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Strange Attractors and Chaotic Behavior of a Mathematical Model for a Centrifugal Filter with Feedback

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Abstract

We present results of the numerical investigation of a mathematical model of a centrifugal filter with a feedback. We demonstrate that the model exhibits complicated behaviour that includes periodic and chaotic regimes. Strange attractors of different shapes occur in the phase space of the system.

AMS Subject Classifications: 37-04, 37M05, 65-04, 65Q99.
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1 Introduction

The present article is a continuation of a previous one [1] on modeling of dynamic processes in centrifugal filters with feedback (CFF).

In [1] such filters were described as a system of curved channels connected in series. Each channel has a constant cross-sectional area and relative displacement with respect to each other. Dusted air flow moves along a curve inside the channel, makes a 180° turn, and is splitting into two parts. One part then moves to the channel with smaller radius of curvature, another one enters the channel with larger radius of curvature. Thus, a filter with feedback allows heavy particles to move constantly toward the channels with larger radii of curvature, come out of the filter and get captured, while clean air flow with the remaining small dust particles comes out of the filter to the atmosphere through an opening in the center of the filter [2].

Cleaning of the dusted air flow in CFF is achieved by the layers of dust moving along equilibrium circular orbits. Here particles of dust are drawn into intensive interaction with each other and coagulate, then move to the higher orbits of dusted flows inside other channels, and finally escape the filter. Thus, effectiveness of the CFF is defined, in essence, by the intensity of the processes of coagulation. Some models of such processes are given in [3].

2 A Three-Level Model of a Stochastic System

A computer model for investigation of dynamic processes in CFF was proposed in [1]. The model describes circulation of the stochastic flow along three mutually connected parallel channels as evolution of a three-level dynamic system. The model is defined by the system of equations

$$\begin{cases} x_{n+1} = x_n - k_{xy} p x_n^2 + k_{yx} q y_n^2 + x_{\text{in}} \\ y_{n+1} = y_n + k_{xy} p x_n^2 - (k_{yx} + k_{yz}) q y_n^2 + k_{zy} r z_n^2 \\ z_{n+1} = z_n + k_{yz} q y_n^2 - (k_{zy} + k_{\text{out}}) r z_n^2. \end{cases} \tag{2.1}$$

Here x, y, z are dynamic variables that define the amount of particles on the levels, $k_{\alpha\beta}$ ($\alpha, \beta = x, y, z$) are transition coefficients that characterize static and dynamic interactions of the levels, p, q, r are distribution coefficients, x_{in} is the number of particles entering the first level; $x, y, z \in \mathbb{R}$, $k_{\alpha\beta}, p, q, r \in (0, 1)$, and $x_{\text{in}} = \text{const} \in \mathbb{R}^+$.

The fact that the system (2.1) contains two groups of coefficients, $k_{\alpha\beta}$ and p, q, r, is explained by its engineering origin and has a physical interpretation. Namely, the coefficients $k_{\alpha\beta}$ describe relative cross displacement of the channels connected in series and define a fraction of the flow moving from one channel to another. The coefficients p, q, r describe a distribution of particles along the width of a channel such that transition of particles between channels is defined by the product of coefficients from both groups. Therefore, each equation in (2.1) is bilinear with respect to $(px_n) \cdot (k_{xy} x_n)$, [1].

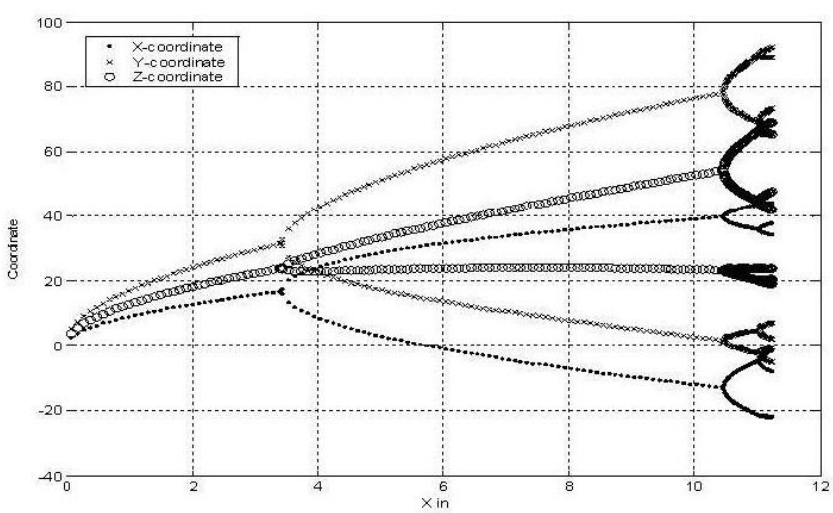
In general, solutions of the system (2.1) can be found only numerically.

