31] considered the Schwarz problem for some higher-order equations and proved the solvability of the problem. Schwarz problem for a general linear elliptic complex partial differential equation whose leading term is the polyanalytic operator is discussed in [3, 5].

Schwarz Problem. Find $w \in W^{k,p}(\mathbb{D})$ as a solution to the k-th order complex differential equation

$$\frac{\partial^k w}{\partial \bar{z}^k} + \sum_{j=1}^{k} q_{1j}(z) \frac{\partial^k w}{\partial \bar{z}^{k-j} \partial z^j} + \sum_{j=1}^{k} q_{2j}(z) \frac{\partial^k \overline{w}}{\partial z^{k-j} \partial \bar{z}^j} + \sum_{l=0}^{k-1} \sum_{m=0}^{l} \left[a_{ml}(z) \frac{\partial^l w}{\partial \bar{z}^{l-m} \partial z^m} + b_{ml}(z) \frac{\partial^l \overline{w}}{\partial z^{l-m} \partial \bar{z}^m} \right] = f(z) \text{ in } \mathbb{D}, \quad (5.2)$$

where

$$a_{ml}, b_{ml} \in L^p(\mathbb{D}), f \in L^p(\mathbb{D}), \quad (5.3)$$

and q_{1j} and q_{2j}, $j = 1, \ldots, k$, are measurable bounded functions satisfying

$$\sum_{j=1}^{k} (|q_{1j}(z)| + |q_{2j}(z)|) \leq q_0 < 1 \quad (5.4)$$

satisfying the nonhomogeneous Schwarz boundary conditions

$$\operatorname{Re} \frac{\partial^l w}{\partial \bar{z}^l} = \gamma_l \text{ on } \partial\mathbb{D}, \qquad \operatorname{Im} \frac{\partial^l w}{\partial \bar{z}^l}(0) = c_l, \ 0 \leq l \leq k-1, \quad (5.5)$$

where $\gamma_l \in C(\partial\mathbb{D}; \mathbb{R}), c_l \in \mathbb{R}$, $0 \leq l \leq k-1$.

Theorem 5.4. *If the inequality*

$$q_0 \max_{1 \leq j \leq k} \|P_j\|_{L^p(\mathbb{D})} \|(I + \hat{T})^{-1} - K_1\|_{L^p(\mathbb{D})} < 1 \quad (5.6)$$

is satisfied for some $K_1 \in K(L^p(\mathbb{D}))$, $0 < p - 2 < \epsilon$, *then equation* (5.2) *with the boundary conditions* (5.5) *has a solution of the form*

$$w = \widetilde{T}_k g_1 + i \sum_{l=0}^{k-1} \frac{c_l}{l!} (z + \bar{z})^l + \sum_{l=0}^{k-1} \frac{(-1)^l}{2\pi i l!} \int_{\partial\mathbb{D}} \gamma_l(\zeta) \frac{\zeta + z}{\zeta - z} (\zeta - z + \overline{\zeta - z})^l \frac{d\zeta}{\zeta}, \quad (5.7)$$

where $g_1 \in L^p(\mathbb{D})$, $p > 2$, *is a solution of the singular integral equation*

$$(I + \hat{\Pi} + \hat{K})g_1 = \tilde{f}, \tag{5.8}$$

where

$$\hat{\Pi}g = \sum_{j=1}^{k}(q_{1j}\Pi_j g + q_{2j}\overline{\Pi_j g}) = \sum_{j=1}^{k}(q_{1j}(P_j + T_{-j,j})g + q_{2j}\overline{(P_j g + T_{-j,j}g)})$$
$$= \hat{P}g + \hat{T}g \tag{5.9}$$

and

$$\hat{K}g = \sum_{l=0}^{k-1}\sum_{m=0}^{l}\left(a_{ml}\frac{\partial^m \widetilde{T}_{k-l+m}g}{\partial z^m} + b_{ml}\frac{\partial^m \overline{\widetilde{T}_{k-l+m}g}}{\partial \bar{z}^m}\right). \tag{5.10}$$

