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Computation of Formal Solutions of Systems of Linear Difference Equations

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Abstract

We present a step by step algorithm which allows to compute a formal fundamental solution for certain systems of first order linear difference equations.

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1 Introduction

In many applications, one is concerned with higher order difference equations, or even more generally with first order systems of difference equations, in an independent variable n that varies within either the set of natural numbers or the set of integers. In the elementary cases when the equations are linear and its coefficients are constants, fundamental solutions can be computed explicitly, but even when these coefficients are polynomials, no explicit formulas for solutions are known, in general. On the other hand, given initial conditions, the difference equation can serve as a *recursion formula* for computing the values of the corresponding solution, but little can be said about its behavior as $n \to \infty$. In this article we in a sense take the opposite approach and compute "solutions" that have a known behavior at infinity, but since these "solutions" involve power series (in $1/n$) whose radius of convergence may be equal to zero, we refer to them as *formal solutions*. According to known results (see the discussion below), these formal solutions determine the asymptotic behavior of any solution of the underlying equation, except for the nontrivial question of how a solution of an initial value problem is related to these formal solutions. *In any case, however, one may read off from the formal solutions what kind of asymptotic behavior a solution may have as $n \to \infty$. So our*

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results should be interesting even for experts who are not so familiar with the concept of formal solutions.

Whenever working with power series, it is more natural to denote the independent variable by z instead of n, regarding z as a complex variable, but in our context this is just a trivial change of notation. More importantly, since we are concerned with formal power series anyway, we may just as well allow such series to come up in the coefficients of our difference equation, too. In addition, we shall apply various kinds of *transformations* to a given system, and some of these are such that the transformed system has a coefficient matrix that is a formal power series *in a root of* $1/z$. Therefore, we choose to consider such systems right from the beginning. For these reasons, we consider a d–dimensional *formal system of difference equations* of the following form:

$$x(z+1) \;=\; \hat{A}(z)\, x(z), \qquad \hat{A}(z) \;=\; p(z^{1/q}) I \;+\; \sum_{n=0}^{\infty} z^{-n/q}\, A_n\,, \tag{1.1}$$

for which the following additional assumptions hold:

- $p(z) \in \mathbb{C}[z]$ is a scalar polynomial without constant term that may be identically zero, but otherwise is of degree $\deg p(z) \le q$. Hence the (rational) powers of z occurring in the term $p(z^{1/q})I$ have exponents at most equal to 1.

- The dimension $d = 1$ is also considered as a particularly simple case. For $d \ge 2$, however, we shall assume that not all the matrices A_n are diagonal, since otherwise the system decouples into d scalar equations.

- If $\deg(p(z)) < q$ holds, then we assume in addition that the matrix A_0 is not equal to a scalar multiple of the identity matrix. As shall be made clear later, a general system can be put into this form by some elementary transformation. Also note that this case cannot occur in dimension $d = 1$.

- We assume the formal determinant of $\hat{A}(z)$ not to be the zero series, which is a natural assumption for difference equations. However, observe that no assumption is made concerning the radius of convergence of the power series occurring in (1.1). Only occasionally we shall speak of a *convergent system*, meaning to say that this power series has a positive, but otherwise arbitrary, radius of convergence.

Starting in 1882, after *H. Poincaré* developed the notion of asymptotic representation, *J. Horn* [12], *G. D. Birkhoff* [3,4], and *Birkhoff and Trjitzinsky* [5], as well as many others, applied Poincaré's concept and proved so-called *fundamental existence theorems* for convergent systems (1.1). This is to say that, given a formal fundamental solution of a form described later, they showed existence of solutions which are analytic and have the formal one as their asymptotic expansion as $z \to \infty$ in a sufficiently small sector of the complex plane. After *J. Ecalle* presented his concept of multisummability, *G. Immink*, *B. Braaksma*, and *B. Faber* in various publications [6, 7, 10, 13] showed that formal solutions even for nonlinear systems are, under relatively weak assumptions, multisummable in all but a discrete set of directions. This, however, shall not be discussed further in this article. Instead, we shall concentrate on the question of how to compute a set of formal fundamental solutions, even for formal systems (1.1). *A. Duval* [9] and *C. Praagman* [14] have shown existence of formal solutions using algebraic tools, while *H. L. Turrittin* [15] gave such a proof in a more algorithmic manner, but left out many details. Our treatment here is similar to that in the article of *Chen and Fahim* [8], but differs from their approach in the situation of a nilpotent highest term: Here, we proceed analogously to results of the first author [1, 2] for systems of linear ordinary differential equations. In detail, we shall present a completely algorithmic approach, which enables us to compute explicitly a formal fundamental solution (2.1) of (1.1), and which gives rise to recursion formulas which can in principle be implemented in computer algebra packages. We use techniques which are analogous to those for the computation of formal solutions of systems of meromorphic differential equations, and which have been simplified in this case by the first author in [1, 2].

This paper is organized as follows: First, we discuss the form of the formal fundamental solutions whose existence is to be shown later. In Section 3 we introduce formal transformations of several different kinds, while in the next one we recall briefly some properties of matrix equations that are needed later on. In Section 5 we compute formal solutions for systems in case $0 < \deg p(z) = q$, whereas in Section 6 we discuss cases when a system can be transformed into one that is a direct sum of smaller systems (for which the formal fundamental solutions can be computed individually). In the next, main, section we treat the remaining cases, showing that one can still make transformations to a new system that, in some sense, is easier to solve than the previous one. We conclude with a summary of our results in the final section.

