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Existence and Approximation of Solutions of Boundary Value Problems on Time Scales

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Abstract

Existence and approximation of solutions for a class of boundary value problems on time scales of the type

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = f(t, y(t)), \quad t \in [a, b]_{\mathbb{T}},$$
$$c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \quad d_1 y(b) + d_2 y^{[\Delta]}(b) = 0,$$

is established. For the existence theory, we develop the method of upper and lower solutions. To approximate the solution, we develop the generalized method of quasilinearization. We show that under suitable conditions on f, there exists a bounded monotone sequence of solutions of linear problems which converges monotonically and rapidly to solution of the original problem.

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1 Introduction

In this paper, we study existence and approximation of solutions of boundary value problems of the type

$$\begin{aligned} &-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = f(t, y(t)), \quad t \in [a, b]_{\mathbb{T}}, \\ &c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \quad d_1 y(b) + d_2 y^{[\Delta]}(b) = 0, \end{aligned} \tag{1.1}$$

where $a \in \mathbb{T}_\kappa$, $b \in \mathbb{T}^\kappa$, $c_1, c_2, d_1, d_2 \in \mathbb{C}$ such that $|c_1| + |c_2| \neq 0$, $|d_1| + |d_2| \neq 0$ and $p(t) > 0$ on $[a, b]_{\mathbb{T}}$, $\mathbb{T}$ is a so-called time scale, that is, any nonempty closed subset of $\mathbb{R}$. The notations $\mathbb{T}^\kappa$, $\mathbb{T}_\kappa$ and $[a, b]_{\mathbb{T}}$ stand for:

$$\mathbb{T}^\kappa = \begin{cases} \mathbb{T} \setminus \{t_1\}, & \text{if } \mathbb{T} \text{ has a left-scattered maximum } t_1; \\ \mathbb{T}, & \text{otherwise;} \end{cases}$$

$$\mathbb{T}_\kappa = \begin{cases} \mathbb{T} \setminus \{t_2\}, & \text{if } \mathbb{T} \text{ has a right-scattered minimum } t_2; \\ \mathbb{T}, & \text{otherwise;} \end{cases}$$

$$[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Traditionally, researchers have assumed that a dynamical process is only continuous or only discrete. However, many of them contain both continuous and discrete elements simultaneously. Thus, traditional mathematical modeling techniques, such as differential or difference equations, provide a limited understanding of these types of models. A simple example of this hybrid continuous-discrete behavior appears in many natural populations: for example, insects that lay their eggs at the end of the season just before the generation dies out, with the eggs laying dormant, hatching at the start of the next season giving rise to a new generation. See Ref. [8] for different examples of species which follow this behavior.

The theory of time scales was introduced by Stefan Hilger [11] to unify the theories of continuous and discrete calculus. Many results for differential equations can be easily extended to the corresponding results for difference equations while some results seem to be completely different from their continuous counterparts. The study of dynamic equations on time scales reveals such discrepancies and helps avoid proving results twice, that is, once for differential equations and once again for difference equations. There are also applications of dynamic equations on time scales to other applied sciences such as quantum mechanics, electrical engineering, neural networks, heat transfer and combinatorics etc.

Since many economic models are dynamic models with continuous and discrete time, therefore, the results of time scales are directly applicable to economics as well. For example, money is invested and accounted at specific (discrete) times but the growth due to the investment is continuous in time.

Recently, existence theory for positive solutions of BVPs on time scales has received much attention from many authors, see for example [1, 4, 6, 9, 14] for two-point (BVPs)

and [2, 10, 12] for multi-point BVPs. However, very little work has been done on the method of upper and lower solutions and the quasilinearization technique [3, 5, 13]. In this paper, we develop the method of upper and lower solutions to obtain an existence and a uniqueness result. Then we approximate the solutions of a class of nonlinear time scale BVPs (1.1). We obtain not only the existence of solutions but also a method to approximate the solutions. We show that under suitable conditions on f, there exists a monotone sequence of solutions of linear problems that converges uniformly and rapidly to unique solution of the original nonlinear problem.

For completeness, we recall the following concepts related to the notion of time scales.

Definition 1.1. Let $t \in \mathbb{T}^\kappa$ and $f : \mathbb{T} \to \mathbb{R}$. The delta derivative $f^\Delta(t)$ of f at t is defined to be the number (if it exists) with the property that for given $\varepsilon > 0$ there is a neighborhood U of t such that

$$| \, [f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s] \, | \leq \varepsilon \mid \sigma(t) - s \mid,$$

for all $s \in U$.

