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Multiplicity Results of Positive Solutions for Nonlinear Three-Point Boundary Value Problems on Time Scales

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Abstract

Let $\mathbb{T}$ be a time scale such that $0, T \in \mathbb{T}$. In this paper, we consider the nonlinear second-order three-point boundary value problem

$$u^{\Delta\nabla}(t) + q(t)f(t, u(t), u^{\Delta}(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0, \quad \alpha u(\eta) = u(T),$$

where $\beta, \gamma \geq 0, \beta + \gamma > 0, \eta \in (0, \rho(T))_{\mathbb{T}}, 0 < \alpha < T/\eta$, and $d = \beta(T - \alpha\eta) + \gamma(1 - \alpha) > 0$. By using a fixed point theorem due to Avery and Peterson, sufficient conditions are obtained for the existence of three positive solutions of the above problem. The interesting point is the nonlinear term f which is involved with the first order derivative explicitly.

AMS Subject Classifications: 34B15, 39A10.
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1 Introduction

Going back to its founder Stefan Hilger (1988), the study of dynamic equations on time scales is a fairly new area of mathematics. Motivating the subject is the notion that

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dynamic equations on time scales can build bridges between differential equations and difference equations. Now, the study of time scales theory has led to many important applications, for example, in the study of insect population models, neural networks, heat transfer, quantum mechanics, epidemic, crop harvest and stock market [3, 5, 6, 8]. Atici et al present a dynamic optimization problem from economics [3], by constructing a time scale model, they told us time scale calculus could allow exploration of a variety of situations in economics.

Very recently, the existence problems of positive solutions for three-point boundary value problems, especially on time scales, have attracted many authors' attention and concern [1, 2, 5–7, 9–14]. But the nonlinear term f is not involved with the first-order delta derivative. Many difficulties occur when the nonlinear term f is involved with the first-order delta derivative.

In [11], Sun and Li considered the existence of positive solution of the problem

$$u^{\Delta\nabla}(t) + a(t)f(t, u(t)) = 0, \quad t \in (0, T)_{\mathbb{T}},$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0, \quad \alpha u(\eta) = u(T)$$

by using fixed point theorem in cones. The authors obtained the existence of at least one, two and three positive solutions of the above problems.

In [7], by a new fixed point theorem, Guo and Ge gave sufficient conditions for the existence of at least one solution to the three-point boundary value problem

$$u''(t) + f(t, u, u') = 0, \quad 0 < t < 1,$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta).$$

Motivated by those works, in this paper we study the existence of multiple positive solutions for the second-order nonlinear three-point dynamic equation on time scales

$$u^{\Delta\nabla}(t) + q(t)f(t, u(t), u^{\Delta}(t)) = 0, \quad t \in (0, T)_{\mathbb{T}}, \tag{1.1}$$

$$\beta u(0) - \gamma u^{\Delta}(0) = 0, \quad \alpha u(\eta) = u(T), \tag{1.2}$$

where $\beta, \gamma \geq 0, \beta + \gamma > 0, \eta \in (0, \rho(T))_{\mathbb{T}}, 0 < \alpha < T/\eta$, and $d = \beta(T - \alpha\eta) + \gamma(1 - \alpha) > 0$. We assume $a : [0, T]_{\mathbb{T}} \to [0, \infty)$ is ld-continuous, with $a\,(t_0) > 0$ for at least one $t_0 \in (\eta, T)_{\mathbb{T}}$; $f : [0, T]_{\mathbb{T}} \times [0, \infty) \times \mathbb{R} \to [0, \infty)$ is continuous. We establish the existence of at least three positive solutions for boundary value problem (1.1) and (1.2). To our best knowledge, no work has been done for (1.1) and (1.2) on time scales by using the fixed point theorem of Avery and Peterson.