Our investigation of the model showed that the region in the parameters space for which there exist stable phase trajectories is bounded. The system (2.1) has a stationary solution that can be obtained analytically if x_{in} is small enough. This solution is given by

$$\begin{cases} x_{\text{st}} = \sqrt{\dfrac{x_{\text{in}}}{k_{xy}p}\left(1 + \dfrac{k_{yx}}{k_{yz}}\left(1 + \dfrac{k_{zy}}{k_{\text{out}}}\right)\right)} \\ y_{\text{st}} = \sqrt{\dfrac{x_{\text{in}}}{k_{yz}q}\left(1 + \dfrac{k_{zy}}{k_{\text{out}}}\right)} \\ z_{\text{st}} = \sqrt{\dfrac{x_{\text{in}}}{k_{\text{out}}r}}. \end{cases} \tag{2.2}$$

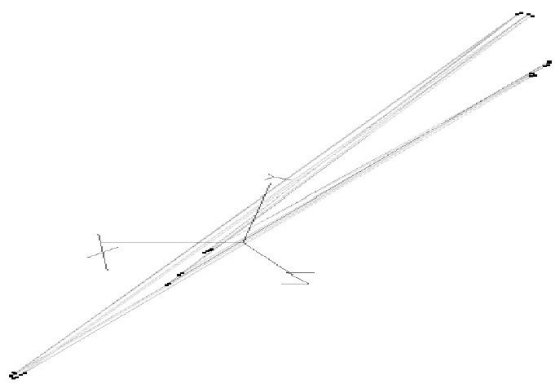
When x_{in} increases, there are two possibilities of further evolution of the system. In the first case it undergoes a cascade of period-doubling bifurcations, and the Feigenbaum scenario of transition to chaos is realized (see Figure 2.1). The ratio of intervals between two successive bifurcations coincides with the universal Feigenbaum constant $\delta_F = 4.669$ with error 1.3%. Attractors consisting of 2^n points were observed in the phase space of the system (see Figure 2.2).

Figure 2.1: Bifurcation diagram scenario of doubling period for the system with $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out}} = 0.4$, $p = 0.08$, $q = 0.02$, $r = 0.015$. The first three bifurcations are shown.



When other values of transition and distribution coefficients were used as bifurcation parameters, we observed other scenarios of evolution of the system (2.1) (see Tables 1 and 2).

Figure 2.2: Attractor consisting of 32 points which occurs after four doubling period bifurcations when parameters are $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out}} = 0.4$, $p = 0.05$, $q = 0.085$, $r = 0.04$, $x_{\text{in}} = 6.44$



In the second case, after the periodic regime, we observed a situation similar to a Hopf bifurcation which leads to a quasi periodic regime and an occurrence of an attractor consisting of two closed curves in the phase space (see Figure 2.3). Further increase of x_{in} leads to a bifurcation resulting in a chaotic regime. In such a case there occurs a strange attractor in phase space (see Figure 2.4). We found out that when a quasi periodic regime is realized, it is possible that a synchronization of the model may occur. In such a situation, a periodic regime in the system occurs with a period depending on the level of synchronization.

Figure 2.3: Attractor occurring at values of parameters $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out}} = 0.4$, $p = 0.065$, $q = 0.03$, $r = 0.03$, $xin = 6.05$.

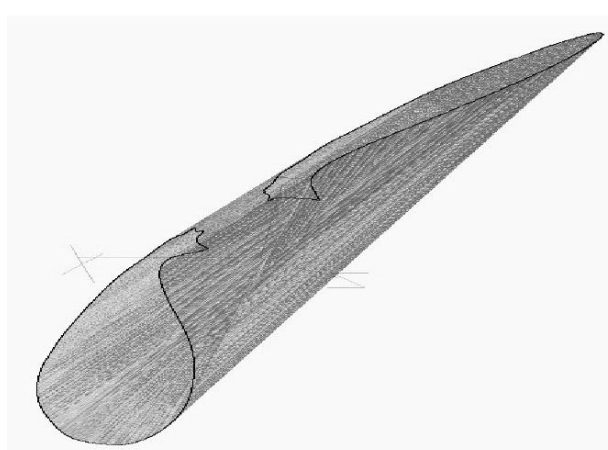


Figure 2.4: Strange attractor occurring at values of parameters $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out}} = 0.4$, $p = 0.008$, $q = 0.005$, $r = 0.0057$, $x_{\text{in}} = 39.65$.

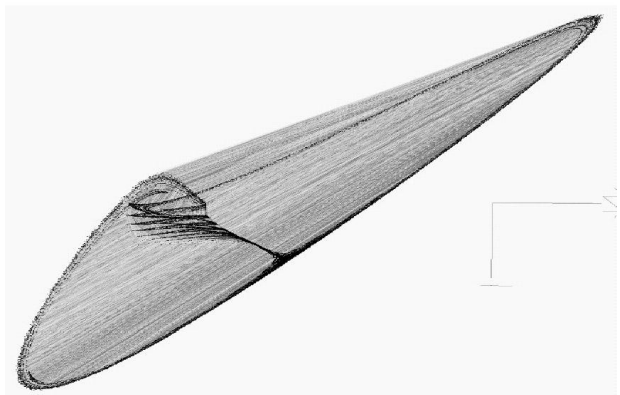


Figure 2.5: Attractor in the synchronization mode. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out}} = 0.4$, $p = 0.008$, $q = 0.005$, $r = 0.0057$, $x_{\text{in}} = 38$.

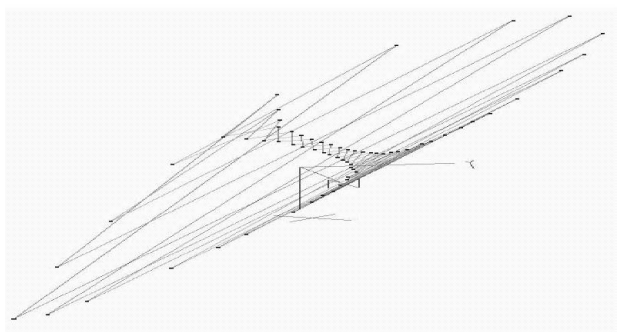


Figure 2.6 presents a bifurcation diagram of chaos development. On the intervals $x_{\text{in}} \in (11.3702, 11.7962)$ and $x_{\text{in}} \in (12.2425, 13.4361)$, there exists hysteresis with respect to the parameter x_{in}.