In Theorem 5.4, the operators $\widetilde{T}_k$ are defined as

$$\widetilde{T}_k f(z) := \frac{(-1)^k}{2\pi(k-1)!}\iint_{\mathbb{D}}(\overline{\zeta - z} + \zeta - z)^{k-1}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar{\zeta}}\frac{1+z\bar{\zeta}}{1-z\bar{\zeta}}\right]d\xi d\eta$$

for $k \in \mathbb{N}$ with $\widetilde{T}_0 f(z) = f(z)$, see [11, 12, 27]. $\partial_z^l \widetilde{T}_k$ are weakly singular integral operators for $0 \le l \le k-1$, while

$$\Pi_k f(z) := \frac{\partial^k}{\partial z^k}\widetilde{T}_k f(z) = \frac{(-1)^k k}{\pi}\iint_{\mathbb{D}}\left[\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1}\frac{f(\zeta)}{(\zeta - z)^2}\right.$$
$$\left. + \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar{\zeta}}\bar{\zeta} - 1\right)^{k-1}\frac{\overline{f(\zeta)}}{(1-z\bar{\zeta})^2}\right] d\xi d\eta \tag{5.11}$$

is a Calderon-Zygmund type strongly singular integral operator. Π_k are shown to be bounded in the space L^p for $1 < p < \infty$ and in particular their L^2 norms are estimated in [4]. These operators are investigated by decomposing them into two parts as $\Pi_k = T_{-k,k} + P_k$, where

$$T_{-k,k}f(z) = \frac{(-1)^k k}{\pi}\iint_{\mathbb{D}}\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1}\frac{f(\zeta)}{(\zeta - z)^2}d\xi d\eta, \tag{5.12}$$

which is investigated extensively in [21, 26].

In [44], the Schwarz problem for the Beltrami equation

$$w_{\bar{z}} + c w_z = f \ \text{ in } \ \mathbb{D}$$
$$\operatorname{Re} w = \gamma, \ \ \operatorname{Im} w(0) = c \ \text{ on } \ \mathbb{D}$$

is solved as a particular form of the above problem.

The Schwarz problem for Poisson equation is explicitly solved in [14].

Open Problem 5.5. The Schwarz problem is not considered for higher-order Poisson equations in $\mathbb{D}$, in unbounded domains and in multiply connected domains.

6 Mixed Type Problems for Complex PDEs

6.1 Mixed Type Problems for Complex Model PDEs

6.1.1 Simply Connected Domain Case

In order to state and solve the mixed problems containing Schwarz, Neumann, Dirichlet and Robin problems, we define the following polyharmonic hybrid Green type functions:

$$H_{m,n}(z,\zeta) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}} G_m(z,\tilde{\zeta})N_n(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta},$$

$$I_{m,n}(z,\zeta) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}} G_m(z,\tilde{\zeta})R_n(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta},$$

$$J_{m,n}(z,\zeta) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}} N_m(z,\tilde{\zeta})R_n(\tilde{\zeta},\zeta)d\tilde{\xi}d\tilde{\eta}$$

which are obtained by convoluting Green, Neumann and Robin functions iteratively, [8].

The integral operators

$$B^{k,l}_{a,b,m-(a+b),n-m}f(z) := \partial_z^k\partial_{\bar{z}}^l\tilde{T}_{n-m}(I_{a,b,m-(a+b)}f(z))$$

are defined in relation to the mixed problems, [8]. The higher-order model differential equation with mixed boundary conditions is discussed in the following theorem.