2 Preliminaries

In a cone P, let ϕ and θ be nonnegative continuous convex functionals, φ a nonnegative continuous concave functional and ψ a nonnegative continuous functional. Then for positive real numbers m_1, m_2, m_3, m_4, we define the following convex sets:

$$P(\phi, m_4) = \{u \in P | \phi(u) < m_4\},$$

$$P(\phi, \varphi, m_2, m_4) = \{u \in P | m_2 \le \varphi(u), \phi(u) \le m_4\},$$

$$P(\phi, \theta, \varphi, m_2, m_3, m_4) = \{u \in P | m_2 \le \varphi(u), \theta(u) \le m_3, \phi(u) \le m_4\},$$

and a closed set

$$R(\phi, \psi, m_1, m_4) = \{u \in P | m_1 \le \psi(u), \phi(u) \le m_4\}.$$

Lemma 2.1 (see [4]). *Let P be a cone in a real Banach space E. Let θ and ϕ be nonnegative continuous convex functionals, φ a nonnegative continuous concave functional and ψ a nonnegative continuous functional satisfying $\psi(\lambda u) \le \lambda \psi(u)$ for $0 \le \lambda \le 1$, such that for some positive numbers M and m_4,*

$$\psi(u) \le \psi(u), \|u\| \le M\phi(u)$$

for all $u \in \overline{P(\phi, m_4)}$. Suppose that

$$A : \overline{P(\phi, m_4)} \to \overline{P(\phi, m_4)}$$

is completely continuous and there exist positive numbers m_1, m_2 and m_3 with $m_1 < m_2$ such that

- (S1) $\{u \in P(\phi, \theta, \varphi, m_2, m_3, m_4) | \varphi(u) > m_2\} \ne \emptyset$*, and* $\varphi(Au) > m_2$ *for* $u \in P(\phi, \theta, \varphi, m_2, m_3, m_4)$*;*
- (S2) $\varphi(Au) > m_2$ *for* $u \in P(\phi, \varphi, m_2, m_4)$ *with* $\theta(Au) > m_3$*;*
- (S3) $0 \notin R(\phi, \psi, m_1, m_4)$ *and* $\psi(Au) < m_1$ *for* $u \in P(\phi, \psi, m_1, m_4)$ *with* $\psi(u) = m_1$*.*

Then A has at least three fixed points $u_1, u_2, u_3 \in \overline{P(\phi, m_4)}$, such that

$$\phi(u_i) \le m_4 \ for \ i = 1, 2, 3; \quad m_2 < \varphi(u_1);$$

$$m_1 < \psi(u_2) \ with \ \varphi(u_2) < m_2; \quad \psi(u_3) < m_1.$$

3 Main Results

Now we define the real Banach space $E = C^{\Delta}([0, T]_{\mathbb{T}})$ as the set of all Δ-differentiable functions with continuous Δ-derivative on $[0, T]_{\mathbb{T}}$ with the norm

$$\|u\| = \max\{\|u\|_0, \|u^{\Delta}\|_1\}, \quad u \in E,$$

where

$$\|u\|_0 := \sup\{|u(t)| : t \in [0,T]_{\mathbb{T}}\}, \quad \|u^\Delta\|_1 := \sup\{|u^\Delta(t)| : t \in [0,T]_{\mathbb{T}^\kappa}\}.$$

We define the cone $P \subset E$ by

$$\begin{aligned} P \;=\; & \{u \in E : u(t) \text{ is nonnegative and concave on } [0,T]_{\mathbb{T}}, \\ & \beta u(0) - \gamma u^\Delta(0) = 0,\ \alpha u(\eta) = u(T)\big\}\,. \end{aligned}$$

The following lemma will play an important role in the proof of our main results.

Lemma 3.1. *For $u \in P$, there exists a constant $M > 0$ such that*

$$\max_{t\in[0,T]_{\mathbb{T}}} |u(t)| \le M \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)|.$$

Proof. We first consider the case $\beta > 0$. In this case, by the concavity of u, we have

$$u(t) - u(0) \le t u^\Delta(0) \le T \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)|.$$

From the boundary condition $\beta u(0) - \gamma u^\Delta(0) = 0$, we know $u(0) = \dfrac{\gamma}{\beta} u^\Delta(0)$, thus

$$\max_{t\in[0,T]_{\mathbb{T}}} |u(t)| \le \frac{\gamma}{\beta} u^\Delta(0) + T \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)| \le \left(\frac{\gamma}{\beta} + T\right) \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)|.$$

Next we consider the case $\beta = 0$. First, by

$$u(t) = u(T) - \int_t^T u^\Delta(s)\Delta s,$$

we have

$$\max_{t\in[0,T]_{\mathbb{T}}} |u(t)| \le u(T) + T \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)|. \tag{3.1}$$