Figure 2.6: Bifurcation diagram of system (2.1) when parameters are $k_{xy} = 0.5$, $k_{yx} = 0.1$, $k_{yz} = 0.8$, $k_{zy} = 0.2$, $k_{\text{out}} = 0.4$, $p = 0.05$, $q = 0.05$, $r = 0.045$.

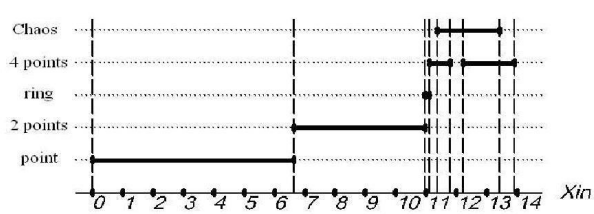


Table 1: The scenario realized for values of parameters $k_{xy} = 0.5$, $k_{yx} = 0.1$, $k_{yz} = 0.8$, $k_{zy} = 0.2$, $k_{\text{out}} = 0.4$, $p = 0.05$, $q = 0.05$, $r = 0.045$.

Bifurcation parameter x_{in}	Description of an attractor
$x_{\text{in}} \in (0.6.6433)$	Stationary regime, 1-point attractor.
$x_{\text{in}} \in (6.6433, 10.9702)$	Periodic regime, 2-point attractor.
$x_{\text{in}} \in (10.9702, 11.1045)$	quasi-periodic regime, attractor is a torus (as on Figure 2.3).
$x_{\text{in}} \in (11.3702, 13.4378)$	Chaotic regime, strange attractor. For $x_{\text{in}} \in (11.3702, 11.7962)$ it coexists with 4-points attractor. For $x_{\text{in}} > 11.7962$ only the strange attractor remains. At the beginning of the interval the strange attractor is a line with a large number of folds which, in fact, exhibits a complicated structure similar to Hénon attractor. (Figure 2.7). Gradually it is getting thicker and folds are getting larger. There often occur synchronizations of different orders.
$x_{\text{in}} \in (12.2425, 13, 9261)$	Periodic regime with a period 4 4-point attractor. On the interval it coexists with a strange attractor.
$x_{\text{in}} > 13.9261$	"Blowup"

Figure 2.7: Strange attractor in the system. The parameters are $k_{xy} = 0.5$, $k_{yx} = 0.1$, $k_{yz} = 0.8$, $k_{zy} = 0.2$, $k_{\text{out}} = 0.4$, $p = 0.05$, $q = 0.05$, $r = 0.045$, $x_{\text{in}} = 11.6$.

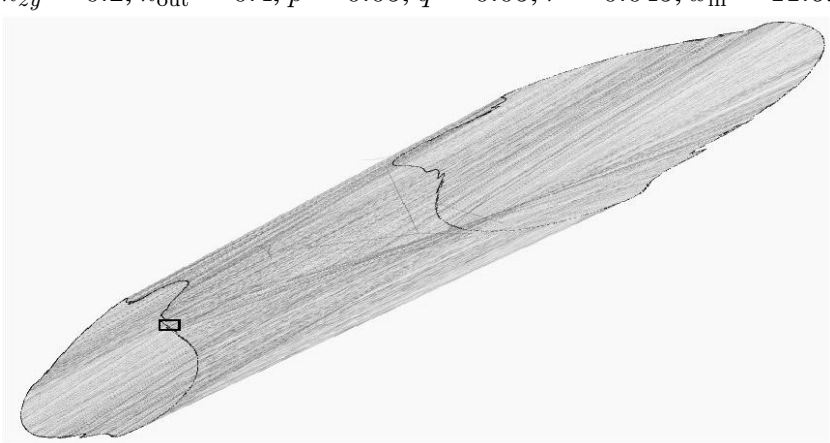


Figure 2.8: Strange attractor. The parameters are $k_{xy} = 0.5$, $k_{yx} = 0.1$, $k_{yz} = 0.8$, $k_{zy} = 0.2$, $k_{\text{out}} = 0.4$, $p = 0.05$, $q = 0.05$, $r = 0.045$, $x_{\text{in}} = 12$.

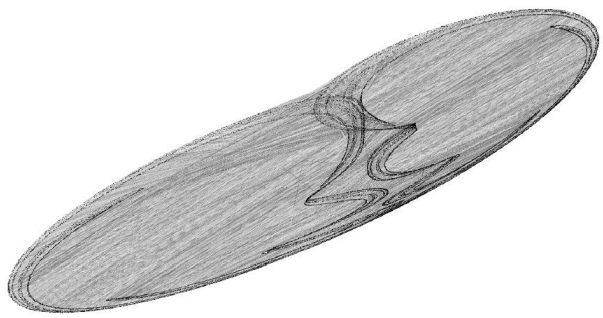
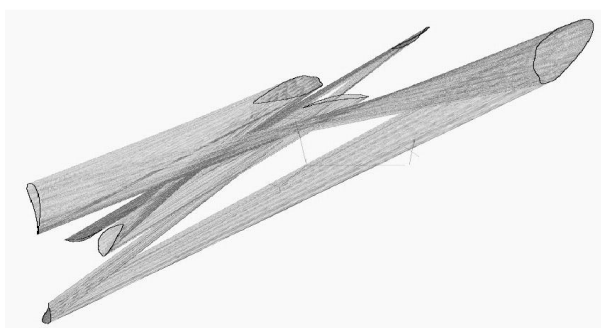


Figure 2.9: Attractor in a shape of 8 rings for quasiperiodic regime. Parameters are:$k_{xy} = 0.1$, $_{kyx} = 0.1$, $k_{yz} = 0.1$, $k_{zy} = 0.1$, $k_{\text{out}} = 0.2$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 1.536$.