Theorem 6.1. *The mixed problem for model equation*

$$\partial_z^m\partial_{\bar{z}}^n w = f \ \textit{ in } \ \mathbb{D}, \ \ n \geq m \ ,$$

$$\operatorname{Re}\partial_{\bar{z}}^{\mu} w = 0 \ \textit{ on } \partial\mathbb{D} \ , \quad \operatorname{Im}\partial_{\bar{z}}^{\mu} w(0) = 0 \ , \ 0 \leq \mu \leq n-m-1 \ ,$$

$$\partial_z^{\mu}\partial_{\bar{z}}^{\mu+n-m} w = 0, \ 0 \leq \mu \leq a-1 \ \textit{ on } \ \partial\mathbb{D}$$

$$\partial_{\nu_z}(\partial_z^{\mu+a}\partial_{\bar{z}}^{\mu+n-m+a})w = 0, \ 0 \leq \mu \leq b-1 \ \textit{ on } \ \partial\mathbb{D}$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} \partial_{\zeta}^{\mu+a}\partial_{\bar{\zeta}}^{\mu+n-m+a} w(\zeta)\frac{d\zeta}{\zeta} = 0, \ 0 \leq \mu \leq b-1 \ \textit{ on } \ \partial\mathbb{D}$$

$$\partial_z^{\mu+a+b}\partial_{\bar{z}}^{\mu+n-m+a+b} w + \partial_{\nu_z}(\partial_z^{\mu+a+b}\partial_{\bar{z}}^{\mu+n-m+a+b})w = 0,$$

$$0 \leq \mu \leq m-a-b-1 \ \textit{ on } \ \partial\mathbb{D}$$

for $f \in L^p(\mathbb{D})$, *is uniquely solvable iff*

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \partial_{\nu_z} N_{n-m+a-\mu}(z,\zeta)\tilde{f}(\zeta)d\xi d\eta \ = 0 \tag{6.1}$$

for a suitable $\tilde{f}$. The solution is

$$w(z) = B^{0,0}_{a,b,m-a-b,n-m} f(z),$$

where $(\partial_z \partial_{\bar{z}})^{m-a-b} \tilde{f} = f$ satisfy the conditions

$$\partial_z^{\mu+a+b} \partial_{\bar{z}}^{\mu+n-m+a+b} \tilde{f} + \partial_{\nu_z} (\partial_z^{\mu+a+b} \partial_{\bar{z}}^{\mu+n-m+a+b}) \tilde{f} = 0, \tag{6.2}$$

$$0 \le \mu \le m-a-b-1 \text{ on } \partial\mathbb{D}.$$

Remark 6.2. The problem given in Theorem 6.1 covers some mixed problems given in [16, 18] for bi-Poisson equation and in [30, 48] for inhomogeneous polyanalytic and polyharmonic equations with homogeneous boundary conditions cases.

6.2 Mixed Type Problems for Complex Elliptic Linear PDEs

We consider the following mixed problem for higher-order complex differential equation of arbitrary order [8].

Problem M. Find $w \in W^{m+n,p}(\mathbb{D})$ as a solution to the equation

$$\partial_z^m \partial_{\bar{z}}^n w + \sum_{\substack{k+l=m+n \\ (k,l)\neq(m,n)}} \left(q^{(1)}_{kl}(z) \partial_z^k \partial_{\bar{z}}^l w + q^{(2)}_{kl}(z) \partial_z^l \partial_{\bar{z}}^k \bar{w} \right)$$

$$+ \sum_{0\le k+l<m+n} \left(a_{kl}(z) \partial_z^k \partial_{\bar{z}}^l w + b_{kl}(z) \partial_z^l \partial_{\bar{z}}^k \bar{w} \right) = f(z) \tag{6.3}$$

satisfying boundary conditions

$$\operatorname{Re} \partial_{\bar{z}}^{\mu} w = 0 \text{ on } \partial\mathbb{D} \,,$$

$$\operatorname{Im} \partial_{\bar{z}}^{\mu} w(0) = 0 \,, \quad 0 \le \mu \le n-m-1 \tag{6.4}$$

$$\partial_z^{\mu} \partial_{\bar{z}}^{\mu+n-m} w = 0, \ 0 \le \mu \le a-1 \text{ on } \partial\mathbb{D} \tag{6.5}$$

$$\partial_{\nu_z} (\partial_z^{\mu+a} \partial_{\bar{z}}^{\mu+n-m+a}) w = 0, \ 0 \le \mu \le b-1 \text{ on } \partial\mathbb{D} \tag{6.6}$$