2 Formal Fundamental Solutions

By definition, a formal fundamental solution of (1.1) is a $d \times d$ matrix of the following form:

$$\hat{X}(z) \;=\; \hat{F}(z)\,\big(\Gamma(z)\big)^{\Lambda}\,\mathrm{e}^{P(z)}\,z^{D}\,, \tag{2.1}$$

consisting of a formal invertible matrix power series* $\hat{F}(z)$ in $z^{-1/q}$, for some $q \in \mathbb{N}$, a constant diagonal matrix Λ of rational numbers with common denominator q (and $\Gamma(z)$ denoting the well-known Gamma function), a diagonal matrix $P(z)$ of polynomials in $z^{1/q}$ of degree less than or equal to q without constant term, and a constant matrix D that may even be chosen in Jordan canonical form. Moreover, the matrices Λ, $P(z)$, and D

*Observe that the term *invertible* refers to the fact that $\det \hat{F}(z)$ is not the zero series – however, observe that the inverse matrix need not be a power series, but in general is a formal Laurent series.

all commute with one another, but in general do not commute with $\hat{F}(z)$. By definition, $\hat{X}(z)$ is a formal solution of (1.1) if, and only if

$$\hat{A}(z) \;=\; \hat{X}(z+1)\,\hat{X}(z)^{-1} \;=\; \hat{F}(z+1)\,z^{\Lambda}\,\mathrm{e}^{P(z+1)-P(z)}\,(1+1/z)^{D}\,\hat{F}(z)^{-1}\,, \quad (2.2)$$

with the second identity using the commutativity of the matrices involved. Observe that for any $r \in \mathbb{C}$ we have

$$(z+1)^r \;-\; z^r \;=\; z^r \sum_{k=1}^{\infty} \binom{r}{k} z^{-k}\,, \qquad |z| \;>\; 1\,.$$

Therefore, due to the fact that the matrix $P(z)$ only involves powers of z with exponents at most 1, one can verify that the term $\mathrm{e}^{P(z+1)-P(z)}$ can be expanded into a Laurent series in $z^{-1/q}$ that does not involve terms with positive rational powers of z. In addition, using a *matrix version* of the binomial coefficients, namely[†]

$$\binom{D}{k} \;=\; \frac{1}{k!}\,D\,(D-1)\ldots(D-k+1)\,,$$

we can expand the term $(1+1/z)^D$ into the (matrix) binomial series as

$$(1+1/z)^D \;=\; \sum_{k=0}^{\infty} \binom{D}{k} z^{-k}\,, \qquad |z| \;>\; 1\,.$$

Accordingly, both sides of (2.2) are of the same (formal) nature, namely are formal Laurent series[‡] in $z^{-1/q}$. So by comparing coefficients one can always verify whether or not a given $\hat{X}(z)$ is a solution of (1.1) – however, it is nontrivial to show existence of $\hat{X}(z)$ for which (2.2) holds, and this is what this article is about!

Remark 2.1. Instead of $(\Gamma(z))^{\Lambda}$, one may in the definition of formal fundamental solutions also use the more elementary term $z^{z\Lambda}$ – due to Stirling's formula for the asymptotic expansion of the Gamma function, the quotient of the two terms formally gives a power series in z^{-1} times $\mathrm{e}^{-z\Lambda}\,z^{-(1/2)\Lambda}$, which may be absorbed into $\hat{F}(z)$, resp. $\mathrm{e}^{P(z)}\,z^{D}$. This is a form which is theoretically more satisfactory, since it avoids using a transcendental function which by itself solves a difference equation of the form we study here. On the other hand, the term $(z+1)^{(z+1)\Lambda} z^{-z\Lambda}$, which would come up frequently in later calculations, is not so easy to use, and that is why we use formal solutions of the form (2.1). Correspondingly, we shall in the next section define pole transformations using the Gamma function instead of z^z.

[†]Here and throughout, for a square matrix A and a complex number α we shall write $A-\alpha$ instead of $A-\alpha\,I$.

[‡]By definition, a formal Laurent series is one with terminating principal part, and a power series part whose radius of convergence may vanish.

In order to prove existence of, or even to compute a formal fundamental solution (2.1), we shall apply finitely many (formal) transformations of various kinds, all of which are discussed in the next section. Each of these transformations simplifies a given system in one way or another, such that in the end we obtain a system of which we can compute a formal fundamental solution directly. Such systems are discussed in Section 5.

3 Formal Transformations

In this section we consider a formal system with a coefficient matrix that is an arbitrary formal Laurent series in $z^{-1/q}$; i. e., we assume

$$x(z+1) \;=\; \hat{A}(z)\,x(z)\,, \qquad \hat{A}(z) \;=\; \sum_{n=-m}^{\infty} z^{-n/q}\,A_n\,, \qquad A_{-m} \neq 0\,, \tag{3.1}$$

where m may be an arbitrary integer number, and m/q is referred to as the *formal pole order* of the system. To exclude trivial cases, we shall assume that not all coefficients A_n are diagonal matrices, since otherwise the system decouples into d one-dimensional equations. Note that every system (1.1) is of the form (3.1), with $m = \deg p(z)$, and we shall explain that the first transformation introduced below may be used to reduce a general system to the form (1.1).

Given any $d \times d$ invertible matrix $\hat{T}(z)$ and setting $x(z) = \hat{T}(z)\,y(z)$, we observe that $x(z)$ is a solution of (3.1) if, and only if, $y(z)$ solves the *transformed equation*

$$y(z+1) \;=\; \hat{B}(z)\,y(z)\,, \qquad \hat{B}(z) \;=\; \hat{T}(z+1)^{-1}\,\hat{A}(z)\,\hat{T}(z)\,. \tag{3.2}$$

For a general *transformation matrix* $\hat{T}(z)$, the new coefficient matrix $\hat{B}(z)$ may not again be of the form (3.1), and that is why we are going to restrict ourselves and consider very particular kinds of transformations which we shall now present:

1. If $\hat{T}(z) = \big(\Gamma(z)\big)^r I$ for some $r \in \mathbb{Q}$, we call $\hat{T}(z)$ a *pole transformation*, since then we have $\hat{B}(z) = z^{-r}\,\hat{A}(z)$. Therefore, unlike in the case of differential equations, the pole order of $\hat{A}(z)$ has no special meaning when computing formal solutions of difference equations! Instead, for a general system (3.1) it is of importance to determine the *maximal number* $k \geq -m$ for which the matrices A_n, with $-m \leq n < k$ all are scalar multiples of the identity matrix. If we then apply a pole transformation with value $-r = \min\{k/q,\, m/q+1\}$, we obtain a transformed system that, after a change of notation, is of the form (1.1). *In particular, observe that in dimension* $d = 1$ *we can always transform a given system into one of this form, and in addition have that* $\deg p(z) = q$.

2. For $\hat{T}(z) = \exp[\tilde{p}(z^{1/q})]I$, with a scalar polynomial $\tilde{p}(z) \in \mathbb{C}[z]$ of degree less than or equal to q, we speak of an *exponential shift*. In this case, $\hat{B}(z) \;=$

$\exp[-\tilde{p}((1+z)^{1/q}) + \tilde{p}(z^{1/q})]\,\hat{A}(z)$, and such transformations shall be used to normalize the term $p(z^{1/q})$ in systems of the form (1.1) – for details, compare the proof of Proposition 5.1.

