Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 4, Number 2, pp. 159–177 (2009)
http://campus.mst.edu/adsa

Summability of Solutions of the Heat Equation with Inhomogeneous Thermal Conductivity in Two Variables

Werner Balser
Universität Ulm, Abteilung Angewandte Analysis
D-89069 Ulm, Germany
werner.balser@uni-ulm.de

Michèle Loday-Richaud
Université d'Angers, LAREMA
2 boulevard Lavoisier 49 045 Angers cedex 01, France
michele.loday@univ-angers.fr

Abstract

We investigate Gevrey order and 1-summability properties of the formal solution of a general heat equation in two variables. In particular, we give necessary and sufficient conditions for the 1-summability of the solution in a given direction. When restricted to the case of constants coefficients, these conditions coincide with those given by D. A. Lutz, M. Miyake, R. Schäfke in a 1999 article [8], and we thus provide a new proof of their result.

AMS Subject Classifications: 35C10, 35C20, 35K05, 40B05.
Keywords: Heat equation, Gevrey series, 1-summability.

1 The Problem

A formal solution of the classical heat initial conditions problem

$$\begin{cases} \partial_t u - \partial_z^2 u = 0 \\ u(0,z) = \varphi(z) \end{cases} \tag{1.1}$$

in one dimensional spatial variable z reads in the form

$$\begin{aligned} \widehat{u}(t,z) &= \exp\left(t\,\partial_z^2 t\right)\varphi(z) \\ &= \sum_{j\geq 0} \frac{t^j}{j!}\varphi^{(2j)}(z) \end{aligned}$$

Received February 28, 2009; Accepted June 26, 2009
Communicated by Martin Bohner

provided that all derivatives $\varphi^{(2j)}$ exist[1]. When $\varphi \in \mathcal{O}(D_p)$ is holomorphic in a disc D_ρ with center 0 and radius ρ and hence satisfies, for any $r < \rho$, estimates of the form

$$\left|\varphi^{(2j)}(z)\right| \leq C\, K^{2j}\, \Gamma(1+2j),$$

for all $j \geq 0$ and positive constants C and K, on D_r, then $\widehat{u}(t,z) \in \mathcal{O}(D_\rho)[[t]]$ is a series of Gevrey type of order 1 in t for all $z \in D_\rho$ (in short, a 1-Gevrey series). The Gevrey estimates are locally uniform with respect to z in D_ρ. These conditions are optimal as shown by the following example: Let us consider $\varphi(z) = \dfrac{1}{1-z} = \sum_{n\geq 0} z^n$ so that $\varphi^{(2j)}(0) = \Gamma(1+2j)$. The corresponding solution $\widehat{u}(t,z)$ is of exact Gevrey order 1 and, in particular, is divergent. It turns out that it is actually 1-summable in all direction but $\mathbb{R}^+$ in the sense of Definition 3.1 below, that is, 1-summable in t uniformly with respect to z near 0.

In 1999, D. Lutz, M. Miyake and R. Schäfke [8] gave necessary and sufficient conditions on φ for $\widehat{u}$ to be 1-summable in a given direction $\arg t = \theta$. Various works have been done towards the summability of divergent solutions of partial differential equations with constant coefficients (cf. [1, 3, 4, 14] etc.) or variable coefficients (cf. [6, 10, 11, 16, 18] etc.) in two variables. In [9], S. Malek has investigated the case of linear partial differential equations with constant coefficients in more variables; in [17], S. Ōuchi considers some partial differential equations associated with a vector field in finitely many variables.

In this article we are interested in the very general heat initial conditions problem with inhomogeneous thermal conductivity and internal heat generation

$$\begin{cases} \partial_t\, u - a(z)\, \partial_z^2\, u = q(t,z) \qquad a(z) \in \mathcal{O}(D_\rho) \\ u(0,z) = \varphi(z) \in \mathcal{O}(D_\rho). \end{cases} \tag{1.2}$$

The heat equation describes heat propagation under thermodynamics and Fourier laws. The coefficient $a(z)$, named thermal diffusivity, is related to the thermal conductivity κ by the formula $a = \dfrac{\kappa}{c\rho}$ where c is the capacity and ρ the density of the medium. We assume that $a(z)$ and $\varphi(z)$ are analytic on a neighborhood of $z = 0$. The internal heat input q may be smooth or not. An important case is the case with no internal heat generation corresponding to a homogeneous heat equation:

$$\begin{cases} \partial_t\, u - a(z)\, \partial_z^2\, u = 0 \qquad a(z) \in \mathcal{O}(D_\rho) \\ u(0,z) = \varphi(z) \in \mathcal{O}(D_\rho). \end{cases} \tag{1.3}$$

In case of an isotropic and homogeneous medium, κ, c, ρ and hence a are constants. An adequate choice of units allows then to assume $a = 1$ and the equation reduces to the reference heat equation $\partial_t u - \partial_z^2 u = 0$.

[1] We denote $\widehat{u}$, with a hat, to emphasize the possible divergence of the series $\widehat{u}$.

Actually, for notational convenience, we consider the problem in the form

$$\boxed{\left(1 - a(z)\,\partial_t^{-1}\partial_z^2\right)\widehat{u} = \widehat{f}(t,z)}\,,\ a(z) \in \mathcal{O}(D_\rho) \text{ and } \widehat{f}(t,z) \in \mathcal{O}(D_\rho)[[t]], \quad (1.4)$$

where $\partial_t^{-1}\,\widehat{u}$ stands for the anti-derivative $\displaystyle\int_0^t \widehat{u}(s,z)ds$ of $\widehat{u}$ with respect to t which vanishes at $t = 0$.

Problem (1.4) is equivalent to Problem (1.2) with $q(t,z) = \partial_t \widehat{f}(t,z)$ and $\varphi(z) = \widehat{f}(0,z)$. Moreover, it reduces to the homogeneous case (1.3) if and only if the inhomogeneity $\widehat{f}$ does not depend on t.

From now, we denote $D = 1 - a(z)\,\partial_t^{-1}\partial_z^2$ and, given a series $\widehat{u} \in \mathcal{O}(D_\rho)[[t]]$, we denote

$$\widehat{u}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} u_{j,*}(z) = \sum_{n\geq 0} \widehat{u}_{*,n}(t)\frac{z^n}{n!} = \sum_{j,n\geq 0} u_{j,n}\frac{t^j}{j!}\frac{z^n}{n!}.$$

Since $\big(\mathcal{O}(D_\rho)[[t]], \partial_t, \partial_z\big)$ is a differential algebra and $a(z) \in \mathcal{O}(D_\rho)$ the operator D acts inside $\mathcal{O}(D_\rho)[[t]]$. More precisely, we can state the following.

Proposition 1.1. *The map*

$$D:\ \mathcal{O}(D_\rho)[[t]] \longrightarrow \mathcal{O}(D_\rho)[[t]]$$

is a linear isomorphism.