Definition 1.2. Let $t \in \mathbb{T}_\kappa$ and $f : \mathbb{T} \to \mathbb{R}$. The nabla derivative of f at t is defined to be the number $f^\nabla(t)$ (if it exists) with the property that for each $\varepsilon > 0$ there is a neighborhood $U = (t - \delta, t + \delta) \cap \mathbb{T}$ (for some $\delta > 0$) of t such that

$$|[f(\rho(t)) - f(s)] - f^\nabla(t)(\rho(t) - s)| \leq \varepsilon|\rho(t) - s| \text{ for all } s \in U.$$

The relationship between Δ and ∇ derivatives is sometimes given by

$$f^\nabla(t) = f^\Delta(\rho(t)) \text{ for all } t \in \mathbb{T}.$$

We note that the homogeneous boundary value problem

$$\begin{aligned} &- [p(t)y^\Delta(t)]^\nabla + q(t)y(t) = 0, \quad t \in [a, b]_\mathbb{T}, \\ &c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \quad d_1 y(b) + d_2 y^{[\Delta]}(b) = 0, \end{aligned} \tag{1.2}$$

has only the trivial solution provided

$$D = \phi^{[\Delta]}(b)\psi(b) - \phi(b)\psi^{[\Delta]}(b) = \delta\phi^{[\Delta]}(b) + \gamma\phi(b) \neq 0, \tag{1.3}$$

where $\{\phi, \psi\}$ is a fundamental set of solutions of (1.2). Hence, for any continuous function h, the corresponding nonhomogeneous linear problem

$$\begin{aligned} &- [p(t)y^\Delta(t)]^\nabla + q(t)y(t) = h(t), \quad t \in [a, b]_\mathbb{T}, \\ &c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \quad d_1 y(b) + d_2 y^{[\Delta]}(b) = 0, \end{aligned}$$

has a unique solution y given by

$$y(t) = \int_{\rho(a)}^{b} G(t, s)h(s)\nabla s,$$

where

$$G(t,s) = \frac{1}{D}\begin{cases}\phi(t)\psi(s), & \rho(a) \le t \le s \le \sigma(b),\\ \phi(s)\psi(t), & \rho(a) \le s \le t \le \sigma(b),\end{cases}$$

is Green's function.

2 Upper and Lower Solutions Method

Let $C_{\mathrm{rd}}([a,b]_{\mathbb{T}},\mathbb{R})$ be the set of all functions $f(t,y)$ such that $f(\cdot,y)$ is rd-continuous for each $y \in \mathbb{R}$ and $f(t,\cdot)$ is continuous for each $t \in [a,b]_{\mathbb{T}}$. By $C^2_{\mathrm{rd}}([a,b]_{\mathbb{T}})$, we mean the set of all functions $y : \mathbb{T} \to \mathbb{R}$ such that

$$C^2_{\mathrm{rd}}([a,b]_{\mathbb{T}}) = \{y : y,\ y^{\Delta} \in C([a,b]_{\mathbb{T}}) \text{ and } y^{\Delta\nabla} \in C_{\mathrm{rd}}([a,b]_{\mathbb{T}})\}.$$

We recall the concept of lower and upper solutions [5, 13].

Definition 2.1. A function $\alpha \in C^2_{\mathrm{rd}}([\rho(a),\sigma(b)]_{\mathbb{T}})$ is said to be a *lower solution* of (1.1) if

$$\begin{aligned} -[p(t)\alpha^{\Delta}(t)]^{\nabla} + q(t)\alpha(t) &\le f(t,\alpha(t)), \quad t \in [a,b]_{\mathbb{T}},\\ c_1\alpha(\rho(a)) - c_2\alpha^{[\Delta]}(\rho(a)) \le 0, &\quad d_1\alpha(b) + d_2\alpha^{[\Delta]}(b) \le 0. \end{aligned}$$

Definition 2.2. A function $\beta \in C^2_{\mathrm{rd}}([\rho(a),\sigma(b)]_{\mathbb{T}})$ is said to be an *upper solution* of (1.1) if

$$\begin{aligned} -[p(t)\beta^{\Delta}(t)]^{\nabla} + q(t)\beta(t) &\ge f(t,\beta(t)), \quad t \in [a,b]_{\mathbb{T}},\\ c_1\beta(\rho(a)) - c_2\beta^{[\Delta]}(\rho(a)) \ge 0, &\quad d_1\beta(b) + d_2\beta^{[\Delta]}(b) \ge 0. \end{aligned}$$

For example, for the boundary value problem,

$$y^{\Delta\Delta}(t) = y^2 - t, \quad t \in [0,1]_{\mathbb{T}}, \quad y(0) = 0, \quad y(1) = 0,$$

the functions $\alpha = 0$, $\beta(t) = t + 1$ on $[0,1]_{\mathbb{T}}$, are lower and upper solutions.