From the boundary condition $\alpha u(\eta) = u(T)$ and $d = \gamma(1-\alpha) > 0$, we get

$$(1-\alpha)u(T) = \alpha u(\eta) - \alpha u(T) \le \alpha(T-\eta)| \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)|. \tag{3.2}$$

In view of (3.1) and (3.2), we obtain

$$\max_{t\in[0,T]_{\mathbb{T}}} |u(t)| \le \left(\frac{\alpha(T-\eta)}{1-\alpha} + T\right) \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)| = \left(\frac{T-\alpha\eta}{1-\alpha}\right) \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^\Delta(t)|.$$

The proof is complete. □

For the sake of applying Lemma 2.1, we let the nonnegative continuous concave functional φ, the nonnegative continuous convex functional θ, ϕ, and the nonnegative continuous functional ψ be defined on the cone P by

$$\varphi(u)=\min_{t\in[\eta,T]_{\mathbb{T}}}|u(t)|,\ \phi(u)=\max_{t\in[0,T]_{\mathbb{T}^\kappa}}|u^{\Delta}(t)|,\ \psi(u)=\theta(u)=\max_{t\in[0,T]_{\mathbb{T}}}|u(t)|,\ u\in P.$$

For convenience, we introduce the following notations.

$$S = \max\left\{\frac{\beta}{d}\int_0^T (T-s)\,q(s)\nabla s,\int_0^T q(s)\nabla s+\frac{\alpha\beta}{d}\int_0^\eta(\eta-s)\,q(s)\nabla s\right\},$$

$$L = \frac{\min\{1,a\}(\beta\eta+\gamma)}{d}\int_\eta^T (T-s)\,q(s)\nabla s,$$

$$N = \frac{\beta T+\gamma}{d}\int_0^T (T-s)\,q(s)\nabla s,$$

$$r=\min\left\{\frac{\alpha(T-\eta)}{T-\alpha\eta},\frac{\alpha\eta}{T},\frac{\eta}{T}\right\},\quad a=\frac{1}{\min\{1,\alpha\}(T-\eta)(\beta\eta T+\gamma\eta)},$$

$$\begin{aligned} b &= daT^2+a\left[\beta(T^2-\alpha\eta^2)+\gamma(1-\alpha)\sigma(0)\right]T\\ &\quad +a\left[\gamma(\beta(T^2-\alpha\eta^2)-\gamma(T-\alpha\eta)\sigma(0)\right]+\frac{1}{r},\end{aligned}$$

$$c = d(T+\sigma(T))+\beta(T^2-\alpha\eta^2)+\gamma(1-\alpha)\sigma(0).$$

Theorem 3.2. *If there exist positive numbers m_1,m_2 and m_4 with $m_1<m_2\le m_4/c$ such that*

(C1) $f(t,u,v)\le m_4/S$ *for* $(t,u,v)\in[0,T]_{\mathbb{T}}\times[0,Mm_4]\times[-m_4,m_4]$*;*

(C2) $f(t,u,v)>m_2/L$ *for* $(t,u,v)\in[\eta,T]_{\mathbb{T}}\times[m_2,bm_2]\times[-m_4,m_4]$*;*

(C3) $f(t,u,v)<m_1/N$ *for* $(t,u,v)\in[0,T]_{\mathbb{T}}\times[0,m_1]\times[-m_4,m_4]$,

then the problem (1.1) *and* (1.2) *has at least three positive solutions u_1,u_2, u_3 satisfying*

$$\max_{t\in[0,T]_{\mathbb{T}^\kappa}}|u_i^{\Delta}(t)|\le m_4\ \textit{for}\ i=1,2,3;$$

$$m_2<\min_{t\in[\eta,T]_{\mathbb{T}}}u_1(t);\quad m_1<\max_{t\in[0,T]_{\mathbb{T}}}u_2(t),\ with\ \min_{t\in[\eta,T]_{\mathbb{T}}}u_2(t)<m_2;$$

$$\max_{t\in[0,T]_{\mathbb{T}}}u_3(t)<m_1.$$

Proof. Define an operator $A : P \to E$ by

$$\begin{aligned} Au\,(t) &= -\int_0^t (t-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s \\ &\quad +\frac{\beta t+\gamma}{d}\int_0^T (T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s \\ &\quad -\frac{\alpha(\beta t+\gamma)}{d}\int_0^{\eta} (\eta-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s. \end{aligned}$$

From [11, Lemma 2.2], we have $Au \in P$. By a standard argument, it is easy to see that $A : P \to P$ is completely continuous. We now show that all the conditions of Lemma 2.1 are satisfied.