3. As the main type of transformations, we use *formally q–analytic transformations*

$$\hat{T}(z) \;=\; \sum_{n=0}^{\infty} z^{-n/q}\,T_n\,, \qquad \det T_0 \neq 0\,.$$

Note that the inverse is again a transformation of the same type. As an especially simple case we may have that $T_n = 0$ for all $n \geq 1$, and then we shall occasionally speak of a *constant transformation*. In order to avoid dealing with the inverse matrix, we shall always rewrite the transformation equation in the form

$$\hat{T}(z+1)\,\hat{B}(z) \;=\; \hat{A}(z)\,\hat{T}(z)\,. \tag{3.3}$$

One may directly verify that[§]

$$\hat{T}(z+1) \;=\; \hat{T}(z) \;+ \sum_{n=q+1}^{\infty} z^{-n/q}\,\tilde{T}_n\,, \quad \tilde{T}_n \;=\; \sum_{j=1}^{n-q} \binom{-j/q}{(n-j)/q} T_j\,, \tag{3.4}$$

and this implies for a matrix $\hat{A}(z)$ as in (3.1) that the *transformed matrix* $\hat{B}(z)$ is of the same form, with coefficients that we denote by B_n. Inserting into (3.3) and comparing coefficients, we obtain

$$\sum_{\nu=0}^{n+m}(T_\nu\,B_{n-\nu} \;-\; A_{n-\nu}\,T_\nu) \;=\; -\sum_{\nu=q+1}^{n+m} \tilde{T}_\nu\,B_{n-\nu} \qquad \forall\, n \geq -m\,. \tag{3.5}$$

For a system of the form (1.1) we have $m \leq q$, and $A_n = a_n\,I$ for $n \leq -1$. Hence we conclude from (3.5) that

$$\sum_{\nu=0}^{n+m} T_\nu\,(B_{n-\nu} \;-\; a_{n-\nu}) \;=\; 0 \qquad (-m \leq n \leq -1)\,.$$

Using the invertibility of T_0, we inductively obtain $B_n = a_n\,I$ for $n \leq -1$, so the new system is again of the form (1.1), and the polynomial $p(z)$ remains unchanged. So we may say that $p(z)$ *is a formal q-analytic invariant*. In particular, we observe that (3.5) is satisfied for $-m \leq n \leq -1$, and for other n simplifies to

$$\sum_{\nu=0}^{n}(T_\nu\,B_{n-\nu} \;-\; A_{n-\nu}\,T_\nu) \;=\; -\sum_{\nu=q+1}^{n+m} \tilde{T}_\nu\,B_{n-\nu} \qquad \forall\, n \geq 0\,. \tag{3.6}$$

[§]In order to simplify notation we use the following convention: $\binom{\alpha}{r} = 0$ if $\alpha \in \mathbb{C}, r \in \mathbb{Q}\backslash\mathbb{N}_{\geq 0}$.

Note that this is also correct for the case of $p(z) \equiv 0$. The importance of these formally q-analytic transformations lies in the fact that for systems (3.1), for which the coefficient A_0 has more than one eigenvalue, we show existence of such a transformation for which $\hat{B}(z)$ *is the direct sum of smaller systems*. So in other words, we can partially decouple a given system unless A_0 has one eigenvalue only.

4. Transformations of the type $\hat{T}(z) = \text{diag}[z^{r_1}, \ldots, z^{r_d}]$, with $r_i \in \mathbb{Q}$, shall be named *shearing transformations*. These transformations are needed when we are left with a system (1.1) whose coefficient A_0 has one eigenvalue only. Observe that such a shearing transformation, when applied to a system of the form (3.1), leads to a system that may again be written in this form, but with q and m changed accordingly. However, a system of the form (1.1) will, in general, be transformed by a shearing transformation to one which no longer is of this form, since off-diagonal terms may occur that involve positive (rational) powers of z. *If this is not the case, then the transformed system can again be written in the form* (1.1)*, with a possibly different value of q, say: $\tilde{q}$, which is a multiple of q. The polynomial $p(z)$ is accordingly changed to $\tilde{p}(z)$, so that $p(z^{1/q}) = \tilde{p}(z^{1/\tilde{q}})$. However, observe that the transformed matrix $\hat{B}(z)$ may be so that B_0 is a scalar multiple of the identity matrix, and a pole transformation then is applied to produce yet another matrix that then satisfies all the requirements made for systems* (1.1). For more details on this, refer to Section 7.

4 Matrix Equations

The following results are well known, and are used here to solve matrix equations of a certain form; for proofs a reader may, e. g., refer to [2].

Lemma 4.1. *Suppose that $A \in \mathbb{C}^{d_1 \times d_1}$ and $B \in \mathbb{C}^{d_2 \times d_2}$, for $d_1, d_2 \in \mathbb{N}$, have disjoint spectra, i. e., do not have an eigenvalue in common. Then for every $C \in \mathbb{C}^{d_1 \times d_2}$ the matrix–equation*

$$A\,X - X\,B = C \tag{4.1}$$

possesses a unique solution $X \in \mathbb{C}^{d_1 \times d_2}$.

In case A and B have eigenvalues in common the situation is more difficult – however, we only need to deal with A and B being two Jordan blocks, i. e., matrices of the form $J = \lambda I + N, \lambda \in \mathbb{C}$, where N is the nilpotent matrix with ones in all places of the first superdiagonal, and zeros elsewhere.

Lemma 4.2. *Suppose that $J_1 \in \mathbb{C}^{d_1 \times d_1}$ and $J_2 \in \mathbb{C}^{d_2 \times d_2}$ are two Jordan blocks having the same eigenvalue, and assume $d_1 \geq d_2$ (resp. $d_2 \geq d_1$). Then for every $C \in \mathbb{C}^{d_1 \times d_2}$*

there exists a unique matrix $B \in \mathbb{C}^{d_1 \times d_2}$ having nonzero entries in the last row (resp. first column) only, such that the matrix equation

$$J_1 X - X J_2 = C - B \tag{4.2}$$

has a solution $X \in \mathbb{C}^{d_1 \times d_2}$. which is unique within the set of matrices X having zero entries in the first row (resp. last column).