Proof. The operator D is linear. A series $\widehat{u}(t,z) = \displaystyle\sum_{j\geq 0} \frac{t^j}{j!} u_{j,*}(z)$ is a solution of Problem (1.4) if and only if

$$u_{j,*}(z) = f_{j,*}(z) + a(z)\,u''_{j-1,*}(z) \quad \text{for all } j \geq 0 \text{ starting from } u_{-1,*}(z) \equiv 0. \quad (1.5)$$

Consequently, to any $\widehat{f}(t,z) \in \mathcal{O}(D_\rho[[t]]$ there is a unique solution $\widehat{u}(t,z) \in \mathcal{O}(D_\rho[[t]]$, which proves that D is bijective. □

In Section 2 we show that the inhomogeneity $\widehat{f}(t,z)$ and the unique solution $\widehat{u}(t,z)$ are together 1-Gevrey.

In Section 3 we prove necessary and sufficient conditions for $\widehat{u}$ to be 1-summable in a given direction $\arg t = \theta$. The conditions are valid in the case when either $a(0) \neq 0$ or $a'(0) \neq 0$. When $a(z) = O(z^2)$ an easy counter-example shows that even the rationality of $\widehat{f}(t,z)$ is insufficient.

In Section 4 we discuss the accessibility of our necessary and sufficient conditions. Indeed, the conditions are given not only in terms of the data $\widehat{f}$ but also in terms of the first two terms $\widehat{u}_{*,0}$ and $\widehat{u}_{*,1}$ of the solution $\widehat{u}$ itself.

In the particular case $a = 1$ our conditions coincide with those of [8]. We thus provide a new proof of the result of [8].

2 Gevrey Properties

In this article, we consider t as the variable and z as a parameter. The classical notion of a series of Gevrey type of order 1 is extended to z-families as follows.

Definition 2.1 (1-Gevrey series)**.** A series $\widehat{u}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} u_{j,*}(z) \in \mathcal{O}(D_\rho)[[t]]$ is said of *Gevrey type of order 1* if there exist $0 < r \leq \rho,\ C > 0,\ K > 0$ such that for all $j \geq 0$ and $|z| \leq r$ we have

$$|u_{j,*}(z)| \leq C\, K^j\, \Gamma(1+2j).$$

In other words, $\widehat{u}(t,z)$ is 1-Gevrey in t, uniformly in z on a neighbourhood of $z=0$.

We denote by $\mathcal{O}(D_\rho)[[t]]_1$ the subset of $\mathcal{O}(D_\rho)[[t]]$ made of the series which are of Gevrey type of order 1.

Proposition 2.2. $\big(\mathcal{O}(D_\rho)[[t]]_1, \partial_t, \partial_z\big)$ *is a differential algebra stable under* ∂_t^{-1} *and* ∂_z^{-1}.

Proof. Since the proposition is true for $\mathcal{O}(D_\rho)[[t]]$ instead of $\mathcal{O}(D_\rho)[[t]]_1$ it is sufficient to prove that $\mathcal{O}(D_\rho)[[t]]_1$ is stable under multiplication, derivation and anti-derivation.

- *Multiplication.* Given $\widehat{u}(t,z)$ and $\widehat{v}(t,z)$ in $\mathcal{O}(D_\rho)[[t]]_1$, we can assume that $\widehat{u}(t,z)$ and $\widehat{v}(t,z)$ satisfy the conditions of Definition 2.1 with the same constants r, C, K. Their product $\widehat{w}(t,z) = \sum_{k\geq 0} \frac{t^k}{k!} w_{k,*}(z)$ is given by

$$w_{k,*}(z) = \sum_{j+\ell=k} \frac{k!}{j!\,\ell!} u_{j,*}(z) v_{\ell,*}(z).$$

Then,

$$\begin{aligned}
\sup_{|z|<r} |w_{k,*}(z)| &\leq C^2 K^k \sum_{j+\ell=k} \frac{k!}{j!(k-j)!}\,(2j)!\,(2k-2j)! \\
&= C^2 K^k (2k)! \sum_{j+\ell=k} C_k^j \Big/ C_{2k}^{2j} \quad (C_k^j \text{ is a binomial coefficient}) \\
&\leq C^2 K^k (2k)!(k+1) \ \ (\text{since } C_{2k}^{2j} \geq C_{k+j}^{2j} \geq C_k^j \text{ for all } j \leq k).
\end{aligned}$$

Hence, for any $K' > K$ there exist $C' > 0$ such that

$$\sup_{|z|<r} |w_{k,*}(z)| \leq C'\, K'^k\, \Gamma(1+2k).$$

- *Derivation* ∂_t. From $\partial_t \widehat{u}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} u_{j+1,*}(z)$ we deduce

$$\sup_{|z|<r} \left|u_{j+1,*}(z)\right| \leq C\, K\, K^j\, \Gamma(1+2(j+1)) \leq C'\, K'^j\, \Gamma(1+2j)$$

  for any $K' > K$ and convenient corresponding $C' > 0$.

- *Derivation* ∂_z. Given $r' < r$, we can apply the Cauchy Integral Formula to $\partial_z \widehat{u}_{j,*}(z)$ on the circle $|t-z| = r - r'$ for all j and all $z \in D_{r'}$:

$$\partial_z u_{j,*}(z) = \frac{1}{2\pi i} \int_{|z'-z|=r-r'} \frac{u_{j,*}(z')}{(z'-z)^2} dz'.$$

  Hence, the inequalities

$$\sup_{|z|<r'} |\partial_z u_{j,*}(z)| \leq C'\, K^j\, \Gamma(1+2j) \quad \text{with} \quad C' = \frac{C}{r-r'} \quad \text{for all } j.$$

- *Anti-derivations*. Stability under anti-derivations ∂_t^{-1} and ∂_z^{-1} is straightforward.

The proof is complete. □

Note that stability under ∂_z would not be guaranteed without the condition "there exist $r \leq \rho$ ..." in Definition 2.1.

It results from Proposition 2.2 that the operator $D = 1 - a(z)\partial_t^{-1}\partial_z^2$ acts inside the space $\mathcal{O}(D_\rho)[[t]]_1$.

Because the main result of this section (Theorem 2.6) is set up using Nagumo norms on $\mathcal{O}(D_\rho)$, we begin with a recall of their definition and main properties and we refer to [15] or to [5] for more details.

Definition 2.3 (Nagumo norms)**.** Let $f \in \mathcal{O}(D_\rho)$, $p \geq 0$, $0 < r \leq \rho$ and let $d_r(z) = r - |z|$ denote the Euclidian distance of $z \in D_r$ to the boundary of the disc D_r. The Nagumo norm $\|f\|_{p,r}$ of f is defined by

$$\boxed{\|f\|_{p,r} = \sup_{|z|<r} \left|f(z) d_r(z)^p\right|}.$$

Proposition 2.4 (Properties of Nagumo norms)**.** *One has the following properties:*

1. $\|\cdot\|_{p,r}$ *is a norm on* $\mathcal{O}(D_\rho)$*;*
2. *For all* $z \in D_r$, $|f(z)| \leq \|f\|_{p,r} d(z)^{-p}$*;*
3. $\|f\|_{0,r} = sup_{z\in D_r}|f(z)|$ *is the usual sup-norm on* D_r*;*

4. $\|fg\|_{p+q,r} \leq \|f\|_{p,r}\|g\|_{q,r}$;
5. $\|f'\|_{p+1,r} \leq e(p+1)\|f\|_{p,r}$.