Besides the aforementioned scenarios of chaos development in the system, it is possible to observe their various combinations. An example is given in Table 2.

Further increase of x_{in} when the system is in the chaotic regime leads to a change of shape of the chaotic attractor (see Figure 3.2). In such a case we observed alternation of the regimes, that is, chaotic behavior of the system on some intervals changes to periodic. In Figure 3.3 some compression of line segments connecting points of the phase trajectory corresponds to the above alternation. In such a case we observed multiple attractors and hysteresis with respect to the parameter x_{in}.

3 Change of the Algorithm in the Iteration Process

The common iteration procedure was implemented in the algorithm described above: Values of dynamic variables x, y, z that determine the amount of particles on the level

Table 2: The scenario realized for values of parameters $k_{xy} = 0.1$, $k_{yx} = 0.1$, $k_{yz} = 0.1$, $k_{zy} = 0.1$, $k_{\text{out}} = 0.2$, $p = 1$, $q = 1$, $r = 1$.

Bifurcation parameter x_{in}	Description of an attractor
$x_{\text{in}} \in (0, 0.6846)$	Stationary regime, 1-point attractor.
$x_{\text{in}} \in (0.6846, 1.3301)$	Periodic regime, 2-point attractor.
$x_{\text{in}} \in (1.3301, 1.5408)$	Periodic regime, 4-point.
$x_{\text{in}} \in (1.5023, 1.5252)$	Periodic regime, 8-point.
$x_{\text{in}} \in (1.5252, 1.5385)$	Quasi-periodic regime, eight annuli attractor (Figure 2.9).
$x_{\text{in}} \in (1.5385, 1.5408)$	Synchronization, 64-point attractor, points are located in eight domains.
$x_{\text{in}} \in (1.5408, 1.5446)$	128-point attractor. The points are are concentrated in eight domains.
$x_{\text{in}} \in (1.5446, 1.5456)$	64-point attractor. The points are are concentrated in eight domains.
$x_{\text{in}} \in (1.5456, 1.5561)$	Chaotic regime, a strange attractor.

were fixed after each iteration. They change only after the new values are calculated. In a modified model discussed below, another algorithm was implemented.

In the modified algorithm we took into account the change of values of the variables that have been calculated in order to evaluate values of other variables in the process of given iteration. Such a "bifurcation" of the algorithm results in changes in the evolution of the system.

As before, for small x_{in}, the system admits a stationary solution defined by equation (2.2). When x_{in} increases, the system leaves the stationary state. In such a case, a bifurcation similar to Hopf bifurcation occurs often. There is also a quasi periodic regime with two aliquant frequencies and attractor in the shape of an annulus.

For the values of parameters $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$, we observed an interesting scenario of the chaos development in the system: From the stationary state after doubling period bifurcation, the periodic regime with an attractor consisting of two points occurs, from which, in its turn, a quasiperiodic regime and an attractor of two annuli come up (see Figure 3.5). When x_{in} increases, the "annuli" deform and lose their stability (see Figure 3.6), and the phase trajectory is attracted to the stable attractor consisting of one "annulus" (see Figure 3.7). We observed a multiple number of attractors in the system, since the attractor "one annulus" had occurred before the attractor "two annuli" lost its stability, and each attractor had its own region of attraction.

When the bifurcation parameter x_{in} decreases, the attractor "annulus" loses its stability and the phase trajectory is attracted to the periodic attractor consisting of two points (see Figure 3.8).

Further increase of x_{in} leads to occurrence of a chaotic attractor "annulus". There

occur folds on the "annulus", whose number and size increase with the increasing of x_{in}, and at some moment a strange attractor comes up (see Figure 3.11).

The bifurcation diagram of the above scenario is given in Figure 3.1.

Figure 3.1: Strange attractor. Parameters are: $k_{xy} = 0.1$, $k_{yx} = 0.1$, $k_{yz} = 0.1$, $k_{zy} = 0.1$, $k_{\text{out}} = 0.2$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 1.546$.

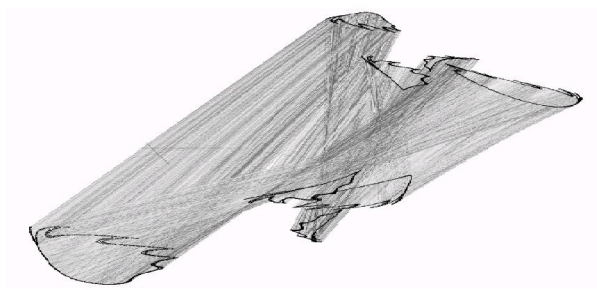


Figure 3.2: Strange attractor with alternation. Parameters are $k_{xy} = 0.1$, $k_{yx} = 0.1$, $k_{yz} = 0.1$, $k_{zy} = 0.1$, $k_{\text{out}} = 0.2$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 1.55$.

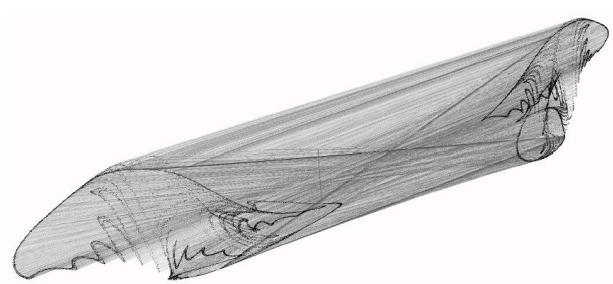


Figure 3.3: Bifurcation diagram for the second variant of dynamic system. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{yz} = 0.2$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$.