Firstly, we show that if (C1) is satisfied, then

$$A : \overline{P(\phi, m_4)} \to \overline{P(\phi, m_4)}. \tag{3.3}$$

For $u \in \overline{P(\phi, m_4)}$, we have $\phi(u) = \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |u^{\Delta}(t)| \leq m_4$. With Lemma 3.1, we have $\max_{t\in[0,T]_{\mathbb{T}}} |u(t)| \leq Mm_4$. Then condition (C1) implies that $f(t,u(t),u^{\Delta}(t)) \leq m_4/S$ for $t \in [0,T]_{\mathbb{T}}$. On the other hand, for $u \in P$, we have $Au \in P$. Then Tu is concave on $[0,T]_{\mathbb{T}}$, so we have

$$\begin{aligned} \gamma(Au) &= \max_{t\in[0,T]_{\mathbb{T}^\kappa}} |(Au)^{\Delta}(t)| \\ &= \max\left\{|(Au)^{\Delta}(0)|, |(Au)^{\Delta}(T)|\right\} \\ &= \max\left\{\left|\frac{\beta}{d}\int_0^T (T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s \right.\right. \\ &\quad \left. -\frac{\alpha\beta}{d}\int_0^{\eta} (\eta-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right|, \\ &\quad \left|-\int_0^T q(s)f(s,u(s),u^{\Delta}(s))\nabla s + \frac{\beta}{d}\int_0^T (T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s \right. \\ &\quad \left.\left. -\frac{\alpha\beta}{d}\int_0^{\eta} (\eta-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right|\right\} \\ &\leq \max\left\{\frac{\beta}{d}\int_0^T (T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s, \right. \\ &\quad \left. \int_0^T q(s)f(s,u(s),u^{\Delta}(s))\nabla s + \frac{\alpha\beta}{d}\int_0^{\eta} (\eta-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right\} \\ &\leq \frac{m_4}{S}\max\left\{\frac{\beta}{d}\int_0^T (T-s)\,q(s)\nabla s, \int_0^T q(s)\nabla s + \frac{\alpha\beta}{d}\int_0^{\eta} (\eta-s)\,q(s)\nabla s\right\} \\ &= m_4. \end{aligned}$$

Therefore, (3.3) holds.

We choose

$$\begin{aligned}\overline{u}(t) &= -dam_2t^2 + a\left[\beta(T^2-\alpha\eta^2)+\gamma(1-\alpha)\sigma(0)\right]m_2t\\ &\quad +a\left[\gamma(T^2-\alpha\eta^2)-\gamma(T-\alpha\eta)\sigma(0)\right]m_2 \text{ for } t\in[0,T]_{\mathbb{T}}.\end{aligned}$$

It is easy to see that $\beta\overline{u}(0)-\gamma\overline{u}^{\Delta}(0)=0$, $\alpha\overline{u}(\eta)=\overline{u}(T)$, $\overline{u}(t)\geq 0$ and is concave on $[0,T]_{\mathbb{T}}$, so $\overline{u}\in P$. Again

$$\varphi(\overline{u})=\min_{t\in[\eta,T]_{\mathbb{T}}}|\overline{u}(t)|=\min\{1,\alpha\}(T-\eta)(\beta\eta T+\gamma T+\gamma\eta-\gamma\sigma(0))am_2>m_2,$$

$$\begin{aligned}\theta(\overline{u}) &= \max_{t\in[0,T]_{\mathbb{T}}}|\overline{u}(t)|\leq dam_2T^2+a\left[\beta(T^2-\alpha\eta^2)+\gamma(1-\alpha)\sigma(0)\right]m_2T\\ &\quad +a\left[\gamma(T^2-\alpha\eta^2)-\gamma(T-\alpha\eta)\sigma(0)\right]m_2=bm_2,\end{aligned}$$

$$\phi(\overline{u})\leq\left[d(T+\sigma(T))+\beta(T^2-\alpha\eta^2)+\gamma(1-\alpha)\sigma(0)\right]m_2=cm_2\leq m_4.$$