5 Elementary Solvable Systems

The following proposition is analogous to the case of a linear system of ODE with a regular singularity; its proof follows the same line as the corresponding one in the book of *F. R. Gantmacher* [11]. More precisely, we shall be concerned with a system (1.1) with $\deg p(z) = q$. In particular we wish to recall that in dimension $d = 1$, a formal equation can always be made to satisfy this assumption by means of a pole transformation! As we shall see in the proof, the differences of the eigenvalues of the matrix A_0 which are integer multiples of $1/q$ shall play a special role: Let the spectrum of A_0 be the set $\{\lambda_1, \ldots, \lambda_\mu\}$, with the enumeration chosen according to the following rules:

- Any two eigenvalues λ_j, λ_k are said to be equivalent modulo q, once their difference is an integer multiple of $1/q$. This is an equivalence relation on the set of eigenvalues, and we assume the enumeration of the spectrum be so that equivalent eigenvalues come consecutively.
- In addition, we enumerate the eigenvalues so that, within each equivalence class, the real parts of the eigenvalues λ_k are weakly increasing. So each equivalence class of eigenvalues is of the form $\{\lambda + k_\nu/q : \nu = j, \ldots \ell\}$, with $\lambda \in \mathbb{C}$ and integer values $0 = k_j < \ldots < k_\ell$.

Observe that these rules do not uniquely determine the ordering of the eigenvalues, but this shall not be relevant here. However, since we shall later on observe that an exponential shift shall change A_0 to a matrix $A_0 + \lambda$, with $\lambda \in \mathbb{C}$, it is important to keep in mind that we may choose the same enumeration of the elements of the shifted spectrum!

In terms of the spectrum of A_0, we now define a diagonal matrix

$$K = \text{diag}\,[k_1\, I_{s_1}, \ldots, k_\mu\, I_{s_\mu}]$$

with s_j being the algebraic multiplicity of the eigenvalue λ_j, and the k_j being as follows:

- Suppose that $\{\lambda_j, \ldots, \lambda_\ell\}$ is one of the equivalence classes of eigenvalues of A_0. Then we set $k_j = 0$ and define $k_\nu = q\,(\lambda_\nu - \lambda_j)$ for $\nu = j+1, \ldots, \ell$. Hence the entries k_j all are integers, and we have that $\lambda_\nu = \lambda_j + k_\nu/q$, for $j \le \nu \le \ell$.

Note that the matrix K does not change once A_0 is replaced by $A_0 + \lambda$, for arbitrary $\lambda \in \mathbb{C}$, since then the spectrum of A_0 changes accordingly. Along with the ordering of the spectrum of A_0, we also consider two block structures for $d \times d$ matrices: In the first, coarser block structure, the diagonal blocks correspond in size and ordering to the equivalence classes of the spectrum, while in the second, finer one, their sizes are determined by the multiplicity of each eigenvalue.

With these preparations, we are now ready to formulate the following result.

Proposition 5.1. *Let a formal system* (1.1) *be given and assume* $\deg p(z) = q$. *Then there exists a formal fundamental solution of the form* (2.2), *with*

- $\Lambda = I$ *and* $P(z) = q(z^{1/q})\, I$, $q(z) \in \mathbb{C}[z]$, $\deg q(z) = q$,
- $\hat{F}(z) = \hat{T}(z)\, z^{q^{-1}K}$, *with a formally* q*-analytic matrix* $\hat{T}(z)$ *and* K *as defined above,*
- *a constant matrix* D *that is diagonally blocked in the coarser block structure induced by the spectrum of* A_0, *with each diagonal block being upper triangularly blocked in the finer block structure.*

Moreover, in the finer one of the two block structures induced by the spectrum of A_0, *each diagonal block of* D *has only one eigenvalue, and these eigenvalues are the same for all such blocks belonging to the same diagonal block of the coarser block structure.*

Proof. To construct the formal fundamental solution, we first use an exponential shift to reduce the polynomial $p(z)$ in (1.1) to a monomial. Since any exponential shift is equivalent to a succession of finitely many such shifts with $\tilde{p}(z)$ itself being a monomial, we begin by observing that the shift $\mathrm{e}^{a\,z}$ may be used to multiply the coefficient matrix $\hat{A}(z)$ by e^{-a}, and thus we may from now on assume that the highest coefficient of $p(z)$ is equal to 1. Observing that for $1 \le \mu \le q-1$ we have

$$\mathrm{e}^{-a\,((z+1)^{\mu/q} - z^{\mu/q})} \;=\; 1 \;-\; \mu\, a/q\, z^{\mu/q-1} \;+\; \dots\,,$$

we find that such shifts can be used to, one after the other, remove all lower order terms from the polynomial $p(z)$. Accordingly, we shall from now on assume $p(z^{1/q}) = z$. Also note that these exponential shifts shall change the coefficient A_0 of (1.1) into $A_0 + \lambda$, for some $\lambda \in \mathbb{C}$, hence the matrix K, as well as the two block structures defined in terms of the spectrum of A_0, are left invariant! Next, we observe that a constant transformation $\hat{T}(z) \equiv T_0$ may be employed in order to reduce to an equation with a coefficient A_0 that is of the form

$$A_0 \;=\; A_{01} \oplus \ldots \oplus A_{0\mu}\,,$$

where each block A_{0k} has only one eigenvalue, and all these eigenvalues are mutually distinct. In fact, we may even assume that all the blocks A_{0j} are arranged in accordance with our preselected enumeration of the spectrum of A_0, meaning that A_{0j} has eigenvalue λ_j, for all $j = 1, \dots, \mu$. Note that then the block structure of A_0 agrees with the finer one of the two block structure that were defined above. Moreover, by definition of the matrix K, its diagonal entries are constant within each one of its corresponding blocks. Within any diagonal block of the coarser block structure, the diagonal entries of K increase strictly when going from one (fine) diagonal block to the next. In this situation, we intend to find a formally q-analytic transformation $\hat{T}(z)$ with leading term $T_0 = I$, to obtain a transformed equation with fundamental solution $\hat{Y}(z) = \Gamma(z)\, z^{q^{-1}K} z^D$, with a so far undetermined matrix D of the form described above – if we did so, then the proof is completed, since the exponential shifts we used before commute with $\hat{T}(z)$ and $z^{q^{-1}K}$. The coefficient matrix of this transformed equation then necessarily is of the form

$$\hat{B}(z) \;=\; \hat{Y}(z+1)\,\hat{Y}(z)^{-1} \;=\; z + z^{q^{-1}K}\left(q^{-1}K + D + O(1/z)\right) z^{-q^{-1}K} \,.$$