Theorem 2.3 (Comparison result)**.** *Assume that α and β are lower and upper solutions of the BVP* (1.1)*. If $f(t,y) \in C_{\mathrm{rd}}[[\rho(a),\sigma(b)]_{\mathbb{T}} \times \mathbb{R}]$ is decreasing in y for each $t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$, then $\alpha(t) \le \beta(t)$ on $[\rho(a),\sigma(b)]_{\mathbb{T}}$.*

Proof. Let $v(t) = \alpha(t) - \beta(t)$, $t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$. Then $v \in C^2_{\mathrm{rd}}([\rho(a),\sigma(b)]_{\mathbb{T}})$ and using the definitions of upper and lower solutions, we have

$$\begin{aligned} -[p(t)v^{\Delta}(t)]^{\nabla} + q(t)v(t) &= -[p(t)(\alpha(t)-\beta(t))^{\Delta}]^{\nabla} + q(t)(\alpha(t)-\beta(t))\\ &= -[p(t)\alpha^{\Delta}(t)]^{\nabla} + [p(t)\beta^{\Delta}(t)]^{\nabla} + q(t)\alpha(t) - q(t)\beta(t)\\ &= \{-[p(t)\alpha^{\Delta}(t)]^{\nabla} + q(t)\alpha(t)\} - \{-[p(t)\beta^{\Delta}(t)]^{\nabla} + q(t)\beta(t)\}\\ &\le f(t,\alpha(t)) - f(t,\beta(t)), \quad t \in [a,b]_{\mathbb{T}}. \end{aligned}$$

The boundary conditions imply that

$$\begin{aligned} c_1 v(\rho(a)) - c_2 v^{[\Delta]}(\rho(a)) &= c_1[\alpha(\rho(a)) - \beta(\rho(a))] - c_2[\alpha(\rho(a)) - \beta(\rho(a))]^{[\Delta]} \\ &= [c_1\alpha(\rho(a)) - c_2\alpha^{[\Delta]}(\rho(a))] - [c_1\beta(\rho(a)) - c_2\beta^{[\Delta]}(\rho(a))] \\ &\leq 0 \end{aligned}$$

and

$$\begin{aligned} d_1 v(b) + d_2 v^{[\Delta]}(b) &= d_1[\alpha(b) - \beta(b)] + d_2[\alpha^{[\Delta]}(b) - \beta^{[\Delta]}(b)] \\ &= [d_1\alpha(b) + d_2\alpha^{[\Delta]}(b)] - [d_1\beta(b) + d_2\beta^{[\Delta]}(b)] \leq 0. \end{aligned}$$

Hence, we have the inequalities

$$\begin{aligned} &-[p(t)v^{\Delta}(t)]^{\nabla} + q(t)v(t) \leq f(t,\alpha(t)) - f(t,\beta(t)), \quad t \in [a,b]_{\mathbb{T}} \\ &c_1 v(\rho(a)) - c_2 v^{[\Delta]}(\rho(a)) \leq 0, \quad d_1 v(b) + d_2 v^{[\Delta]}(b) \leq 0. \end{aligned} \tag{2.1}$$

Suppose that the conclusion of the theorem is not true, then $v(t)$ has a positive maximum say M at some $t_0 \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. If $t_0 = \rho(a)$, then

$$v(\rho(a)) > 0 \text{ and } v^{\Delta}(\rho(a)) \leq 0. \tag{2.2}$$

The boundary condition $c_1 v(\rho(a)) - c_2 v^{[\Delta]}(\rho(a)) \leq 0$ implies that

$$c_1 v(\rho(a)) \leq c_2 v^{[\Delta]}(\rho(a)) = c_2 p(\rho(a)) v^{\Delta}(\rho(a)) < 0,$$

which leads to $v(\rho(a)) < 0$, a contradiction to (2.2). If $t_0 = \sigma(b)$, then

$$v(\sigma(b)) > 0 \text{ and } v^{\Delta}(b) \geq 0. \tag{2.3}$$

The boundary condition $d_1 v(b) + d_2 v^{[\Delta]}(b) \leq 0$ implies that

$$d_1 v(b) \leq -d_2 v^{[\Delta]}(b) = -d_2 p(b) v^{\Delta}(b) < 0,$$

which is a contradiction. Hence $t_0 \neq \sigma(b)$. Therefore, $t_0 \in (\rho(a), \sigma(b))$.

Now, choose $t_0 = \max\{e \in (\rho(a), \sigma(b)) : v(e) = M\}$. Then $v(t) < v(t_0)$ for each $t \in (\rho(a), \sigma(b))$. Firstly, we show that t_0 cannot be left-dense and right-scattered simultaneously. Suppose that t_0 is left-dense and right-scattered, that is, $\rho(t_0) = t_0 < \sigma(t_0)$. Since, $v(\sigma(t_0)) < v(t_0)$, it follows that $v^{\Delta}(t_0) < 0$. Also, $v^{\Delta}(t_0) = \lim_{t \to t_0^-} v^{\Delta}(t)$, therefore, $\lim_{t \to t_0^-} v^{\Delta}(t) < 0$ which implies that there exists a neighborhood $N_\varepsilon(t_0)$ such that $v^{\Delta}(t) < 0$ for each $t \in N_\varepsilon(t_0)$, that is, $v(t) \geq v(t_0)$ for each $t \in N_\varepsilon(t_0)$, a contradiction. Hence, t_0 is not simultaneously left-dense and right-scattered. Therefore, by [14, Lemma (2)], we obtain