Remark 2.5. Inequality 5 is the most important property. Note that the same index r occurs on both sides of the inequality 5. One gets thus an estimate for the derivative f' in terms of f without having to shrink the domain D_r. In order to explain where the constant e comes from we sketch the proof of Property 5.

Proof of Lemma 2.4, Property 5. Let $z \in D_r$. Using the Cauchy integral formula we can write

$$|\, f'(z)\,| = \frac{1}{2\pi}\left|\int_{|z'-z|=R} \frac{f(z')}{(z'-z)^2}dz'\right|, \quad 0 < R < d_r(z)$$

and then,

$$\begin{aligned} |\, f'(z)\,| &\leq \frac{1}{R}\max_{|z'-z|=R}|\, f(z')\,| \\ &\leq \|\, f\,\|_{p,r}\,\frac{1}{R}\max_{|z'-z|=R} d_r(z')^{-p} \\ &= \|\, f\,\|_{p,r}\,\frac{1}{R}\,\big(d_r(z)-R\big)^{-p}. \end{aligned}$$

For $p > 0$, choosing $R = \dfrac{d_r(z)}{p+1}$ and using the inequality $\Big(1-\dfrac{1}{p+1}\Big)^{-p} < e$, we obtain

$$\begin{aligned} |\, f'(z)\,| &\leq \|\, f\,\|_{p,r}\, d_r(t)^{-p-1}(p+1)\Big(1-\frac{1}{p+1}\Big)^{-p} \\ &\leq (p+1)\, e\,\|\, f\,\|_{p,r}\, d_r(z)^{-p-1}. \end{aligned}$$

Hence, the result:

$$\|f'\|_{p+1,r} = \sup_{|z|<r}\big|\, f'(z)d_r(z)^{p+1}\big| \leq (p+1)\, e\,\|\, f\,\|_{p,r}.$$

For $p = 0$, choosing $R = \dfrac{d_r(z)}{c}$ for any constant $c > 1$, we obtain $\|f'\|_{1,r} \leq c\,\|f\|_{0,r}$ and, in particular, $\|f'\|_{1,r} \leq e\,\|f\|_{0,r}$. □

Theorem 2.6. *Recall $D = 1 - a(z)\,\partial_t^{-1}\partial_z^2$, where $a(z) \in \mathcal{O}(D_\rho)$. The map*

$$D : \begin{cases} \mathcal{O}(D_\rho)[[t]]_1 & \longrightarrow \quad \mathcal{O}(D_\rho)[[t]]_1 \\ \widehat{u}(t,z) & \mapsto \quad \widehat{f}(t,z) = D\widehat{u}(t,z) \end{cases}$$

is a linear isomorphism.

Proof. It results from Proposition 2.2 that $D\big(\mathcal{O}(D_\rho)[[t]]_1\big) \subset \mathcal{O}(D_\rho)[[t]]_1$ and from Proposition 1.1 that D is linear and injective. We are left to prove that D is also surjective.

Let $\widehat{f}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} f_{j,*}(z) \in \mathcal{O}(D_\rho)[[t]]_1$. The coefficients $f_{j,*}(z)$ satisfy

$$\begin{cases} \bullet f_{j,*}(z) \in \mathcal{O}(D_\rho) \text{ for all } j \geq 0. \\ \bullet \text{ There exist } 0 < r \leq \rho,\ C > 0,\ K > 0 \text{ such that for all } j \geq 0 \text{ and } |z| \leq r \\ \quad |f_{j,*}(z)| \leq CK^j \Gamma(1+2j) \end{cases}$$

and we look forward to similar conditions on the coefficients $u_{j,*}(z)$ of $\widehat{u}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} u_{j,*}(z)$.

From the recurrence relation (1.5), the relation

$$\frac{u_{j,*}(z)}{\Gamma(1+2j)} = \frac{f_{j,*}(z)}{\Gamma(1+2j)} + a(z)\frac{u''_{j-1,*}(z)}{\Gamma(1+2j)}$$

starting from $u_{-1,*}(z) \equiv 0$ holds for all $j \geq 0$. Applying the Nagumo norms of indices $(2j, r)$ and Properties 4 and 5 of Proposition 2.4 we get

$$\begin{aligned} \frac{\|u_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)} &\leq \frac{\|f_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)} + \|a(z)\|_{0,r} \frac{\|u''_{j-1,*}(z)\|_{2j,r}}{\Gamma(1+2j)} \\ &\leq \quad '' \quad + \|a(z)\|_{0,r}\, e^2\, \frac{\|u_{j-1,*}(z)\|_{2j-2,r}}{\Gamma\big(1+(2j-2)\big)}. \end{aligned}$$

Denote $g_j = \frac{\|f_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)}$ and $\alpha = \|a(z)\|_{0,r}\, e^2$ and consider the numerical sequence

$$\begin{cases} v_{-1} = 0 \\ v_j = g_j + \alpha\, v_{j-1} \ \text{ for all } j \geq 0. \end{cases}$$

By construction, $\frac{\|u_{j,*}(z)\|_{2j,r}}{\Gamma(1+2j)} \leq v_j$ for all $j \geq 0$. Let us bound v_j as follows. By assumption, $0 \leq g_j \leq \frac{CK^j\Gamma(1+2j)}{\Gamma(1+2j)} r^{2j} = C(Kr^2)^j$ for all j and the series $g(X) = \sum_{j\geq 0} g_j X^j$ is convergent. Due to the recurrence relation defining the v_j's the series

$v(X) = \sum_{j\geq 0} v_j X^j$ satisfy $(1-\alpha X)v(X) = g(X)$. It is then convergent and there exist constants $C' > 0, K' > 0$ such that $v_j \leq C' K'^j$ for all j. Hence,

$$\|u_{j,*}(z)\|_{2j,r} \leq C' K'^j \Gamma(1+2j) \quad \text{for all } j \geq 0.$$

We deduce a similar estimate on the sup-norm by shrinking the domain D_r. Indeed, let $0 < r' < r$. For all $j \geq 0$ and $z \in D_{r'}$,

$$\begin{aligned} |u_{j,*}(z)| &= \left| u_{j,*}(z) d_r(z)^{2j} \frac{1}{d_r(z)^{2j}} \right| \\ &\leq \frac{1}{(r-r')^{2j}} \left| u_{j,*}(z) d_r(z)^{2j} \right|. \end{aligned}$$

Hence,

$$\begin{aligned} \sup_{z\in D_{r'}} |u_{j,*}(z)| &\leq \frac{1}{(r-r')^{2j}} \|u_{j,*}\|_{2j,r} \\ &\leq C' \left(\frac{K'}{(r-r')^2} \right)^j \Gamma(1+2j). \end{aligned}$$

The proof is complete. □

3 1-Summability

Still considering t as the variable and z as a parameter, one extends the classical notions of summability to families parameterized by z in requiring similar conditions, the estimates being however uniform with respect to the parameter z. For a general study of series with coefficients in a Banach space we refer to [2]. Among the many equivalent definitions of 1-summability in a given direction $\arg t = \theta$ at $t = 0$ we choose here a generalization of Ramis' definition which states that a series $\widehat{f}$ is 1-summable in the direction θ if there exists a holomorphic function f which is 1-Gevrey asymptotic to $\widehat{f}$ on an open sector $\Sigma_{\theta,>\pi}$ bisected by θ with opening larger than π (cf. [19, Déf 3.1]). There are various equivalent ways of expressing the 1-Gevrey asymptotics. We choose to extend the one which sets conditions on the successive derivatives of f (see [13, p. 171] or [19, Thm 2.4], for instance).