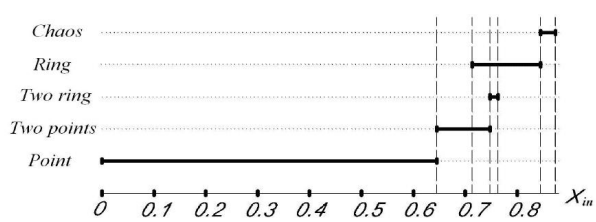


Figure 3.4: Attractor of quasi periodic regime, occurring after stationary state bifurcation. It is seen how phase trajectory moves from stationary state (point in the center), which has lost stability, to the attractor in the shape of ring. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.1$, $k_{yz} = 0.1$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 0.05$, $q = 0.02$, $r = 0.01$, $x_{\text{in}} = 30$.

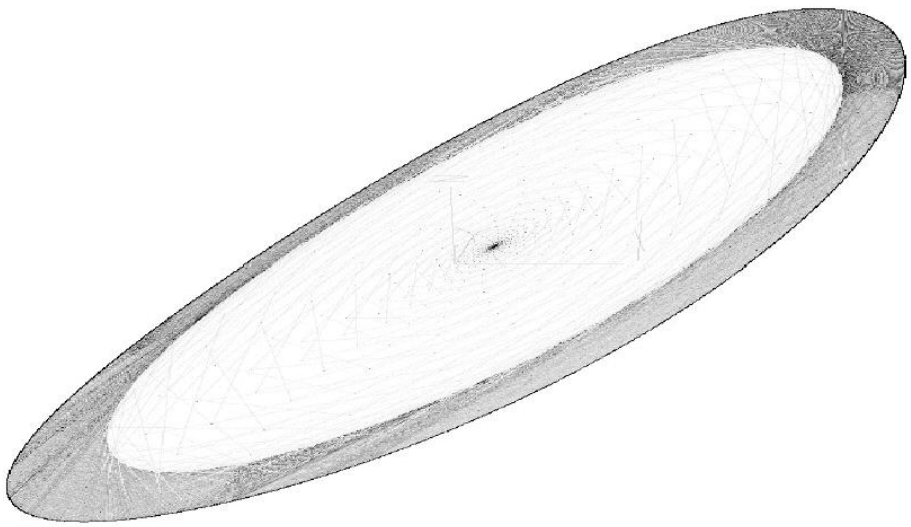


Figure 3.5: Attractor of quasi periodic regime with two aliquant frequencies. It occurs after periodic attractor of two points looses its stability. It is seen how phase trajectory moves on spirals from earlier stable attractor consisting of two points to a new one. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{yz} = 0.2$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 0.748$.

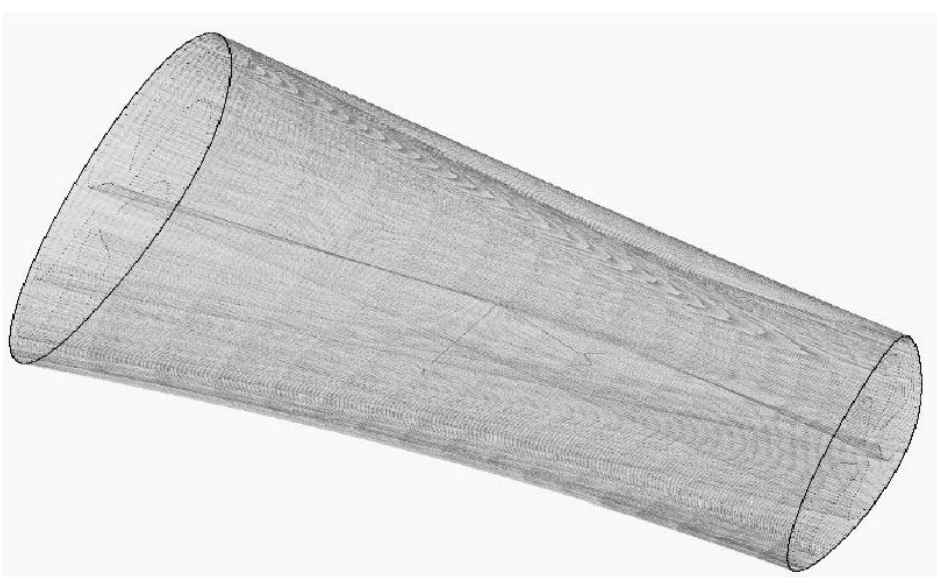


Figure 3.6: Attractor of quasiperiodic regime. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{yz} = 0.2$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 0.763$.