Hence, $\overline{u}\in P(\phi,\theta,\varphi,m_2,bm_2,m_4)$ and

$$\{u\in P(\phi,\theta,\varphi,m_2,bm_2,m_4)|\varphi(u)>m_2\}\neq\emptyset.$$

For $u\in P(\phi,\theta,\varphi,m_2,bm_2,m_4)$, we have $m_2\leq u(t)\leq bm_2$ and $|u^{\Delta}(t)|\leq m_4$ for $t\in[\eta,T]_{\mathbb{T}}$. Hence by condition (C2), one has that $f(t,u(t),u^{\Delta}(t))>m_2/L$ for $t\in[\eta,T]_{\mathbb{T}}$. So by the definition of the functional φ, we see that

$$\begin{aligned}\varphi(Au) &= \min_{t\in[\eta,T]_{\mathbb{T}}}|Au(t)|=\min\{Au(\eta),Au(T)\}\\ &= \min\{1,\alpha\}\left(\frac{\beta\eta+\gamma}{d}\int_0^T(T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right.\\ &\quad\left.-\frac{\beta T+\gamma}{d}\int_0^{\eta}(\eta-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right)\\ &> \min\{1,\alpha\}\left(\frac{\beta\eta+\gamma}{d}\int_0^T(T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right.\\ &\quad\left.-\frac{\beta T+\gamma}{d}\int_0^{\eta}(\eta-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right)\\ &\quad-\frac{T-\eta}{d}\int_0^{\eta}(\beta s+\gamma)q(s)f(s,u(s),u^{\Delta}(s))\nabla s\\ &= \min\{1,\alpha\}\left(\frac{\beta\eta+\gamma}{d}\int_0^T(T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right.\\ &\quad\left.-\frac{\beta\eta+\gamma}{d}\int_0^{\eta}(T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\right)\\ &= \frac{\min\{1,a\}(\beta\eta+\gamma)}{d}\int_{\eta}^T(T-s)\,q(s)f(s,u(s),u^{\Delta}(s))\nabla s\\ &> \frac{\min\{1,a\}(\beta\eta+\gamma)}{d}\int_{\eta}^T(T-s)\,q(s)\frac{m_2}{L}\nabla s=m_2.\end{aligned}$$

Therefore, we get $\varphi(Au) > m_2$ for $u \in P(\phi, \theta, \varphi, m_2, bm_2, m_4)$, and condition (S1) in Lemma 2.1 is satisfied.

Next, it follows from [11, Lemma 2.4] that for any $u \in P(\phi, \varphi, m_2, m_4)$ with $\theta(Au) > bm_2$, we have

$$\varphi(Au) = \min_{t\in[\eta,T]_{\mathbb{T}}} |A(u)| \geq r \max_{t\in[0,T]_{\mathbb{T}}} A(u) \geq r\theta(Au) > m_2.$$

Thus, condition (S2) of Lemma 2.1 is satisfied.

Finally, we assert that (S3) in Lemma 2.1 also holds. Clearly, as $\psi(0) = 0 < m_1$, so $0 \notin R(\phi, \psi, m_1, m_4)$. Assume that $u \in P(\phi, \psi, m_1, m_4)$ with $\psi(u) = m_1$. Then by the condition (C3), we obtain that

$$\begin{aligned}
\psi(Au) &= \max_{t\in[0,T]_{\mathbb{T}}} |(Au)(t)| \\
&= \max\Bigg| -\int_0^t (t-s)\, q(s) f(s, u(s), u^{\Delta}(s))\nabla s \\
&\quad + \frac{\beta t+\gamma}{d}\int_0^T (T-s)\, q(s) f(s, u(s), u^{\Delta}(s))\nabla s \\
&\quad - \frac{\alpha(\beta t+\gamma)}{d}\int_0^{\eta} (\eta-s)\, q(s) f(s, u(s), u^{\Delta}(s))\nabla s\Bigg| \\
&\leq \max_{t\in[0,T]_{\mathbb{T}}} \left|\frac{\beta t+\gamma}{d}\int_0^T (T-s)\, q(s) f(s, u(s), u^{\Delta}(s))\nabla s\right| \\
&< \frac{\beta T+\gamma}{d}\int_0^T (T-s)\, q(s)\frac{m_1}{N}\nabla s = m_1.
\end{aligned}$$