$$v^{\Delta\Delta}(\rho(t_0)) \leq 0. \tag{2.4}$$

Let $v^{\Delta} = f$. Then from (2.4), we have $(f^{\Delta})(\rho(t_0)) \leq 0$. Using the relation $f^{\nabla}(t_0) = f^{\Delta}(\rho(t_0))$, it follows that $f^{\nabla}(t_0) \leq 0$. Hence, $v^{\Delta\nabla}(t_0) \leq 0$ which implies that

$$\alpha^{\Delta\nabla}(t_0) \leq \beta^{\Delta\nabla}(t_0). \tag{2.5}$$

The relation $-\alpha^{[\Delta]\nabla}(t_0) \leq -q(t_0)\alpha(t_0) + f(t_0, \alpha(t_0))$, and the decreasing property of $f(t, y)$ in y, leads to

$$\begin{aligned}\alpha^{[\Delta]\nabla}(t_0) &\geq q(t_0)\alpha(t_0) - f(t_0, \alpha(t_0))\\ &\geq q(t_0)\beta(t_0) - f(t_0, \alpha(t_0))\\ &> q(t_0)\beta(t_0) - f(t_0, \beta(t_0)) \geq \beta^{[\Delta]\nabla}(t_0),\end{aligned}$$

a contradiction to (2.5). Hence there does not exist $t_0 \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ such that $v(t_0) > 0$. Consequently, $v(t) \leq 0$, for every $t_0 \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. □

Corollary 2.4. *Under the hypothesis of the Theorem 2.3, the BVP* (1.1) *has a unique solution.*

The proof of the following result can be found in [14].

Lemma 2.5. *If $f(t, y) \in C_{\mathrm{rd}}([\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R})$ is uniformly bounded on $[\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}$, then the BVP* (1.1) *has a solution.*

Theorem 2.6. *Assume that α and β are lower and upper solutions of the boundary value problem* (1.1) *such that $\alpha \leq \beta$ on $[\rho(a), \sigma(b)]_{\mathbb{T}}$. Assume $f(t, y) \in C_{\mathrm{rd}}[[\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}]$. Then the boundary value problem* (1.1) *has a solution y such that $\alpha \leq y \leq \beta$ on $[\rho(a), \sigma(b)]_{\mathbb{T}}$.*

Proof. Define the following modification F of f

$$F(t, y(t)) = \begin{cases} f(t, \beta(t)) + \dfrac{y(t) - \beta(t)}{1 + |y(t) - \beta(t)|}, & \text{for } y(t) \geq \beta(t)\\ f(t, y(t)), & \text{for } \alpha(t) \leq y(t) \leq \beta(t)\\ f(t, \alpha(t)) + \dfrac{\alpha(t) - y(t)}{1 + |\alpha(t) - y(t)|}, & \text{for } y(t) \leq \alpha(t). \end{cases}$$

Clearly, $F \in C_{\mathrm{rd}}[[\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}]$ and is bounded on $[\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}$. Hence by Lemma 2.5, the modified problem,

$$\begin{aligned} -y^{[\Delta]\nabla}(t) + q(t)y(t) &= F(t, y(t)), t \in [a, b]_{\mathbb{T}}\\ c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) &= 0, d_1 y(b) + d_2 y^{[\Delta]}(b) = 0 \end{aligned} \tag{2.6}$$

has a solution. Using the definition of F, we have

$$F(t, \alpha(t)) = f(t, \alpha(t)) \geq -\alpha^{[\Delta]\nabla}(t) + q(t)\alpha(t), \quad t \in [a, b]_{\mathbb{T}}$$

and

$$F(t,\beta(t)) = f(t,\beta(t)) \le -\beta^{[\Delta]\nabla}(t) + q(t)\beta(t), \quad t \in [a,b]_{\mathbb{T}}$$

which implies that α and β are lower and upper solutions of (2.6). Moreover, we note that any solution y of the modified problem (2.6) such that

$$\alpha(t) \le y(t) \le \beta(t), \quad t \in [\rho(a),\sigma(b)]_{\mathbb{T}}, \tag{2.7}$$

is a solution of (1.1). We need to show that any solution y of (2.6) does satisfy (2.7).