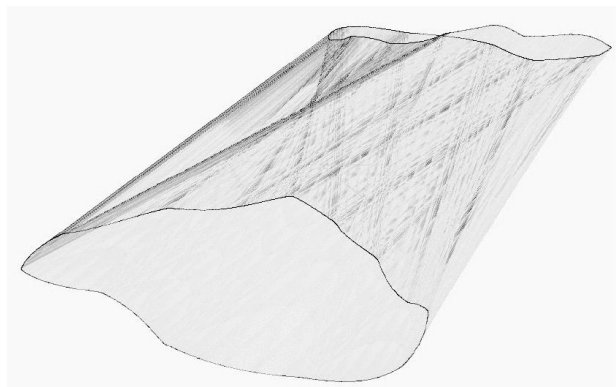
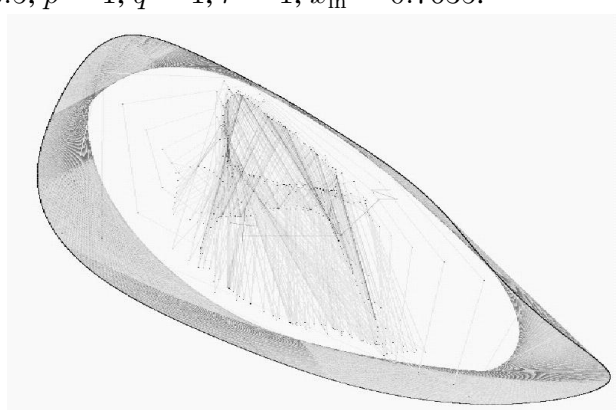


Figure 3.7: Attractor in the shape of a ring occurring after the attractor consisting of two rings lost its stability. It is seen how phase trajectory leaves the area of already unstable two rings and is attracted by one ring. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{yz} = 0.2$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 0.7635$.



The third modification of the system assumes a possibility of additional exit of dust particles from the first channel of a CFF. This possibility was modeled by adding to the system one more transition coefficient $k_{\text{out\_1}}$. This coefficient represents a fraction of dust particles in the first channel that leaves the filter through that opening. The modified system has the form

$$\begin{cases} x_{n+1} = x_n - (k_{xy} + k_{\text{out\_1}})px_n^2 + k_{yx}qy_n^2 + x_{\text{in}} \\ y_{n+1} = y_n + k_{xy}px_n^2 - (k_{yx} + k_{yz})qy_n^2 + k_{zy}rz_n^2 \\ z_{n+1} = z_n + k_{yz}qy_n^2 - (k_{zy} + k_{\text{out}})rz_n^2. \end{cases} \tag{3.1}$$

The stationary solution of the system (3.1) has form

$$\begin{cases} x_{\mathrm{st}} = \sqrt{\dfrac{x_{\mathrm{in}}}{\left[k_{xy} + k_{\mathrm{out\_1}}\left(\dfrac{k_{yx}}{k_{yz}}\left(\dfrac{k_{zy}}{k_{\mathrm{out}}}+1\right)+1\right)\right]p}\left(\dfrac{k_{yx}}{k_{yz}}\left(\dfrac{k_{zy}}{k_{\mathrm{out}}}+1\right)+1\right)} \\ y_{\mathrm{st}} = \sqrt{\dfrac{x_{\mathrm{in}} - k_{\mathrm{out\_1}}px_{\mathrm{st}}^2}{k_{yz}q}\left(\dfrac{k_{zy}}{k_{\mathrm{out}}}+1\right)} \\ z_{\mathrm{st}} = \sqrt{\dfrac{x_{\mathrm{in}} - k_{\mathrm{out\_1}}px_{\mathrm{st}}^2}{k_{\mathrm{out}}r}}. \end{cases}$$

As in the previous cases, we observed two scenario of transition to chaos when x_{in} increases. One way is through cascade of doubling periods bifurcation, another way is through quasi periodic regimes. We observed multiple attractors and alternation of regimes. For illustration we depict attractors that occurred in the system, and scheme of regimes possible in this version of the model.

Figure 3.8: Attractor in periodic regime consisting of two points. It is shown how phase trajectory leaves unstable ring shaped attractor and is attracted to the attractor of two points (shown with arrows). Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{yz} = 0.2$, $k_{zy} = 0.4$, $k_{\mathrm{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$, $x_{\mathrm{in}} = 0.713$.

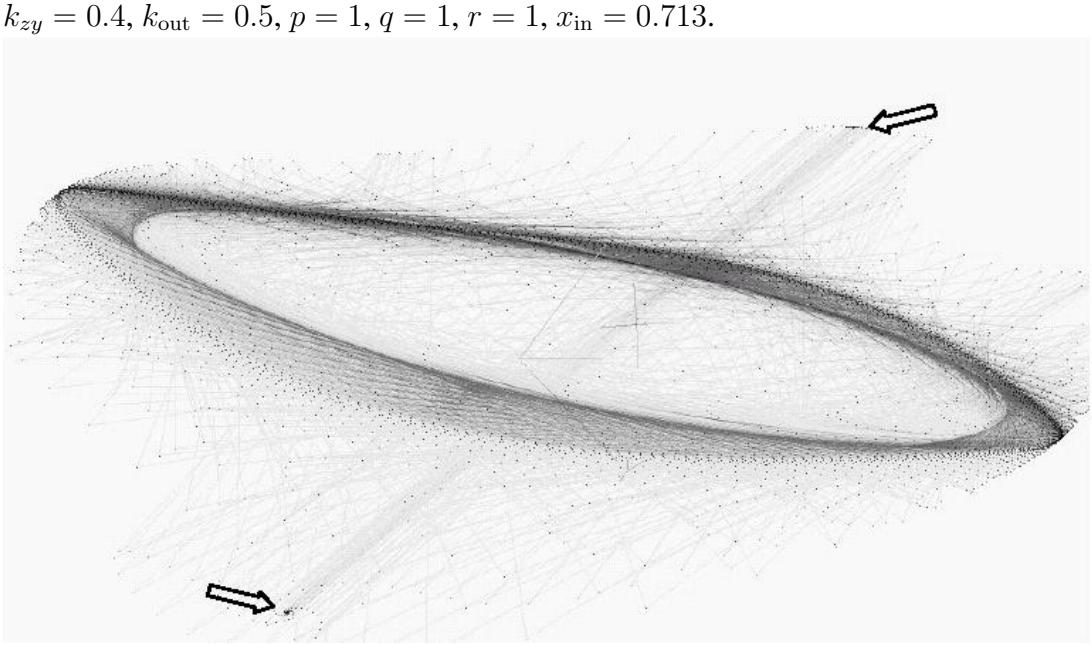


Figure 3.9: Formation of folds in the ring shaped attractor. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{yz} = 0.2$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 0.84$.