Therefore, an application of Lemma 2.1 implies that the BVP (1.1) and (1.2) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,T]_{\mathbb{T}^{\kappa}}} |u_i^{\Delta}(t)| \leq m_4 \ \text{ for } \ i = 1, 2, 3; \quad m_2 < \min_{t\in[\eta,T]_{\mathbb{T}}} u_1(t);$$

$$m_1 < \max_{t\in[0,T]_{\mathbb{T}}} u_2(t) \ \text{ with } \ \min_{t\in[\eta,T]_{\mathbb{T}}} u_2(t) < m_2; \quad \max_{t\in[0,T]_{\mathbb{T}}} u_3(t) < m_1.$$

The proof is complete. □

4 Example

Let $\mathbb{T} = \left[0, \frac{1}{2}\right] \cup \left\{\frac{1}{2} + \frac{1}{2^n}, n = 1, 2, 3\cdots\right\}$, $\beta = 1, \gamma = \frac{1}{2}, \alpha = \frac{1}{2}, T = 1, \eta = \frac{1}{2}$. Now we consider the BVP

$$u^{\Delta\nabla}(t) + f(t, u(t), u^{\Delta}(t)) = 0, \quad t \in (0, 1)_{\mathbb{T}}, \tag{4.1}$$

$$u(0) - \frac{1}{2}u^{\Delta}(0) = 0, \quad \frac{1}{2}u\left(\frac{1}{2}\right) = u(1), \tag{4.2}$$

where

$$f(t,u,v) = \begin{cases} \dfrac{1}{100}t + 13u^3 + \dfrac{\sin v}{100}, & u \le 12, \\ \dfrac{1}{100}t + 22464 + \dfrac{\sin v}{100}, & u > 12. \end{cases}$$

It is easy to see by calculating that $d = 1$, $M = 2$, $r = \dfrac{1}{4}$ and

$$S = \int_0^1 \nabla s + \frac{1}{2}\int_0^{\frac{1}{2}} \left(\frac{1}{2} - s\right) \nabla s = \frac{17}{16},$$

$$L = \frac{\min\{1, \frac{1}{2}\}(\frac{1}{2} + \frac{1}{2})}{\frac{1}{2}} \int_{\frac{1}{2}}^1 (1-s)\,\nabla s = \frac{1}{12},$$

$$N = \frac{\frac{1}{2} + \frac{1}{2}}{\frac{1}{2}} \int_0^1 (1-s)\,\nabla s = \frac{11}{12}.$$

If we choose $m_1 = \dfrac{1}{4}$, $m_2 = 1$, $m_4 = 10^5$, then $f(t,u,v)$ satisfies

(1) $f(t,u,v) < \dfrac{16}{17} \times 10^5 = \dfrac{m_4}{S}$ for

$$(t,u,v) \in [0,1]_{\mathbb{T}} \times [0, 2\times 10^5] \times [-10^5, 10^5];$$

(2) $f(t,u,v) \ge \dfrac{1}{200} + 13 - \dfrac{1}{100} > 12 = \dfrac{m_2}{L}$ for

$$(t,u,v) \in \left[\frac{1}{2}, 1\right]_{\mathbb{T}} \times \left[1, \frac{49}{3}\right] \times [-10^5, 10^5];$$

(3) $f(t,u,v) \le \dfrac{1}{100} + \dfrac{13}{64} + \dfrac{1}{100} < \dfrac{3}{11} = \dfrac{m_1}{N}$ for

$$(t,u,v) \in [0,1]_{\mathbb{T}} \times \left[0, \frac{1}{4}\right] \times [-10^5, 10^5].$$

Then all assumptions of Theorem 3.2 hold. Thus by Theorem 3.2, the problem (4.1) and (4.2) has at least three positive solutions u_1, u_2 and u_3 such that

$$\max_{t\in[0,1]_{T^\kappa}} |u_i^{\Delta}(t)| \le 10^5 \text{ for } i = 1,2,3, \quad 1 < \min_{t\in[\frac{1}{2},1]_{\mathbb{T}}} u_1(t);$$

$$\frac{1}{2} < \max_{t\in[0,1]_{\mathbb{T}}} u_2(t) \text{ with } \min_{t\in[\frac{1}{2},1]_{\mathbb{T}}} u_2(t) < 1; \quad \max_{t\in[0,1]_{\mathbb{T}}} u_3(t) < \frac{1}{2}.$$

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