Set $u(t) = \alpha(t) - y(t), t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$, where y is a solution of (1.1). Then $u \in C^2_{\mathrm{rd}}[\rho(a),\sigma(b)]_{\mathbb{T}}$ and the boundary conditions imply that

$$c_1 u(\rho(a)) - c_2 u^{[\Delta]}(\rho(a)) \le 0, \quad d_1 u(b) + d_2 u^{[\Delta]}(b) \le 0.$$

Assume that

$$\max\{u(t) : t \in [\rho(a),\sigma(b)]_{\mathbb{T}}\} = u(t_0) > 0.$$

We can show as in Theorem 2.3 that $t_0 \ne \rho(a), \sigma(b)$. Moreover, t_0 is not simultaneously left-dense and right-scattered. Consequently,

$$u^{\triangle\nabla}(t_0) \le 0. \tag{2.8}$$

On the other hand, using the definition of the lower solution and that of the modified function, we have

$$\begin{aligned}
-u^{[\triangle]\nabla}(t_0) &= -\alpha^{[\triangle]\nabla}(t_0) + y^{[\triangle]\nabla}(t_0) \\
&\le f(t_0,\alpha(t_0)) - q(t_0)\alpha(t_0) + y^{[\triangle]\nabla}(t_0) \\
&= f(t_0,\alpha(t_0)) - q(t_0)\alpha(t_0) - F(t_0,y(t_0)) + q(t_0)y(t_0) \\
&= f(t_0,\alpha(t_0)) - q(t_0)\alpha(t_0) - f(t_0,\alpha(t_0)) \\
&\quad - \frac{\alpha(t_0) - y(t_0)}{1 + |\alpha(t_0) - y(t_0)|} + q(t_0)y(t_0) \\
&= -q(t_0)u(t_0) - \frac{u(t_0)}{1 + u(t_0)} \\
&= -\left[q(t_0)u(t_0) + \frac{u(t_0)}{1 + u(t_0)}\right] < 0,
\end{aligned}$$

a contradiction to (2.8). Hence $u(t) \le 0,$ on $[\rho(a),\sigma(b)]_{\mathbb{T}}$. Now, set $u(t) = y(t) - \beta(t)$, $t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$, where y is a solution of (1.1). Then $u \in C^2_{\mathrm{rd}}[\rho(a),\sigma(b)]_{\mathbb{T}}$ and the boundary conditions imply that

$$c_1 u(\rho(a)) - c_2 u^{[\Delta]}(\rho(a)) \le 0, \quad d_1 u(b) + d_2 u^{[\Delta]}(b) \le 0.$$

Assume that

$$\max\{u(t) : t \in [\rho(a),\sigma(b)]_{\mathbb{T}}\} = u(t_0) > 0.$$

We can show as in Theorem 2.3 that $t_0 \neq \rho(a), \sigma(b)$. Moreover, t_0 is not simultaneously left-dense and right-scattered. Consequently,

$$u^{\triangle\nabla}(t_0) \leq 0. \tag{2.9}$$

On the other hand, using the definition of the upper solution and that of the modified function, we have

$$\begin{aligned}
u^{[\triangle]\nabla}(t_0) &= y^{[\triangle]\nabla}(t_0) - \beta^{[\triangle]\nabla}(t_0) \\
&\geq y^{[\triangle]\nabla}(t_0) + f(t_0, \beta(t_0)) - q(t_0)\beta(t_0) \\
&= q(t_0)y(t_0) - F(t_0, y(t_0)) + f(t_0, \beta(t_0)) - q(t_0)\beta(t_0) \\
&= q(t_0)y(t_0) - q(t_0)\beta(t_0) - f(t_0, \beta(t_0)) - \frac{y(t_0) - \beta(t_0)}{1 + |y(t_0) - \beta(t_0)|} \\
&\quad + f(t_0, \beta(t_0)) \\
&= q(t_0)(y(t_0) - \beta(t_0)) - \frac{y(t_0) - \beta(t_0)}{1 + |y(t_0) - \beta(t_0)|} \\
&= q(t_0)u(t_0) - \frac{u(t_0)}{1 + u(t_0)} > 0,
\end{aligned}$$

a contradiction to (2.9). Hence $u(t) \leq 0$, $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$. □

3 Generalized Approximation Technique

We develop the approximation scheme and show that under suitable conditions on f, there exists a bounded monotone sequence of solutions of linear problems that converges uniformly to a solution of the original problem. Assume that

(A_1) α and β are lower and upper solutions of the BVP (1.1) such that $\alpha \leq \beta$ on $[\rho(a), \sigma(b)]_{\mathbb{T}}$.