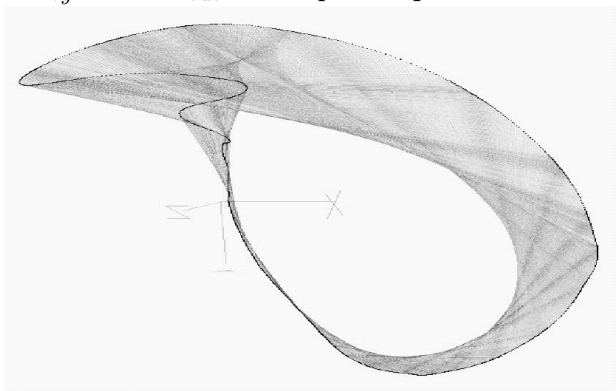


Figure 3.10:

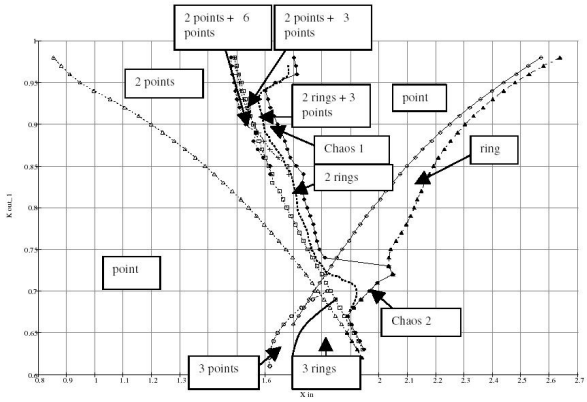


Figure 3.11: Strange attractor, generated after ring shaped attractor. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.2$, $k_{yz} = 0.2$, $k_{zy} = 0.4$, $k_{\text{out}} = 0.5$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 0.87$.

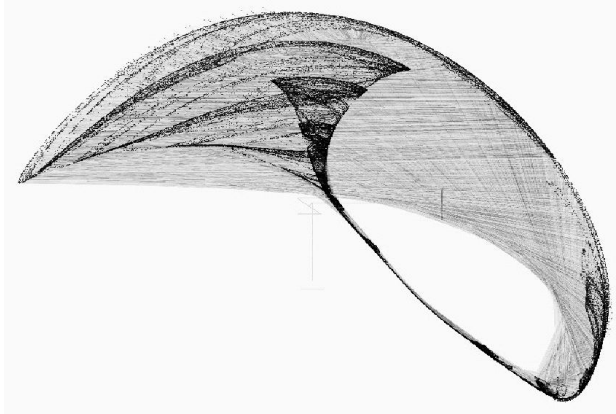


Figure 3.12: Strange attractor in system (3.1). Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out\_1}} = 0.72$, $k_{\text{out}} = 0.4$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 1.99$.

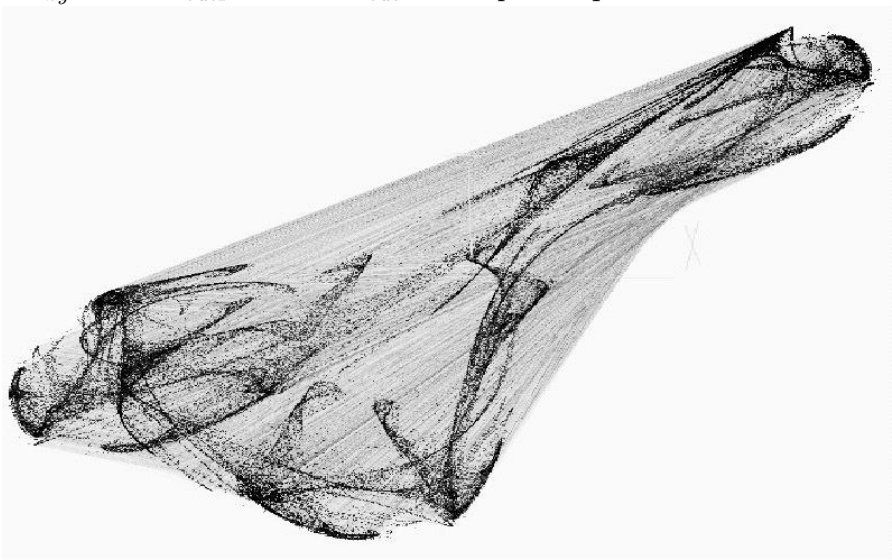


Figure 3.13: Strange attractor in system (3.1). Parameters: $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out\_1}} = 0.95$, $k_{\text{out}} = 0.4$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 1.62$.