Due to the required form of D, we observe that $\hat{B}(z)$ is diagonally blocked in the coarser one of the block structures defined above. Moreover, each diagonal block is upper triangularly blocked with respect to the finer block structure. Aside from the leading term $p(z^{1/q}) = z$, the diagonal blocks begin with a constant term which we set equal to the diagonal blocks of A_0, which implies that the diagonal blocks of D have eigenvalues that satisfy the statement in the proposition. The off-diagonal blocks of $\hat{B}(z)$ begin with a term of the form $z^{-k_{j\ell}}\, D_{j\ell}$, with an undetermined block $D_{j\ell}$, $j < \ell$, of D, and according to the definition of K, $k_{j\ell}$ is a positive integer and in fact equal to the difference of the single eigenvalues of A_{0j} and $A_{0\ell}$. To show existence of a formally q-analytic transformation linking the one system to the other, we first observe that (3.6) holds for $n = 0$, since we have chosen the diagonal blocks of D so that $B_0 = A_0$. We now consider any $n \geq 1$: In this case, observing that $B_{-q} = I$, $B_{-k} = 0$ for $1 \leq k \leq q-1$, we find that (3.6) is equivalent to

$$T_n \left(A_0 - (n/q)\right) \;-\; A_0\, T_n \;+\; B_n \;=\; R_n \,,$$

where the matrix R_n only involves matrices T_m and B_m with $m < n$. Blocking all matrices in the block structure induced by A_0, this in turn can be written as

$$T_n^{(jk)} \left(A_0^{(kk)} - (n/q)\right) \;-\; A_0^{(jj)}\, T_n^{(jk)} \;+\; B_n^{(jk)} \;=\; R_n^{(jk)} \,, \qquad 1 \leq j, k \leq \mu \,. \tag{5.1}$$

According to Lemma 4.1 we can uniquely determine $T_n^{(jk)}$ from this identity provided $\lambda_k - \lambda_j \neq n/q$. This is always the case when the two eigenvalues are not equivalent modulo q, while for equivalent ones this condition is violated only for one value of $n \geq 1$, and then $k > j$ follows, due to the assumptions made on the ordering of the blocks of A_0. So whenever $\lambda_k - \lambda_j \neq n/q$, we solve (5.1) for $T_n^{(jk)}$. In the opposite case, we decide to choose $T_n^{(jk)} = 0$ and determine $B_n^{(jk)}$, i.e. to say, the corresponding block of D, such that (5.1) holds. In this fashion, we get a unique transformation $\hat{T}(z)$ that is as desired. □

Remark 5.2. Observe that, unlike in the case of a regular singularity for a system of differential equations, the formal fundamental solution obtained in the previous proposition involves, in general, a divergent power series $\hat{F}(z)$ – that this is so can already be observed in the most elementary situation of $d = 1$; for details a reader may refer to the proof of Stirling's formula for the Gamma function given in [2, p. 229].

6 The Splitting Lemma

In this and the next section we deal with general systems (1.1) of dimension $d > 1$ and $\deg p(z) < q$. Observe that this includes the case when $p(z)$ vanishes identically, since then it is natural to set $\deg p(z) = -\infty$. Whenever the leading term A_0 of (1.1) has several distinct eigenvalues, we will show existence of a *splitting transformation*, i. e., a formally q-analytic transformation for which the transformed system is partially decoupled, or in other words is a direct sum of systems of smaller dimensions. So this situation is completely analogous to the case of differential equations!

Lemma 6.1 (Splitting Lemma). *Let* (1.1) *be a formal system of dimension* $d > 1$ *with* $\deg p(z) < q$*, and assume that* $A_0 = A_0^{(11)} \oplus A_0^{(22)}$*, such that the two diagonal blocks have no eigenvalue in common. Then there exists a unique formally* q*–analytic transformation of the form*

$$\hat{T}(z) = \begin{bmatrix} I & \hat{T}_{12}(z) \\ \hat{T}_{21}(z) & I \end{bmatrix}, \quad \hat{T}_{ij}(z) = \sum_{n=1}^{\infty} T_n^{(ij)} z^{-n/q}, \tag{6.1}$$

such that the transformed formal system is diagonally blocked in the block structure of A_0.

Proof. Setting $\hat{B}(z) = p(z^{1/q})\, I + \sum_0^{\infty} B_n z^{-n/q}$, $B_0 = A_0$, and $\hat{T}(z) = I + \sum_1^{\infty} T_n z^{-n/q}$, we conclude from (3.2) that

$$T_n\, A_0 \; - \; A_0\, T_n \; = \; A_n \; - \; B_n \; + \; R_n \qquad \forall\, n \geq 1\,, \tag{6.2}$$

where R_n only involves coefficients $T_m,\ B_m$ with $m < n$. Blocking

$$T_n = \begin{bmatrix} 0 & T_n^{(12)} \\ T_n^{(21)} & 0 \end{bmatrix}, \; A_n = \begin{bmatrix} A_n^{(11)} & A_n^{(12)} \\ A_n^{(21)} & A_n^{(22)} \end{bmatrix}, \; B_n = \begin{bmatrix} B_n^{(11)} & 0 \\ 0 & B_n^{(22)} \end{bmatrix}, \quad n \geq 1$$

and inserting into (6.2) leads to

$$\begin{aligned} T_n^{(21)} A_0^{(11)} - A_0^{(22)} T_n^{(21)} &= A_n^{(21)} + R_n^{(21)} \\ B_n^{(11)} &= A_n^{(11)} + \sum_{m=1}^{n-1} A_m^{(12)} T_{n-m}^{(21)}, \quad n \geq 1 \end{aligned}$$

plus two other equations with indices $1, 2$ permuted that are omitted but can be treated in the same way. Since $R_n^{(21)}$ only involves blocks $T_m^{(21)}, B_m^{(11)}$ with $m < n$ we can substitute the second equation into the first one and compute all coefficients $T_n^{(21)}$ uniquely according to Lemma 4.1. □

Applying the splitting lemma repeatedly we obtain an analogous result if A_0 has more than two eigenvalues.