Introduce the notation:

$$\begin{aligned}
C^1_{\mathrm{rd}}([\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R}) = \Big\{ & f : [\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R} \text{ such that} \\
& f(\cdot, y),\ f_y(\cdot, y) \text{ are rd-continuous for each } y \in \mathbb{R}, \\
& f(t, \cdot),\ f_y(t, \cdot) \text{ are continuous for each } t \in [\rho(a), \sigma(b)]_{\mathbb{T}} \Big\},
\end{aligned}$$

$$\begin{aligned}
\bar{\alpha} &= \min\{\alpha(t) : t \in [\rho(a), \sigma(b)]_{\mathbb{T}}\}, \\
\bar{\beta} &= \max\{\beta(t) : t \in [\rho(a), \sigma(b)]_{\mathbb{T}}\}, \\
[\bar{\alpha}, \bar{\beta}] &= \{x \in \mathbb{R} : \bar{\alpha} \leq x \leq \bar{\beta}\} \subset \mathbb{R}.
\end{aligned}$$

Assume that the following holds:

(A$_2$) $f \in C^1_{\mathrm{rd}}([\rho(a),\sigma(b)]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$, $f(t,\cdot)$ is decreasing for each $t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$ and

$$f(t,y) \geq f(t,z) + f_y(t,z)(y-z) \text{ if } t \in [\rho(a),\sigma(b)]_{\mathbb{T}} \text{ and } y,\ z \in [\bar{\alpha},\bar{\beta}]. \quad (3.1)$$

Theorem 3.1. *Assume that* (A$_1$) *and* (A$_2$) *hold. Then there exists a monotone sequence* $\{w_n\}$ *of solutions of linear problems converging uniformly to a unique solution of the BVP* (1.1).

Proof. By the comparison and existence theorems (Theorems 2.3 and 2.6), the conditions (A$_1$) and (A$_2$) ensure the existence of a unique solution y of the BVP (1.1) such that

$$\alpha(t) \leq y(t) \leq \beta(t), \quad t \in [\rho(a),\sigma(b)]_{\mathbb{T}}.$$

Define $g : [\rho(a),\sigma(b)]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$g(t,y,z) = f(t,z) + f_y(t,z)(y-z). \quad (3.2)$$

Then $g(\cdot,y,z)$ is rd-continuous for each $(y,z) \in \mathbb{R} \times \mathbb{R}$, $g(t,\cdot,\cdot)$, $g_y(t,\cdot,\cdot)$ are continuous for each $t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$ and $g(t,\cdot,z)$ is decreasing for each $(t,z) \in [\rho(a),\sigma(b)]_{\mathbb{T}} \times \mathbb{R}$. Moreover, in view of (3.1) and (3.2), g satisfies

$$\begin{cases} f(t,y) \geq g(t,y,z), \\ f(t,y) = g(t,y,y), \end{cases} \quad (3.3)$$

where $t \in [\rho(a),\sigma(b)]_{\mathbb{T}}$ and $y,z \in [\bar{\alpha},\bar{\beta}]$. Now, we develop the iterative scheme to approximate the solution. As an initial approximation, we choose $w_0 = \alpha$ and consider the linear BVP

$$\begin{cases} -y^{[\triangle]\nabla}(t) + q(t)y(t) = g(t,y(t),w_0(t)), \quad t \in [a,b]_{\mathbb{T}}; \\ c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \\ d_1 y(b) + d_2 y^{[\Delta]}(b) = 0. \end{cases} \quad (3.4)$$

Using (3.3) and the definitions of lower and upper solutions, we obtain

$$g(t,w_0(t),w_0(t)) = f(t,w_0(t)) \geq -w_0^{[\triangle]\nabla}(t) + q(t)w_0(t), \quad t \in [a,b]_{\mathbb{T}},$$

$$g(t,\beta(t),w_0(t)) \leq f(t,\beta(t)) \leq -\beta^{[\triangle]\nabla}(t) + q(t)\beta(t), \quad t \in [a,b]_{\mathbb{T}},$$

which imply that w_0 and β are lower and upper solutions of (3.4) respectively. Hence by Theorems 2.3 and 2.6, there exists a unique solution $w_1 \in C^2_{\mathrm{rd}}[\rho(a),\sigma(b)]_{\mathbb{T}}$ of (3.4) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [\rho(a),\sigma(b)]_{\mathbb{T}}.$$

Using (3.3) and the fact that w_1 is a solution of (3.4), we obtain

$$\begin{cases} -w_1^{[\triangle]\nabla}(t) + q(t)w_1(t) = g(t,w_1(t),w_0(t)) \leq f(t,w_1(t)), \quad t \in [a,b]_{\mathbb{T}}; \\ c_1 w_1(\rho(a)) - c_2 w_1^{[\Delta]}(\rho(a)) = 0, \\ d_1 w_1(b) + d_2 w_1^{[\Delta]}(b) = 0, \end{cases} \quad (3.5)$$

which implies that w_1 is a lower solution of (1.1). Using (3.3), (3.5) and the definition of upper solution, we have

$$g(t, w_1(t), w_1(t)) = f(t, w_1(t)) \geq -w_1^{[\Delta]\nabla}(t) + q(t)w_1(t), \quad t \in [a, b]_{\mathbb{T}}$$

and

$$g(t, \beta(t), w_1(t)) \leq f(t, \beta(t)) \leq -\beta^{[\Delta]\nabla}(t) + q(t)\beta(t), \quad t \in [a, b]_{\mathbb{T}},$$

which imply that w_1 and β are lower and upper solutions of

$$\begin{cases} -y^{[\triangle]\nabla}(t) + q(t)y(t) = g(t, y(t), w_1(t)), & t \in [a, b]_{\mathbb{T}}; \\ c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \\ d_1 y(b) + d_2 y^{[\Delta]}(b) = 0. \end{cases} \tag{3.6}$$