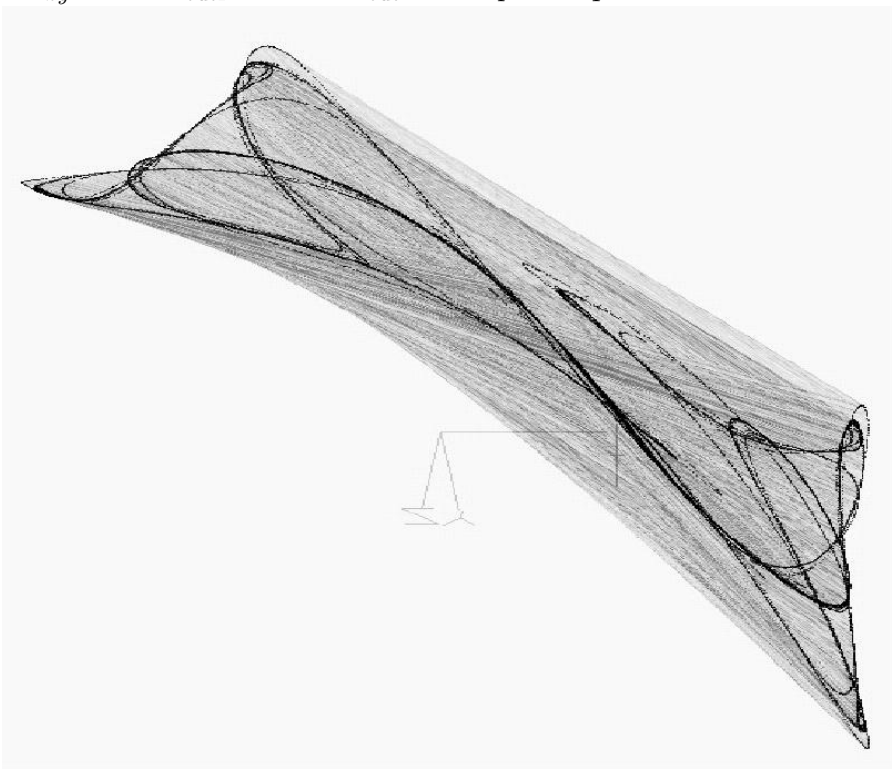


Figure 3.14: Strange attractor of alternation regime. In chaotic regime there are intervals of periodic fluctuations. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out\_1}} = 0.7$, $k_{\text{out}} = 0.4$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 1.8$.

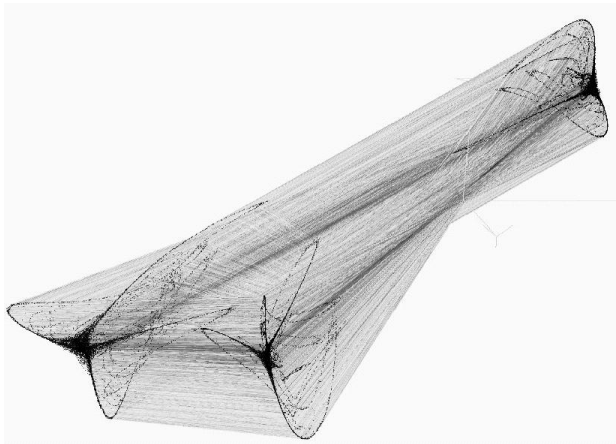


Figure 3.15: Attractor of quasiperiodic regime consisting of three rings. Parameters are: $k_{xy} = 0.5$, $k_{yx} = 0.4$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out\_1}} = 0.65$, $k_{\text{out}} = 0.4$, $p = 1$, $q = 1$, $r = 1$, $k_{yx} = 0.4$, $x_{\text{in}} = 1.8$.

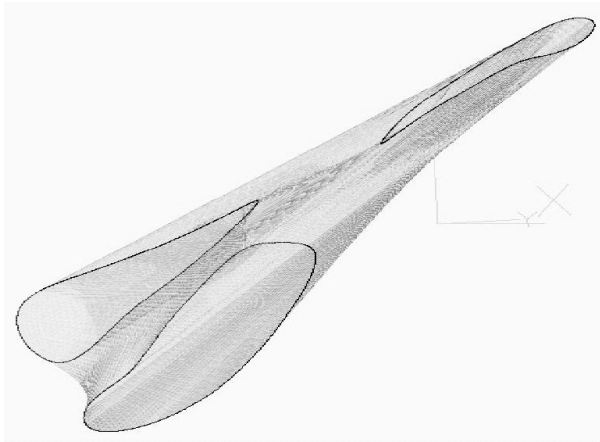
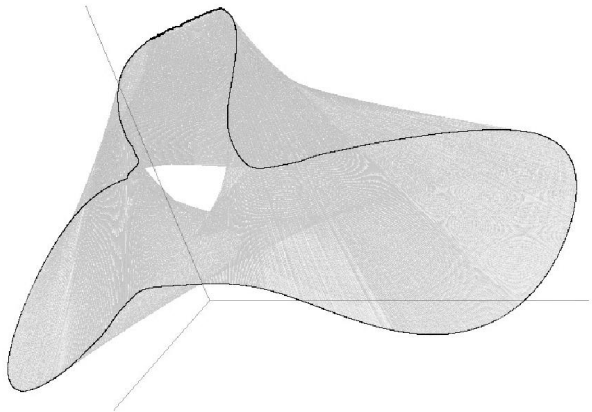


Figure 3.16: Attractor generated after the merge of three rings. Parameters are: $k_{xy} = 0.5$, $k_{yz} = 0.3$, $k_{zy} = 0.3$, $k_{\text{out\_1}} = 0.65$, $k_{\text{out}} = 0.4$, $p = 1$, $q = 1$, $r = 1$, $x_{\text{in}} = 1.8365$.



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[2] A. I. Burov. *Hydraulics of the stratified curvilinear flows in feedback apparatus*. PhD thesis, Leningrad, 1991.

[3] A. N. Herega. Model for coagulation in a circulating flow. In *Proceedings of the Fourth Siberian Congress on Industrial and Applied Mathematics, INPRIM-2000*, Novosibirsk, 2000.