Definition 3.1 (1-summability)**.** A series $\widehat{u}(t,z) \in \mathcal{O}(D_\rho)[[t]]$ is said to be *1-summable in the direction* $\arg t = \theta$ if there exist a sector $\Sigma_{\theta,>\pi}$, a radius $0 < r \leq \rho$ and a function $u(t,z)$ called *1-sum of* $\widehat{u}(t,z)$ *in the direction* θ such that

1. u is defined and holomorphic on $\Sigma_{\theta,>\pi} \times D_r$;

2. For any $z \in D_r$ the map $t \mapsto u(t,z)$ has $\widehat{u}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!}\, u_{j,*}(z)$ as Taylor series at 0 on $\Sigma_{\theta,>\pi}$;

3. For any proper[2] subsector $\Sigma \subset\subset \Sigma_{\theta,>\pi}$ there exist constants $C > 0, K > 0$ such that for all $\ell \geq 0$, all $t \in \Sigma$ and $z \in D_r$

$$\boxed{\left|\partial_t^\ell\, u(t,z)\right| \leq C K^\ell \Gamma(1+2\ell)}\,.$$

We denote by $\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ the subset of $\mathcal{O}(D_\rho)[[t]]$ made of all 1-summable series in the direction $\arg t = \theta$. Actually, $\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ is included in $\mathcal{O}(D_\rho)[[t]]_1$.

For any fixed $z \in D_r$, the 1-summability of the series $\widehat{u}(t,z)$ is the classical 1-summability, and Watson's lemma implies the unicity of its 1-sum, if any exists.

Proposition 3.2. *$\big(\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}, \partial_t, \partial_z\big)$ is a differential $\mathbb{C}$-algebra stable under ∂_t^{-1} and ∂_z^{-1}.*

Proof. Let $\widehat{u}(t,z)$ and $\widehat{v}(t,z)$ be two 1-summable series in direction θ. In Definition 3.1 we can choose the same constants r, C, K both for $\widehat{u}$ and $\widehat{v}$. The product $w(t,z) = u(t,z)v(t,z)$ satisfies Conditions 1 and 2 of Definition 3.1. Moreover,

$$\begin{aligned}
\left|\partial_t^\ell w(t,z)\right| &= \left|\sum_{p=0}^{\ell} C_\ell^p\, \partial_t^p u(t,z)\, \partial_t^{\ell-p} v(t,z)\right| \\
&\leq C^2\, K^\ell\, \Gamma(1+2\ell) \left|\sum_{p=0}^{\ell} \frac{\Gamma(1+\ell)}{\Gamma(1+2\ell)} \frac{\Gamma(1+2p)}{\Gamma(1+p)} \frac{\Gamma\big(1+2(\ell-p)\big)}{\Gamma\big(1+(\ell-p)\big)}\right| \\
&\leq C^2\, K^\ell\, (\ell+1)\, \Gamma(1+2\ell) \\
&\leq C'\, K'^\ell\, \Gamma(1+2\ell) \quad \text{for adequate } C', K' > 0.
\end{aligned}$$

This proves Condition 3 of Definition 3.1 for $w(t,z)$, that is, stability of $\mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ under multiplication.

Stability under ∂_t, ∂_t^{-1} or ∂_z^{-1} is straightforward. Stability under ∂_z is obtained using the Integral Cauchy Formula on a disc $D_{r'}$ with $r' < r$. □

Note that the 1-sum $u(t,z)$ of a 1-summable series $\widehat{u}(t,z) \in \mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta}$ may be analytic with respect to z on a disc D_r smaller than the common disc D_ρ of analyticity

[2] In this context a subsector Σ of a sector Σ' is said a proper subsector and one denotes $\Sigma \subset\subset \Sigma'$ if its closure in $\mathbb{C}$ is contained in $\Sigma' \cup \{0\}$.

of the coefficients $\widehat{u}_{j,*}(z)$ of $\widehat{u}(t,z) = \sum_{j\geq 0} \frac{t^j}{j!} u_{j,*}(z)$. With respect to t, the 1-sum $u(t,z)$ is analytic on a sector supposedly open and containing a closed sector $\overline{\Sigma}_{\theta,\pi}$ bisected by θ with opening π; there is no control on the angular opening except that it must be larger than π and no control on the radius of this sector except that it must be positive. Thus, the 1-sum $u(t,z)$ is well defined as a section of the sheaf of analytic functions in (t,z) on a germ of closed sector of opening π (i.e., a closed interval $\overline{I}_{\theta,\pi}$ of length π on the circle S^1 of directions issuing from 0, cf. [12, 1.1] or [7, I.2]) times $\{0\} \subset \mathbb{C}_z$. We denote $\mathcal{O}_{\overline{I}_{\theta,\pi}\times\{0\}}$ the space of such sections.

Corollary 3.3. *The operator of 1-summation*

$$\mathcal{S} : \begin{cases} \mathcal{O}(D_\rho)\{\{t\}\}_{1,\theta} & \longrightarrow & \mathcal{O}_{\overline{I}_{\theta,\pi}\times\{0\}} \\ \widehat{u}(t,z) & \mapsto & u(t,z) \end{cases}$$

is a homomorphism of differential $\mathbb{C}$-algebras for the derivations ∂_t and ∂_z and it commutes with ∂_t^{-1} and ∂_z^{-1}.

Theorem 3.4. *Let a direction $\arg t = \theta$ issuing from 0 and a series $\widehat{f}(t,z) \in \mathcal{O}(D_\rho)[[t]]$ be given. Recall $D = 1 - a(z)\partial_t^{-1}\partial_z^2$ with $a(z) \in \mathcal{O}(D_\rho)$ and assume that either $a(0) \neq 0$ or $a(0) = 0$ and $a'(0) \neq 0$. Then, the unique solution $\widehat{u}(t,z)$ of Equation*

$$D\widehat{u} = \widehat{f} \tag{1.4}$$

in $\mathcal{O}(D_\rho)[[t]]$ is 1-summable in the direction θ if and only if $\widehat{u}_{*,0}(t), \widehat{u}_{*,1}(t)$ and $\widehat{f}(t,z)$ are 1-summable in the direction θ. Moreover, the 1-sum $u(t,z)$, if any, satisfies Equation (1.4) *in which $\widehat{f}(t,z)$ is replaced by the 1-sum $f(t,z)$ of $\widehat{f}(t,z)$ in direction θ.*

Proof. • We first place ourselves in the case $a(0) \neq 0$. Denote $a(z) = \sum_{n\geq 0} a_n z^n$. As a preliminary remark we notice that, by identification of equal powers of z in Equation

$$\left(1 - a(z)\,\partial_t^{-1}\partial_z^2\right) \sum_{n\geq 0} \widehat{u}_{*,n}(t)\,\frac{z^n}{n!} = \sum_{n\geq 0} \widehat{f}_{*,n}(t)\,\frac{z^n}{n!}, \tag{1.4}$$

we get

$$\begin{cases} \widehat{u}_{*,0}(t) - a_0\,\partial_t^{-1}\widehat{u}_{*,2}(t) = \widehat{f}_{*,0}(t) \\ \widehat{u}_{*,1}(t) - a_1\,\partial_t^{-1}\widehat{u}_{*,2}(t) - a_0\,\partial_t^{-1}\widehat{u}_{*,3}(t) = \widehat{f}_{*,1}(t) \\ \text{and so on} \ldots \end{cases}$$

so that each $\widehat{u}_{*,n}(t)$ is uniquely and linearly determined from $\widehat{u}_{*,0}(t),\ \widehat{u}_{*,1}(t)$ and $\widehat{f}(t,z)$.