Hence by Theorems 2.3 and 2.6, there exists a unique solution $w_2 \in C^2_{\mathrm{rd}}[\rho(a), \sigma(b)]_{\mathbb{T}}$ of (3.6) such that

$$w_1(t) \leq w_2(t) \leq \beta(t) \quad \text{on} \quad [\rho(a), \sigma(b)]_{\mathbb{T}}.$$

Continuing in the above fashion, we obtain a bounded monotone sequence $\{w_n\}$ of solutions of linear problems satisfying

$$w_0(t) \leq w_1(t) \leq w_2(t) \leq w_3(t) \leq \ldots \leq w_n(t) \leq \beta(t), \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}},$$

where the element w_n of the sequence $\{w_n(t)\}$ is a solution of the linear problem

$$\begin{cases} -y^{[\triangle]\nabla}(t) + q(t)y(t) = g(t, y(t), w_{n-1}(t)), & t \in [a, b]_{\mathbb{T}}; \\ c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \\ d_1 y(b) + d_2 y^{[\Delta]}(b) = 0 \end{cases}$$

and is given by

$$w_n(t) = \int_{\rho(a)}^{b} G(t, s) g(s, w_n(s), w_{n-1}(s)) \Delta s.$$

Since $[\rho(a), \sigma(b)]_{\mathbb{T}}$ is compact and the convergence is monotone and bounded, $\{w_n(t)\}$ converges uniformly to some function y, see [3, 7]. Note that

$$g(t, w_n(t), w_{n-1}(t)) \to g(t, y(t), y(t)) = f(t, y(t)) \quad \text{as} \quad n \to \infty.$$

Passing to the limit, we obtain

$$y(t) = \int_{\rho(a)}^{b} G(t, s) f(s, y(s)) \Delta s,$$

that is, y is a solution of (1.1). $\square$

Now we generalize the result (Theorem 3.1) by allowing weaker hypothesis on the nonlinearity f. Choose an auxiliary function $\phi \in C^1_{\mathrm{rd}}([\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ such that $\phi_x(t, \cdot)$ is increasing for each fixed $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and

$$\phi(t, y) \geq \phi(t, z) + \phi_y(t, z)(y - z), \text{ if } t \in [\rho(a), \sigma(b)]_{\mathbb{T}},\ y,\ z \in [\bar{\alpha}, \bar{\beta}]. \tag{3.7}$$

Define $\tilde{F} = f + \phi$. Then $\tilde{F} \in C^1_{\mathrm{rd}}([\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$. Assume that the following holds:

(A_3)

$$f(t, y) \geq \tilde{F}(t, z) + \tilde{F}_y(t, z)(y - z) - \phi(t, y), \text{ if } t \in [\rho(a), \sigma(b)]_{\mathbb{T}},\ y,\ z \in [\bar{\alpha}, \bar{\beta}]. \tag{3.8}$$

Theorem 3.2. *Assume that* (A_1) *and* (A_3) *hold. Then there exists a monotone sequence* $\{w_n\}$ *of solutions of linear problems converging uniformly to a unique solution of the BVP* (1.1).

Proof. For fixed $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and $y,\ z \in [\bar{\alpha}, \bar{\beta}]$, using the increasing property of $\phi_y(t, \cdot)$, we obtain

$$\phi(t, y) - \phi(t, z) \geq \phi_y(t, \bar{\beta})(y - z), \text{ if } y \geq z. \tag{3.9}$$

Substituting (3.9) in (3.8), we obtain

$$f(t, y) \geq f(t, z) + (\tilde{F}_y(t, z) - \phi_y(t, \bar{\beta}))(y - z), \tag{3.10}$$

where $y \geq z$,$t \in [\rho(a), \sigma(b)]_{\mathbb{T}},\ y,\ z \in [\bar{\alpha}, \bar{\beta}]$. Define $\tilde{g} : [\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{g}(t, y, z) = f(t, z) + (\tilde{F}_y(t, z) - \phi_y(t, \bar{\beta}))(y - z). \tag{3.11}$$