Theorem 6.2. *Let* (1.1) *be a formal system of dimension* $d > 1$ *with* $\deg p(z) < q$*, and assume that* $A_0 = \text{diag}[A_0^{(11)}, \dots, A_0^{(\mu\mu)}]$*, such that any two blocks have no eigenvalue in common. Then there exists a unique formally* q*–analytic transformation* $\hat{T}(z)$ *with diagonal blocks all equal to* I *and off–diagonal ones having no constant term, such that the transformed formal system is diagonally blocked in the block structure of* A_0*.*

So the results of the previous sections show that we are left with discussing systems for which $\deg p(z) < q$, while A_0 only has one eigenvalue. This we shall do in the next section with help of shearing transformations.

7 Shearing Transformations

In the following we have to investigate formal systems (1.1) with $\deg p(z) < q$, whose leading term A_0 only has one eigenvalue, so that the splitting lemma does not apply. Without loss of generality we may assume A_0 to be in Jordan canonical form, since otherwise we can apply a constant transformation. According to our normalizing assumption we have that A_0 is not equal to a multiple of the unit matrix, hence we may assume that

$$A_0 \;=\; \lambda\, I \;+\; N_A\,, \qquad N_A \;=\; N_1 \oplus \ldots \oplus N_\mu \;\neq\; 0\,, \qquad \lambda \in \mathbb{C}\,, \quad \mu \in \mathbb{N}\,, \quad (7.1)$$

with nilpotent Jordan blocks N_j of dimensions d_j, which we assume to be ordered so that $d_1 \ge \ldots \ge d_\mu \ge 1$. The treatment of these cases shall be completely analogous to the case of differential equations studied in [2]); in particular we shall use the same order relation for nilpotent matrices introduced there:

> **(Order relation for nilpotent matrices)** Given any two nilpotent matrices $N_A, N_B \in \mathbb{C}^{d\times d}$, we say that N_B is superior to N_A, if for some $n \ge 1$ we have rank $N_A^m =$ rank N_B^m for $1 \le m \le n-1$, and rank $N_A^n <$ rank N_B^n.

Note that this indeed defines a (partial) order relation on the set of $d \times d$ nilpotent matrices, with the nilpotent Jordan block, i. e., the matrix with ones above the diagonal and zeros everywhere else, being a maximal element. The strategy that we shall follow is to find transformations that will produce a transformed equation with superior nilpotent matrix N_B, and in order to achieve this goal, we shall first arrange finitely many coefficients $A_1, \ldots A_{n_0}$ to have a special form. To do so, it shall be convenient to block both the transformation as well as the two systems in the block structure induced by N_A:

Lemma 7.1. *Let a formal system* (1.1) *with leading term as in* (7.1) *be given. Then for every* $n_0 \in \mathbb{N}$ *there exists a terminating* q*–analytic transformation* $\hat{T}(z) = I + [T_{ij}(z)]$*, blocked in the block structure of* N_A*, such that the coefficient matrix of the transformed system has the form* $\hat{B}(z) = p(z^{1/q})I + A_0 + [\hat{B}_{ij}(z)]$*, with*

$$T_{ij}(z) \;=\; \sum_{n=1}^{n_0} T_n^{(ij)} z^{-n/q}\,, \qquad \hat{B}_{ij}(z) \;=\; \sum_{n=1}^{\infty} B_n^{(ij)} z^{-n/q}\,,$$

and so that for $1 \le n \le n_0$ *the coefficients* $B_n^{(ij)}$

1. *have all zero columns except for the first one in case* $1 \le i \le j \le \mu$*,*

2. *have all zero rows except for the last one in case* $1 \le j < i \le \mu$*.*

Furthermore the transformation $\hat{T}(z)$ *is unique if we require for* $1 \le n \le n_0$ *that*

1. *all* $T_n^{(ij)}$ *have vanishing last column in case* $1 \le i \le j \le \mu$*,*

2. *all* $T_n^{(ij)}$ *have vanishing first row in case* $1 \le j < i \le \mu$*.*

Proof. Insertion into (3.2) implies the following recursion formula for the coefficients of the blocks of $\hat{A}(z), \hat{B}(z), \hat{T}(z)$:

$$T_n^{(ij)} N_j - N_i T_n^{(ij)} = -B_n^{(ij)} + R_n^{(ij)}, \quad n \ge 1, \quad 1 \le i, j \le \mu, \tag{7.2}$$

where $R_n^{(ij)}$ only involves blocks of T_m, B_m with $m < n$. For $n \le n_0$ Lemma 4.2 implies existence of a unique matrix $B_n^{(ij)}$ with nonzero entries in the first column (resp. last row), such that (7.2) possesses a solution $T_n^{(ij)}$. This solution is unique if we require its last column (resp. first row) to vanish. Which case applies depends upon the position of the block. In case $n > n_0$ we have $T_n^{(ij)} \equiv 0$ and choose $B_n^{(ij)}$ so that (7.2) holds. □

Remark 7.2. We say that (1.1) with A_0 as in (7.1), $\mu \ge 1$, is *normalized up* to n_0, if all coefficients $A_n^{(ij)}$, for $1 \le n \le n_0$, have nonzero entries only in the first column resp. last row (in case $i \le j$ resp. $i > j$). If (1.1) is normalized up to n_0, and in addition all $A_n^{(ij)}$ with $i \ne j$ vanish for $1 \le n \le n_0$, then we say that (1.1) is *reduced up* to n_0. We mention briefly that a system (1.1) normalized up to n_0 can be normalized up to $\tilde{n}_0 > n_0$ using a transformation $T(z)$ with coefficients $T_n = 0$ for $1 \le n \le n_0$; therefore the corresponding coefficients A_n remain unchanged.

Next, we will apply shearing transformations for systems (1.1) normalized up to some n_0 in order to get a transformed system with leading term $B_0 = \lambda I + N_B$ and superior N_B.