— The condition is necessary by Proposition 3.2. Indeed, if $\widehat{u}$ is 1-summable then so are $\widehat{u}_{*,0}(t)=\widehat{u}(t,0),\ \widehat{u}_{*,1}(t)=\dfrac{1}{z}\big(\widehat{u}(t,z)-\widehat{u}_{*,0}(t)\big)\Big|_{z=0}$ and $\widehat{f}=Du$.

— Prove that the condition is sufficient. Assume that $\widehat{u}_{*,0}(t),\ \widehat{u}_{*,1}(t)$ and $\widehat{f}(t,z)$ are 1-summable in direction θ.

Set $\widehat{u}(t,z)=\widehat{u}_{*,0}(t)+z\,\widehat{u}_{*,1}(t)+\partial_z^{-2}\widehat{v}(t,z)$ and $\widehat{w}=\partial_t^{-1}\,\widehat{v}$.
With these notations Equation (1.4) becomes

$$\Big(1-\frac{1}{a(z)}\partial_t\partial_z^{-2}\Big)\,\widehat{w}(t,z)=\widehat{g}(t,z),\quad \text{where } \widehat{g}=\frac{1}{a(z)}(\widehat{u}_{*,0}+z\widehat{u}_{*,1}-\widehat{f}) \tag{3.1}$$

and it suffices to prove that $\widehat{w}$ is 1-summable in direction θ when $\widehat{g}$ is. To this end, we proceed through a fixed point method as follows.

Setting $\widehat{w}(t,z)=\sum_{p\geq 0}\widehat{w}_p(t,z)$, Equation (3.1) reads

$$\begin{aligned}
&\widehat{w}_0-\frac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_0 &&=\widehat{g}\\
&+\widehat{w}_1-\frac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_1\\
&+\cdots\\
&+\widehat{w}_p-\frac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_p\\
&+\cdots
\end{aligned}$$

and we choose the solution given by the system

$$\left\{\begin{array}{lcl}
\widehat{w}_0 & = & \widehat{g}\\
\widehat{w}_1 & = & \dfrac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_0\\
\cdots & & \\
\widehat{w}_p & = & \dfrac{1}{a(z)}\partial_t\partial_z^{-2}\,\widehat{w}_{p-1}\\
\cdots. & &
\end{array}\right. \tag{3.2}$$

We can check that, for all $p\geq 0$, the formal series $\widehat{w}_p(t,z)$ are of order $O(z^{2p})$ in z and consequently, the series $\widehat{w}(t,z)=\sum_{p\geq 0}\widehat{w}_p(t,z)$ itself makes sense as a formal series in t and z.

Let $w_0(t,z)$ denote the 1-sum of $\widehat{w}_0=\widehat{g}$ in direction θ and, for all $p>0$, let $w_p(t,z)$ be determined as the solution of System (3.2) in which all $\widehat{w}_p$ are replaced by w_p. All w_p are defined on a common domain $\Sigma_{\theta,>\pi}\times D_{\rho'}$.

To end the proof, we show that the series $\sum_{p\geq 0} w_p(t,z)$ is convergent and that its sum $w(t,z)$ is the 1-sum of $\widehat{w}(t,z)$ in direction θ.

The 1-summability of $\widehat{w}_0$ implies that there exists $0 < r' < \rho'$ with the following property. For any proper subsector $\Sigma \subset\subset \Sigma_{\theta,>\pi}$ (cf. Footnote[2]), there exist constants $C' > 0$, $K' > 0$ such that, for all $\ell \geq 0$ and $(t,z) \in \Sigma \times D_{r'}$, the function w_0 satisfy the condition

$$\left|\partial_t^\ell w_0(t,z)\right| \leq C' K'^\ell\, \Gamma(1+2\ell).$$

Denote $B = \max_{z\in D_r} \left|\frac{1}{a(z)}\right|$. From $w_1 = \frac{1}{a(z)}\, \partial_t \partial_z^{-2} w_0$, we deduce that

$$\begin{aligned}
\left|\partial_t^\ell w_1\right| &= \left|\frac{1}{a(z)}\, \partial_t^{\ell+1}\partial_z^{-2}\, w_0\right| \\
&\leq B \max_{z\in D_r} \left|\partial_t^{\ell+1} w_0\right| \frac{|z|^2}{2!} \\
&\leq C'\, K'^{\ell+1}\, \Gamma\big(1+2(\ell+1)\big) \frac{B|z|^2}{2!}
\end{aligned}$$

and, by recursion, that

$$\left|\partial_t^\ell w_p(t,z)\right| \leq C' K'^{\ell+p}\, \Gamma\big(1+2(\ell+p)\big) \frac{(B|z|^2)^p}{(2p)!} \quad \text{for all } p \geq 0. \tag{3.3}$$

This implies

$$\begin{aligned}
\sum_{p\geq 0} \left|\partial_t^\ell w_p(t,z)\right| &\leq C'\, K'^\ell\, \Gamma(1+2\ell) \sum_{p\geq 0} C_{2\ell+2p}^{2p} \big(K'\, B\, |z|^2\big)^p \\
&\leq C'\, (4K')^\ell\, \Gamma(1+2\ell) \sum_{p\geq 0} \big(4K' B |z|^2\big)^p \\
&\qquad \text{since } C_{2\ell+2p}^{2p} \leq \sum_{k=0}^{2\ell+2p} C_{2\ell+2p}^{k} = 2^{2\ell+2p}.
\end{aligned}$$

Denote $L = 4K'Br^2$ and choose r so small that $L < 1$. Denote $C = C' \sum_{p\geq 0} L^p < \infty$ and $K = 4K'$. Then,

$$\sum_{p\geq 0} \left|\partial_t^\ell w_p(t,z)\right| \leq C K^\ell \Gamma(1+2\ell) \quad \text{on } \Sigma \times D_r. \tag{3.4}$$

In particular, for $\ell = 0$, the series $\sum w_p(t,z)$ is normally convergent on $\Sigma \times D_r$. Consequently, its sum $w(t,z)$ exists and is analytic on $\Sigma \times D_r$. This proves Condition 1

of Definition 3.1 if we choose as sector $\Sigma \subset \Sigma_{\theta,>\pi}$ a sector bisected by θ with opening larger than π.

For all $\ell \geq 1$, the series $\sum \partial_t^\ell w_p(t,z)$ is also normally convergent on $\Sigma \times D_r$ so that the series $\sum w_p(t,z)$ can be derivated termwise infinitely many times with respect to t and the estimates (3.4) imply

$$\left|\partial_t^\ell w(t,z)\right| \leq CK^\ell \Gamma(1+2\ell) \quad \text{on } \Sigma \times D_r \tag{3.5}$$

which proves Condition 3 of Definition 3.1.