Then $\tilde{g}(\cdot, y, z)$ is rd-continuous for each $(y, z) \in \mathbb{R} \times \mathbb{R}$, $\tilde{g}(t, \cdot, \cdot)$, $\tilde{g}_y(t, \cdot, \cdot)$ are continuous for each $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and $\tilde{g}(t, \cdot, z)$ is decreasing for each $(t, z) \in [\rho(a), \sigma(b)]_{\mathbb{T}} \times \mathbb{R}$. Moreover, in view of (3.10) and (3.11), $\tilde{g}$ satisfies

$$\begin{cases} f(t, y) \geq \tilde{g}(t, y, z),\ y \geq z, \\ f(t, y) = \tilde{g}(t, y, y), \end{cases} \tag{3.12}$$

where $t \in [\rho(a), \sigma(b)]_{\mathbb{T}}$ and $y, z \in [\bar{\alpha}, \bar{\beta}]$. We choose $w_0 = \alpha$ and consider the linear BVP

$$\begin{cases} -y^{[\triangle]\nabla}(t) + q(t)y(t) = \tilde{g}(t, y(t), w_0(t)), \quad t \in [a, b]_{\mathbb{T}}; \\ c_1 y(\rho(a)) - c_2 y^{[\Delta]}(\rho(a)) = 0, \\ d_1 y(b) + d_2 y^{[\Delta]}(b) = 0. \end{cases} \tag{3.13}$$

Using (3.12) and the definitions of lower and upper solutions, we obtain

$$\tilde{g}(t, w_0(t), w_0(t)) = f(t, w_0(t)) \geq -w_0^{[\triangle]\nabla}(t) + q(t)w_0(t), \quad t \in [a, b]_{\mathbb{T}},$$

$$\tilde{g}(t, \beta(t), w_0(t)) \leq f(t, \beta(t)) \leq -\beta^{[\triangle]\nabla}(t) + q(t)\beta(t), \quad t \in [a, b]_{\mathbb{T}},$$

which imply that w_0 and β are lower and upper solutions of (3.13), respectively. Hence by Theorems 2.3 and 2.6, there exists a unique solution $w_1 \in C^2_{\mathrm{rd}}[\rho(a), \sigma(b)]_{\mathbb{T}}$ of (3.13) such that

$$w_0(t) \leq w_1(t) \leq \beta(t), \quad t \in [\rho(a), \sigma(b)]_{\mathbb{T}}.$$

By the same process as discussed in Theorem 3.1, we obtain a monotone sequence of solutions of linear problems that converges to a unique solution of the original problem (1.1). □

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References

[1] Ravi P. Agarwal and Donal O'Regan. Nonlinear boundary value problems on time scales. *Nonlinear Anal.*, 44(4, Ser. A: Theory Methods):527–535, 2001.

[2] Douglas R. Anderson and Richard I. Avery. An even-order three-point boundary value problem on time scales. *J. Math. Anal. Appl.*, 291(2):514–525, 2004.

[3] F. Merdivenci Atici, P. W. Eloe, and B. Kaymakçalan. The quasilinearization method for boundary value problems on time scales. *J. Math. Anal. Appl.*, 276(1):357–372, 2002.

[4] F. Merdivenci Atici and G. Sh. Guseinov. On Green's functions and positive solutions for boundary value problems on time scales. *J. Comput. Appl. Math.*, 141(1-2):75–99, 2002. Dynamic equations on time scales.

[5] F. Merdivenci Atici and S. Gulsan Topal. The generalized quasilinearization method and three point boundary value problems on time scales. *Appl. Math. Lett.*, 18(5):577–585, 2005.

[6] Richard I. Avery and Douglas R. Anderson. Existence of three positive solutions to a second-order boundary value problem on a measure chain. *J. Comput. Appl. Math.*, 141(1-2):65–73, 2002. Dynamic equations on time scales.

[7] Martin Bohner and Allan Peterson. *Dynamic equations on time scales*. Birkhäuser Boston Inc., Boston, MA, 2001. An introduction with applications.

[8] Freddy B. Christiansen and Tom M. Fenchel. *Theories of Populations in Biological Communities*, volume 20 of *Lecture Notes in Ecological Studies*. Springer-Verlag, Berlin, 1977.

[9] L. Erbe and A. Peterson. Positive solutions for a nonlinear differential equation on a measure chain. *Math. Comput. Modelling*, 32(5-6):571–585, 2000. Boundary value problems and related topics.

[10] T. Gnana Bhaskar. Comparison theorem for a nonlinear boundary value problem on time scales. *J. Comput. Appl. Math.*, 141(1-2):117–122, 2002. Dynamic equations on time scales.

[11] Stefan Hilger. Analysis on measure chains—a unified approach to continuous and discrete calculus. *Results Math.*, 18(1-2):18–56, 1990.

[12] Eric R. Kaufmann. Positive solutions of a three-point boundary-value problem on a time scale. *Electron. J. Differential Equations*, pages No. 82, 11 pp. (electronic), 2003.

[13] Rahmat Ali Khan, Juan J. Nieto, and V. Otero-Espinar. Existence and approximation of solution of three-point boundary value problems on time scales. *J. Difference Equ. Appl.*, 14(7):723–736, 2008.

[14] Christopher C. Tisdell, Pavel Drábek, and Johnny Henderson. Multiple solutions to dynamic equations on time scales. *Comm. Appl. Nonlinear Anal.*, 11(4):25–42, 2004.