Proposition 7.3. *For some* $n_0 \in \mathbb{N}$*, let* (1.1) *be a formal system with leading term* $A_0 = \lambda I + N_A$ *as in* (7.1)*, normalized up to* n_0 *and assume* $\mu \geq 2$*. Assume the existence of* $n_1 \in \mathbb{N} : 1 \leq n_1 \leq n_0$*, so that*

$$\hat{A}_{i\mu}(z) \;=\; \sum_{n=n_1}^{\infty} A_n^{(i\mu)} z^{-n/q}\,, \qquad 1 \leq i \leq \mu - 1\,, \tag{7.3}$$

and not all $A_{n_1}^{(i\mu)}$ *vanish. Then the shearing transformation*

$$T(z) \;=\; \text{diag}[I_{d_1}, \ldots, I_{d_{\mu-1}}, z^{n_1/q} I_{d_\mu}]$$

produces a system with leading term $B_0 = \lambda I + N_B$*, with* N_B *being superior to* N_A*.*

Proof. Defining $C_i = A_{n_1}^{(i\mu)}, 1 \leq i \leq \mu - 1$ one checks easily that the stated shearing transformation produces a transformed system

$$\hat{B}(z) = p(z^{1/q})I + \lambda I + N_B + \sum_{n=1}^{\infty} B_n z^{-n/q} \tag{7.4}$$

with

$$N_B = \begin{bmatrix} N_1 & 0 & \cdots & 0 & C_1 \\ 0 & N_2 & \cdots & 0 & C_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & N_{\mu-1} & C_{\mu-1} \\ 0 & 0 & \cdots & 0 & N_\mu \end{bmatrix}, \quad C_i = \begin{bmatrix} c_{i,1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ c_{i,d_i} & 0 & \cdots & 0 \end{bmatrix}_{d_i \times d_\mu}. \tag{7.5}$$

By assumption we have $C_i \neq 0$ for at least one i and for $n \geq 1$ we find

$$N_B^n = \begin{bmatrix} N_1^n & 0 & \cdots & 0 & C_1^{(n)} \\ 0 & N_2^n & \cdots & 0 & C_2^{(n)} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & N_{\mu-1}^n & C_{\mu-1}^{(n)} \\ 0 & 0 & \cdots & 0 & N_\mu^n \end{bmatrix}, \quad C_i^{(n)} = N_i^{n-1} C_i + C_i^{n-1} N_\mu,\; n \geq 2. \tag{7.6}$$

We conclude that the nth column of $C_i^{(n)}$ equals the first column of C_i, while the followings vanish. Therefore we have rank $N_A^n \leq \text{rank} N_B^n, n \geq 1$ and equality holds iff the columns of $C_i^{(n)}$, for every $1 \leq i \leq \mu - 1$, are linear combinations of the columns of N_i^n. Choosing $n = d_\mu$ we have $N_\mu^n = 0$ and finally rank $N_A^{d_\mu} < \text{rank} N_B^{d_\mu}$ because $C_i^{(d_\mu)} \neq 0$ for at least one i. □

Due to Proposition 7.3 we are left to deal with the following two situations: Either we have $\hat{A}_{i\mu}(z) = O\big(z^{-(n_0+1)/q}\big), 1 \leq i \leq \mu - 1$, or there is a shearing transformation producing a transformed system with leading coefficient $B_0 = \lambda I + N_B$ and superior N_B. Since the nilpotent matrix with rank equal to $d-1$ is a maximal element for our order relation, it follows that after finitely many applications of Proposition 7.3, in combination with constant transformations for the leading terms, we find that either $\mu = 1$ or $\hat{A}_{i\mu}(z) = O\big(z^{-(n_0+1)/q}\big), 1 \le i \le \mu - 1$. Much more can be said, however.

Theorem 7.4. *Let* (1.1) *be a formal system with leading term* $A_0 = \lambda I + N_A$ *as in* (7.1)*, normalized up to* $n_0 \in \mathbb{N}$*. Moreover, let* $\mu \ge 2$*. Then we can find a shearing transformation producing a system with leading term* $B_0 = \lambda I + N_B$ *and* N_B *superior to* N_A*, except when all coefficients* $A_n, 1 \le n \le n_0$ *are already diagonally blocked in the block structure of* A_0*, i.e. when* (1.1) *is reduced up to* n_0*.*

Proof. According to Proposition 7.3 such a shearing transformation always exists, except when condition (7.3) is violated, meaning when all $A_n, 1 \le n \le n_0,$ are lower triangularly blocked in the block structure consisting of two diagonal blocks, with the second one of the same size as N_μ. Next, one may check that a shearing transformation inverse to the one used in Proposition 7.3 shall lead to an equation with superior leading term, except when the coefficients $A_n, 1 \le n \le n_0,$ even are diagonally blocked in the same block structure. Repeating the same argument for the first diagonal block completes the proof. □

According to the previous results we are left to deal with a formal system (1.1) with $\deg p(z) < q$, a leading term A_0 as in (7.1), and such that for some $n_0 \in \mathbb{N}$ that we may choose as large as we want, we have for all $n = 1, \ldots, n_0, 1 \le i, j \le \mu$

$$A_n^{(ii)} = \begin{bmatrix} a_n^{(i,1)} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_n^{(i,d_i)} & 0 & \cdots & 0 \end{bmatrix}_{d_i \times d_i}, \quad A_n^{(ij)} = 0 \ (i \ne j). \tag{7.7}$$

For such a system, with a suitable rational value r satisfying $0 < r \le 1/q$, we are going to show existence of a transformation of the form

$$\left.\begin{aligned} T(z) &= (\Gamma(z))^r \,\text{diag}[T_1(z), \ldots, T_\mu(z)] \\ T_i(z) &= \text{diag}[1, z^{-r}, z^{-2r}, \ldots, z^{-(d_i-1)r}] \end{aligned}\right\} \tag{7.8}$$

which is a shearing transformation combined with a pole transformation, for which the transformed system $\hat{B}(z)$ is again of the form (1.1), with new entries $\tilde{q}$ and $\tilde{p}(z)$, such that either $\tilde{q} = q$ and $\deg \tilde{p}(z) > \tilde{q}$ hold, or so that the leading term B_0 has several eigenvalues. To achieve this goal, it is necessary to discuss the effect of such a transformation: It is natural here to block both $\hat{A}(z)$ and $\hat{B}(z)$ in the block structure defined by A_0, and then we obtain from (3.2) that $\hat{B}_{jk}(z) = z^{-r}\, T_j^{-1}(z+1)\, \hat{A}_{jk}(z)\, T_k(z)$. Accordingly, elements in off-diagonal blocks of $\hat{A}(z)$ get multiplied by factors of the form $z^{r(\tau-\nu)}\,(1+1/z)^{r\tau}$, where ν, τ are nonnegative integers bounded by the number of rows, resp. columns, of this block. Hence, $|r(\tau-\nu)| \le \max_j d_j\, r \le d$ follows. Consequently,

- if we assume that $n_0 \geq d+1$, which we shall do without loss of generality, then the off-diagonal blocks of $\hat{B}(z)$ are formal power series in $z^{-1/\tilde{q}}$, for $\tilde{q}$ being the least common multiple of q and the denominator of r.