Moreover, summing Equations (3.2) for w_p and the 1-sum $g(t,z)$ instead of $\widehat{w}_p$ and $\widehat{g}(t,z)$ we get $w(t,z) = g(t,z) + \dfrac{1}{a(z)} \displaystyle\sum_{p\geq 0} \partial_t \partial_z^{-2} w_p(t,z) = g(t,z) + \dfrac{1}{a(z)} \partial_t \partial_z^{-2} w(t,z)$. Hence, $w(t,z)$ satisfies Equation (3.1) with right-hand side $g(t,z)$ in place of $\widehat{g}(t,z)$.

Finally, the fact that all derivatives of $w(t,z)$ with respect to t are bounded on Σ implies the existence of $\displaystyle\lim_{\substack{t\to 0\\ t\in\Sigma}} \partial_t^\ell w(t,z)$ for all $z \in D_r$ and hence the existence of the Taylor series of w at 0 on Σ for all $z \in D_r$. Since $w(t,z)$ satisfies Equation (3.1), so does its Taylor series. Since Equation (3.1) has a unique formal solution $\widehat{w}(t,z)$, we can conclude that the Taylor expansion of $w(t,z)$ is $\widehat{w}(t,z)$, which proves Condition 2 of Definition 3.1.

This achieves the proof of the 1-summability of $\widehat{u}(t,z)$ in direction θ in the case when $a(0) \neq 0$.

— The fact that the 1-sum $u(t,z)$ of $\widehat{u}(t,z)$ in direction θ satisfies Equation (1.4) with right-hand side the 1-sum $f(t,z)$ of $\widehat{f}(t,z)$ instead of $\widehat{f}(t,z)$ is equivalent to the fact that $w(t,z)$ satisfies Equation (3.1) with right-hand side $g(t,z)$ instead of $\widehat{g}(t,z)$, which we proved above. It is also a consequence of Corollary 3.3.

• In the case when $a(0) = 0$ and $a'(0) \neq 0$, the necessary condition again results from Proposition 3.2. The fact that $u(t,z)$ satisfies Equation (1.4) results from Corollary 3.3. We sketch the proof of the sufficient condition.

Denote $a(z) = zA(z)$ with $A(0) \neq 0$. In this case, identification of equal powers of z shows that $\widehat{u}_{*,0} = \widehat{f}_{*,0}$ and that all $\widehat{u}_{*,n}$ for $n \geq 1$ are uniquely determined by $\widehat{u}_{*,1}$ and $\widehat{f}$. We set again $\widehat{u}(t,z) = \widehat{u}_{*,0} + z\widehat{u}_{*,1} + \partial_t \partial_z^{-2} \widehat{w}$ so that $\widehat{w}$ satisfies the equation

$$\Big(1 - \frac{1}{zA(z)} \partial_t \partial_z^{-2}\Big) \widehat{w}(t,z) = \widehat{g}(t,z) \quad \text{where } \widehat{g} = \frac{1}{A(z)} \Big(\widehat{u}_{*,1} + \frac{\widehat{u}_{*,0} - \widehat{f}}{z}\Big). \tag{3.6}$$

Still, $\widehat{g}$ is a formal power series, assumed to be 1-summable in direction θ and we look for $\widehat{w}$ in the form $\widehat{w} = \displaystyle\sum_{p\geq 0} \widehat{w}_p$ as previously. The operator $\dfrac{1}{z}\partial_z^{-2}$ implies that $\widehat{w}^p = O(z^p)$ instead of $O(z^{2p})$. If we denote $B = \displaystyle\max_{z\in D_r} \frac{1}{|A(z)|}$, then, for all p and ℓ,

$$\left|\partial_t^\ell w_p\right| \leq C' K'^{\ell+p} \Gamma\big(1 + 2(\ell+p)\big) \frac{(B|z|)^p}{p!}$$

and it follows that, for a convenient choice of $r > 0$,

$$\left|\partial_t^\ell w(t,z)\right| \leq C\, K^\ell\, \Gamma(1+2\ell)$$

with $C = C' \sum_{p\geq 0} (4KBr)^p < \infty$ and $K = 4K'$. □

The case of a thermal diffusivity $a(z) = O(z^2)$ gives rise to the conditions $\widehat{u}_{*,0}(t) = \widehat{f}_{*,0}(t)$ and $\widehat{u}_{*,1}(t) = \widehat{f}_{*,1}(t)$ and we could hope for similar necessary and sufficient conditions applying to the inhomogeneity $\widehat{f}(t,z)$ only. This is not the case since the previous proof cannot be extended to that situation. Indeed, the appearance of $\dfrac{\partial_z^{-2}}{z^2}$ instead of ∂_z^{-2} or $\dfrac{\partial_z^{-2}}{z}$ implies that no power of z remains in the estimates (3.3) and we cannot guaranty the convergence of the estimate for $\partial_t^\ell w$.

The counter-example below shows that, even with $\widehat{f}(t,z)$ independent of t and rational, the 1-summability of $\widehat{u}(t,z)$ may fail.

Counter Example 3.5. Consider the heat initial conditions problem (1.4) with

$$a(z) = z^2 \quad \text{and} \quad \widehat{f}(t,z) = \sum_{n\geq 0} z^n = \frac{1}{1-z}.$$

The series $\widehat{f}(t,z)$ is independent of t and is convergent in z near 0 with rational sum. The problem is equivalent to the heat initial conditions problem without internal heat generation

$$\begin{cases} \partial_t \widehat{u} - z^2\, \partial_z^2 \widehat{u} = 0 \\ \widehat{u}(0,z) = \displaystyle\sum_{n\geq 0} z^n. \end{cases} \tag{3.7}$$

In this case, $\widehat{u}_{*,0}(t) = \widehat{f}_{*,0}(t) \equiv 1$, $\widehat{u}_{*,1}(t) = \widehat{f}_{*,1}(t) \equiv 1$ and for all $n \geq 2$, $\widehat{u}_{*,n}(t)$ satisfies

$$\widehat{u}'_{*,n}(t) - n(n-1)\widehat{u}_{*,n}(t) = 0 \quad \text{and} \quad \widehat{u}_{*,n}(0) = n!.$$

Consequently, $\widehat{u}_{*,n}(t) = n!\, e^{n(n-1)t}$.

Suppose $\widehat{u}(t,z)$ is 1-summable in a direction θ with sum $u(t,z)$. Then, since

$$\widehat{u}_{*,n}(t) = \partial_z^n \widehat{u}(t,z)\Big|_{z=0},$$

all $\widehat{u}_{*,n}(t)$ are 1-summable in direction θ with sum $u_{*,n}(t) = \partial_z^n u(t,z)\Big|_{z=0}$. The integral Cauchy formula applied to $\partial_z^n u(t,z)$ at $z=0$ provides estimates of the form

$$|u_{*,n}(t)| = \left| \frac{n!}{2\pi i} \int_{|\zeta|=R<r} \frac{u(t,\zeta)}{\zeta^{n+1}} d\zeta \right| \leq \frac{n!}{2\pi} \frac{C 2\pi R}{R^{n+1}} = C\, k^n\, n!$$

on a sector bisected by θ with opening larger than π. In our case, $\widehat{u}_{*,n}(t) = u_{*,n}(t) = n!\, e^{n(n-1)t}$. The functions $e^{n(n-1)t}$ being unbounded on any sector larger than a half plane such estimates are impossible. Hence, $\widehat{u}(t,z)$ is 1-summable in no direction.