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} \partial_\zeta^{\mu+a} \partial_{\bar{\zeta}}^{\mu+n-m+a} w(\zeta) \frac{d\zeta}{\zeta} = 0, \ 0 \le \mu \le b-1 \text{ on } \partial\mathbb{D},$$

$$\partial_z^{\mu+a+b} \partial_{\bar{z}}^{\mu+n-m+a+b} w + \partial_{\nu_z} (\partial_z^{\mu+a+b} \partial_{\bar{z}}^{\mu+n-m+a+b}) w = 0 \tag{6.7}$$

$$0 \le \mu \le m-a-b-1 \text{ on } \partial\mathbb{D},$$

where

$$a_{kl}, b_{kl}, f \in L^p(\mathbb{D}) \tag{6.8}$$

and $q_{kl}^{(1)}$ and $q_{kl}^{(2)}$, are measurable bounded functions

$$\sum_{\substack{k+l=m+n \\ (k,l)\neq(m,n)}} (|q_{kl}^{(1)}(z)| + |q_{kl}^{(2)}(z)|) \le q_0 < 1 \tag{6.9}$$

and $l < n-m$ for $k+l \ge n-m$, $k+l < m+n$ for $l \ge n-m$, $k+l = m+n$ for $l \ge n-m$.

We transform the Problem M to a singular integral equation.

Lemma 6.3. *The mixed problem* (6.3), (6.4), (6.5), (6.6) *and* (6.7) *is equivalent to the singular integral equation*

$$(I + \hat{M} + \hat{K})g = f, \tag{6.10}$$

if

$$w = B_{a,b,m-a-b,n-m}^{0,0} g,$$

where

$$\hat{M}g = \sum_{\substack{k+l=m+n \\ (k,l)\neq(m,n)}} (q_{kl}^{(1)} B_{a,b,m-a-b,n-m}^{k,l} g + q_{kl}^{(2)} \overline{B_{a,b,m-a-b,n-m}^{k,l} g}),$$

$$\hat{K}g = \sum_{k+l<m+n} \left(a_{kl} B_{a,b,m-a-b,n-m}^{k,l} g + b_{kl} \overline{B_{a,b,m-a-b,n-m}^{k,l} g} \right).$$

Solvability of the problem is given in the next theorem.

Theorem 6.4. *If the inequality*

$$q_0 \max_{k+l=m+n} \|B_{a,b,m-a-b,n-m}^{k,l} f\|_{L^p(\mathbb{D})} \le 1 \tag{6.11}$$

is satisfied, then equation (6.3) *with the conditions* (6.4), (6.5), (6.6) *and* (6.7) *has a solution of the form* $w = B_{a,b,m-a-b,n-m}^{0,0} g$*, where* $g \in L^p(\mathbb{D})$ *is a solution of the singular integral equation* (6.10) *with* $p > 2$ *and* g *satisfies the solvability condition*

$$\frac{1}{\pi} \iint_{\mathbb{D}} \partial_{\nu_z} N_{n-m+a-\mu}(z,\zeta) \tilde{g}(\zeta) d\xi d\eta \ = 0, \tag{6.12}$$

where $(\partial_z \partial_{\bar{z}})^{m-a-b} \tilde{g} = g$ *satisfy the conditions*

$$\partial_z^{\mu+a+b} \partial_{\bar{z}}^{\mu+n-m+a+b} \tilde{g} + \partial_{\nu_z} (\partial_z^{\mu+a+b} \partial_{\bar{z}}^{\mu+n-m+a+b}) \tilde{g} = 0, \tag{6.13}$$

$$0 \le \mu \le m-a-b-1 \ \textit{ on } \ \partial\mathbb{D} \ .$$

Open Problem 6.5. On unbounded domains and multiply connected domains, mixed type problems are not studied for higher-order linear equations and model equations.

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