Hence, to continue our discussion, we may concentrate on a diagonal block $\hat{A}_{jj}(z)$: Within this block, the elements directly above the diagonal (which all are series beginning with a constant term equal to 1) are multiplied by $(1+1/z)^{\nu r}$, for $0 \leq \nu \leq d_j - 1$, hence they all become power series in $z^{-1/\tilde{q}}$ which again have constant term 1. Other elements of this block are identically zero (and remain so under this transformation), except for some of those in the first column, which are multiplied by $z^{\tau r}$, with $1 \leq \tau \leq d_j$. Therefore, the corresponding element of $\hat{B}_{jj}(z)$ is a formal series beginning with a power of the form $z^{\tau r - n_{\tau j}/q}$, for some $n_{\tau j} \geq 1$, except if the series happens to vanish completely, in which case we set $n_{\tau j} = \infty$. We do not want to have any positive powers of z occurring, hence we restrict r so that $\tau\, r \leq n_{\tau j}/q$. On the other hand, the idea is to pick r, if possible, so that equality holds for at least one pair (τ, j). This implies that

- we only need to consider a finite set of rationals r, since $\tau\, r = n_{\tau j}/q$ for some $\tau \in \{1 \leq \tau \leq d_j \leq d_1\}$ and some j implies that the denominator of r is bounded by $d_1\, q$, and the set of such $r \leq 1/q$ indeed is finite.

So let r be the maximal value from this finite set for which $\tau\, r \leq n_{\tau j}/q$ holds for all $1 \leq \tau \leq d_j$, and all $j = 1, \ldots, \mu$. First, assume that $r = 1/q$: Then $\tilde{q} = q$, and the transformed equation is again of the form (1.1), but with $p(z)$ replaced by $z(p(z)+\lambda)$, hence indeed its degree has risen by 1. In the second case of $r < 1/q$, we have $\tau\, r \leq n_{\tau j}/q$ for all $1 \leq \tau \leq d_j$ and $j = 1, \ldots, \mu$, with equality holding for at least one pair (τ, j). Then, the corresponding transformation leads to

$$\hat{B}(z) \;=\; \tilde{p}(z^{1/\tilde{q}}) \;+\; \sum_{n=0}^{\infty} z^{-n/\tilde{q}}\, B_n\,, \tag{7.9}$$

with $\tilde{q}$ as above, $\tilde{p}(z)$ so that $\tilde{p}(z^{1/\tilde{q}}) = z^r\,(p(z^{1/q})+\lambda)$, and $B_0 = B_{01} \oplus \ldots \oplus B_{0\mu}$, with at least one block B_{0j} being of the form

$$B_{0j} \;=\; N_j \;+\; C_j\,,$$

with a nonzero matrix C_j whose entries vanish except for some in the first column. Since $r < 1/q$, we find that the entry in position $(1,1)$ of C_j is equal to zero, so that for C_j to be different from the zero matrix we necessarily have $d_j \geq 2$. Moreover, B_{j0} is a companion matrix and therefore not nilpotent (since $C_j \neq 0$). However, the trace of B_{j0} vanishes, and thus B_0 has more than one eigenvalue.

We summarize the result of the foregoing discussion as follows.

Theorem 7.5. *Let a system* (1.1) *with* $\deg p(z) < q$ *and a leading term* A_0 *as in* (7.1) *be given, and assume that* (7.7) *holds for* $n_0 \geq d+1$. *Then there exists a transformation of type* (7.8) *which produces a transformed matrix* $\hat{B}(z)$ *being as follows:*

- *If* $r = 1/q$*, then* $\hat{B}(z)$ *is as in* (1.1)*, with the same value of* q *but with* $p(z)$ *replaced by a polynomial of higher degree.*
- *If* $0 < r < 1/q$*, then* $\hat{B}(z)$ *is as in* (1.1)*, but with* q *replaced by a larger value* $\tilde{q}$*, with* $p(z)$ *replaced by a different polynomial* $\tilde{p}(z)$*, and the constant term* B_0 *having more than one eigenvalue.*

In particular, observe that in the first one of the two cases, we can again apply the results of this section until either we end with a system as in the second case, or one with a polynomial of degree equal to q*.*

As shall be explained in more detail in the following section, the results of this, together with those of the preceding sections, enable us to show existence of a formal fundamental solution of any linear system of difference equations, and at the same time allow for its computation in an algorithmic manner.

8 Summary

Summing up the results we have obtained in this article, we come to the following conclusion.

Theorem 8.1 (Main Result). *Every formal system of linear difference equations of the form* (3.1) *possesses a formal fundamental solution of the form* (2.1)*, that can be computed by the following algorithmic procedure:*

(a) *Determine the maximal number of leading coefficients in* (3.1) *that are scalar multiples of the unit matrix, and apply a pole transformation to put the coefficient matrix into the form* (1.1)*.*

(b) *If the system is as in* (1.1)*, with* $\deg p(z) = q$*, refer to* Proposition 5.1 *to compute the formal fundamental solution. Note that this in particular covers the case of dimension* $d = 1$*.*

(c) *If the system is as in* (1.1)*, with* $\deg p(z) < q$ *and* A_0 *having more than one eigenvalue, apply* Theorem 6.2 *to compute a formally* q*-meromorphic transformation to obtain a coefficient matrix that is the direct sum of several smaller matrices that then can be treated separately.*

(d) *If the system is as in* (1.1)*, with* $\deg p(z) < q$ *and* A_0 *having only one eigenvalue, apply the results of the previous section to produce a new system that either is as in case* (b)*, or as in case* (c)*.*

In conclusion we wish to say that the computation of a formal fundamental solution for difference equations can be done in the same fashion as for differential equations.

What has not been discussed here is the problem of summation of the formal power series occurring in Theorem 6.2. However, observe that on one hand existing results on (nonlinear) systems of difference equations may be applied to see how these series can be summed, and when one meets the phenomenon of level $1+$. On the other hand, it is very likely that there are more direct ways of examining this question for the relatively easy situation of Theorem 6.2, or the even easier one of Lemma 6.1, but this is not done here.

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