4 Initial Conditions

We end this article with a discussion of how to apply the above result (Theorem 3.4) and we develop the cases when $a(z) = a \in \mathbb{C}^*$ or $a(z) = bz$, $b \in \mathbb{C}^*$.

The formal series $\widehat{f}(t,z)$ is a data of the problem and although its 1-summability may be not obvious we assume that it is known. $\widehat{f}(t,z)$ is not itself the initial conditions but is closely connected to (see Section 1).

The series $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ can, at least theoretically, be computed in terms of $\widehat{f}(t,z)$ from the formula

$$\widehat{u}(t,z) = \sum_{k\geq 0} \left(a\partial_t^{-1}\partial_z^2\right)^k \widehat{f}(t,z)$$

and an explicit computation can be achieved for simple $a(z)$ such as $a(z) = a$ constant, $a(z) = bz$ ($b \in \mathbb{C}^*$) or $a(z) = a + bz$. However, an explicit computation of $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ looks like hopeless for a general $a(z)$.

4.1 Case $a(z) = a \in \mathbb{C}^*$

When a is a constant, the operators a, ∂_t and ∂_z commute and $\left(a\partial_t^{-1}\partial_z^2\right)^k = a^k\partial_t^{-k}\partial_z^{2k}$. From the calculation of $\widehat{u}(t,z) = \sum_{k\geq 0} \left(a\partial_t^{-1}\partial_z^2\right)^k \widehat{f}(t,z)$ we obtain

$$\left\{\begin{array}{rcl} \widehat{u}_{*,0}(t) & = & \displaystyle\sum_{k\geq 0} \frac{t^k}{k!} \sum_{j+n=k} a^n f_{j,2n} \\ \widehat{u}_{*,1}(t) & = & \displaystyle\sum_{k\geq 0} \frac{t^k}{k!} \sum_{j+n=k} a^n f_{j,2n+1}. \end{array}\right. \tag{4.1}$$

Our aim is to characterize the 1-summability of these two series as a property of the inhomogeneity $\widehat{f}$.

• We start with the case where $\widehat{f}(t,z) = \sum_{n\geq 0} f_{0,n}\dfrac{z^n}{n!}$ is independent of t which corresponds to Problem (1.3). For simplicity, we denote $\widehat{f}(z)$. The formulæ (4.1) become

$$\left\{\begin{array}{rcl} \widehat{u}_{*,0}(t) & = & \displaystyle\sum_{k\geq 0} \frac{(at)^k}{k!} f_{0,2k} \\ \widehat{u}_{*,1}(t) & = & \displaystyle\sum_{k\geq 0} \frac{(at)^k}{k!} f_{0,2k+1}. \end{array}\right. \tag{4.2}$$

Define the 2-Laplace transform of $\widehat{f}(z)$ by $\mathcal{L}_z^{[2]}\widehat{f}(\zeta) = \sum_{n\geq 0} f_{0,n}\frac{\zeta^n}{n!}\frac{n!}{[n/2]!}$, where $[n/2]$ stands for the integer part of $n/2$. Then,

$$\mathcal{L}_z^{[2]}\widehat{f}\big((at)^{1/2}\big) = \widehat{u}_{*,0}(t) + (at)^{1/2}\widehat{u}_{*,1}(t)$$

and we may state the following result.

Proposition 4.1. *Suppose $a(z) = a \in \mathbb{C}^*$ and $\widehat{f}(t,z) = \widehat{f}(z)$. Then, the following three assertions are equivalent.*

1. *$\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ are 1-summable in direction θ;*
2. *$\mathcal{L}_z^{[2]}\widehat{f}(z)$ is 2-summable in the directions $\frac{1}{2}(\theta + \arg a)$ mod π;*
3. *$\widehat{f}(z)$ is analytic near 0 and it can be analytically continued to sectors neighbouring the directions $\frac{1}{2}(\theta + \arg a)$ mod π with exponential growth of order 2 at infinity.*

Assertion 3 with $a = 1$ (hence $\arg a = 0$) is how the conditions are formulated in [8] and proved via direct Borel–Laplace estimations. Our method provides thus a new proof of this result.

• Consider now the case of a general $\widehat{f}(t,z)$. The interpretation of the 1-summability of $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ becomes more involved and uses Borel and Laplace transforms of $\widehat{f}(t,z)$ in both variables.

We denote $\mathcal{L}_z$ or $\mathcal{B}_z$ and so on... the 1-Laplace or 1-Borel transform w.r.t. z and so on. These operators are defined here by $\mathcal{L}_z z^n = \zeta^n[n]!$ and $\mathcal{B}_z = \mathcal{L}_z^{-1}$, where $[n]$ denotes the integer part of n. Consider

$$\mathcal{L}_t\mathcal{L}_z\widehat{f}\big(\tau,(a\tau)^{1/2}\big) = \sum_{k\geq 0}\tau^k\sum_{j+n=k} f_{j,2n}a^n + (a\tau)^{1/2}\sum_{k\geq 0}\tau^k\sum_{j+n=k} f_{j,2n+1}a^n$$

and

$$\mathcal{B}_\tau\mathcal{L}_t\mathcal{L}_z\widehat{f}\big(\tau,(a\tau)^{1/2}\big)(t) = \sum_{k\geq 0}\frac{t^k}{k!}\sum_{j+n=k} f_{j,2n}a^n + (at)^{1/2}\sum_{k\geq 0}\frac{t^k}{k!}\sum_{j+n=k} f_{j,2n+1}a^n$$

(the terms in τ^k are divided by $k!$ and the terms in $\tau^{k+1/2}$ by $[k+1/2]! = k!$). Denote $\widehat{F}(t) = \mathcal{B}_\tau\mathcal{L}_t\mathcal{L}_z\widehat{f}\big(\tau,(a\tau)^{1/2}\big)(t^2)$. Then,

$$\widehat{F}(t^{1/2}) = \widehat{u}_{*,0}(t) + (at)^{1/2}\widehat{u}_{*1}(t)$$

and we may state the following result.

Proposition 4.2. *Suppose $a(z) = a \in \mathbb{C}^*$ and $\widehat{f}(t,z)$ general. Then, the series $\widehat{u}_{*,0}(t)$ and $\widehat{u}_{*,1}(t)$ are 1-summable in direction θ if and only if the series $\widehat{F}$ associated with $\widehat{f}$ as above is 2-summable in the directions $\theta/2$ mod π.*

The condition in Proposition 4.1 may be not easy to check but seems reasonable. In Proposition 4.2, the link between $\widehat{f}$ and $\widehat{F}$ is more complicated and the question remains of how to check the 2-summability of $\widehat{F}$ in practice.

4.2 Case $a(z) = bz,\ b \in \mathbb{C}^*$

In this case, $\left(a(z)\partial_t^{-1}\partial_z^2\right)^k = b^k\partial_t^{-k}(z\partial_z^2)^k$ and

$$(z\partial_z^2)^k \cdot \frac{z^n}{n!} = \begin{cases} \dfrac{z^{n-k}}{(n-k)!}\dfrac{(n-1)!}{(n-k-1)!} & \text{if } 0 \le k < n \\ 0 & \text{if } n \le k. \end{cases}$$

From the calculation of $\widehat{u}(t,z) = \sum_{k\ge 0}\left(bz\partial_t^{-1}\partial_z^2\right)^k \widehat{f}(t,z)$ we obtain

$$\begin{cases} \widehat{u}_{*,0}(t) &= \displaystyle\sum_{j\ge 0}\frac{t^j}{j!}\, f_{j,0} = \widehat{f}_{*,0}(t) \\ \widehat{u}_{*,1}(t) &= \displaystyle\sum_{j,k\ge 0} f_{j,k+1}\, b^k\, \frac{t^{j+k}}{(j+k)!}k!. \end{cases} \tag{4.3}$$

Since $\widehat{u}_{*,0}(t) = \widehat{f}_{*,0}(t)$ is 1-summable when so is $\widehat{f}(t,z)$, our aim is now to characterize the 1-summability of the series $\widehat{u}_{*,1}(t)$ as a property of $\widehat{f}$.

• Let us first again place ourselves in the situation of Problem (1.3), where the inhomogeneity $\widehat{f}(t,z) = \sum_{n\ge 0} f_{0,n}\dfrac{z^n}{n!}$ is independent of t. Formulæ (4.3) become

$$\begin{cases} \widehat{u}_{*,0}(t) &= = f_{0,0}; \\ \widehat{u}_{*,1}(t) &= \displaystyle\sum_{k\ge 0} f_{0,k+1}\, b^k\, t^k. \end{cases} \tag{4.4}$$

Thus, $\mathcal{L}_z\widehat{f}(bt) = f_{0,0} + bt\widehat{u}_{*,1}(t)$ and we may state the following results.

Proposition 4.3. *Suppose $a(z) = bz,\ b \in \mathbb{C}^*$ and $\widehat{f}(t,z) = \widehat{f}(z)$. Then, $\widehat{u}_{*,0}(t)$ is a constant and the following three assertions are equivalent.*

1. *$\widehat{u}_{*,1}(t)$ is 1-summable in direction θ;*

2. *$\mathcal{L}_z\widehat{f}(z)$ is 1-summable in the direction $\theta + \arg b$;*

3. *$\widehat{f}(z)$ is analytic near 0 and it can be analytically continued to a sector neighbouring the direction $\theta + \arg b$ with exponential growth of order 1 at infinity.*

• Consider finally the case of a general $\widehat{f}(t,z)$. The Laplace transform of $\widehat{f}$ w.r.t. z reads

$$\mathcal{L}_z\widehat{f}(t,z) = \widehat{f}_{*,0}(t) + z\sum_{j,n\geq 0}\frac{t^j}{j!}\widehat{f}_{j,n+1}z^n.$$

Consider the series $\widehat{g}(t,z) = \mathcal{L}_t\mathcal{L}_z\left[\frac{1}{z}\big(\mathcal{L}_z\widehat{f}(t,z) - \widehat{f}_{*,0}(t)\big)\right]$. We can check that the Borel transform of the series $\widehat{g}(t,bt)$ is equal to $\widehat{u}_{*,1}(t)$ and we may state the following result.

Proposition 4.4. *Suppose $a(z) = bz,\ b \in \mathbb{C}^*$ and $\widehat{f}(t,z)$ general. Then, the series $\widehat{u}_{*,1}(t)$ is 1-summable in direction θ if and only if the Borel transform of $\widehat{g}(t,bt)$ is 1-summable in direction θ.*

The comment following Propositions 4.1 and 4.2 keeps valid.

References

[1] Werner Balser. Divergent solutions of the heat equation: on an article of Lutz, Miyake and Schäfke. *Pacific J. Math.*, 188(1):53–63, 1999.

[2] Werner Balser. *Formal power series and linear systems of meromorphic ordinary differential equations*. Universitext. Springer-Verlag, New York, 2000.

[3] Werner Balser. Multisummability of formal power series solutions of partial differential equations with constant coefficients. *J. Differential Equations*, 201(1):63–74, 2004.

[4] Werner Balser and M. Miyake. Summability of formal solutions of certain partial differential equations. *Acta Sci. Math. (Szeged)*, 65(3-4):543–551, 1999.

[5] M. Canalis-Durand, J. P. Ramis, R. Schäfke, and Y. Sibuya. Gevrey solutions of singularly perturbed differential equations. *J. Reine Angew. Math.*, 518:95–129, 2000.

[6] Masaki Hibino. Borel summability of divergence solutions for singular first-order partial differential equations with variable coefficients. I. *J. Differential Equations*, 227(2):499–533, 2006.

[7] Michèle Loday-Richaud. Stokes phenomenon, multisummability and differential Galois groups. *Ann. Inst. Fourier (Grenoble)*, 44(3):849–906, 1994.

[8] D. A. Lutz, M. Miyake, and R. Schäfke. On the Borel summability of divergent solutions of the heat equation. *Nagoya Math. J.*, 154:1–29, 1999.

[9] Stéphane Malek. On the summability of formal solutions of linear partial differential equations. *J. Dyn. Control Syst.*, 11(3):389–403, 2005.

[10] Stéphane Malek. On the Stokes phenomenon for holomorphic solutions of integro-differential equations with irregular singularity. *J. Dyn. Control Syst.*, 14(3):371–408, 2008.

[11] Stéphane Malek. Gevrey functions solutions of partial differential equations with Fuchsian and irregular singularities. *J. Dyn. Control Syst.*, 15(2):To appear, 2009.

[12] B. Malgrange and J.-P. Ramis. Fonctions multisommables. *Ann. Inst. Fourier (Grenoble)*, 42(1-2):353–368, 1992.

[13] Bernard Malgrange. Sommation des séries divergentes. *Exposition. Math.*, 13(2-3):163–222, 1995.

[14] Masatake Miyake. Borel summability of divergent solutions of the Cauchy problem to non-Kowalevskian equations. In *Partial differential equations and their applications (Wuhan, 1999)*, pages 225–239. World Sci. Publ., River Edge, NJ, 1999.

[15] Mitio Nagumo. Über das Anfangswertproblem partieller Differentialgleichungen. *Jap. J. Math.*, 18:41–47, 1942.

[16] Sunao Ōuchi. Multisummability of formal solutions of some linear partial differential equations. *J. Differential Equations*, 185(2):513–549, 2002.

[17] Sunao Ōuchi. Borel summability of formal solutions of some first order singular partial differential equations and normal forms of vector fields. *J. Math. Soc. Japan*, 57(2):415–460, 2005.

[18] M. E. Pliś and B. Ziemian. Borel resummation of formal solutions to nonlinear Laplace equations in 2 variables. *Ann. Polon. Math.*, 67(1):31–41, 1997.

[19] J.-P. Ramis. Les séries k-sommables et leurs applications. In *Complex analysis, microlocal calculus and relativistic quantum theory (Proc. Internat. Colloq., Centre Phys., Les Houches, 1979)*, volume 126 of *Lecture Notes in Phys.*, pages 178–199. Springer, Berlin, 1980.