Advances in Dynamical Systems and Applications
ISSN 0973-5321, Volume 4, Number 1, pp. 133–150 (2009)
http://campus.mst.edu/adsa

Solvability Criteria for Second Order Generalized Sturm–Liouville Problems at Resonance

Youwei Zhang
Hexi University
Department of Mathematics
Gansu 734000, PR China
ywzhang0288@163.com

Abstract

This paper presents some solvability criteria for the second order nonlinear equation

$$(p(t)u'(t))' - q(t)u(t) = f\left(t, \int_0^t u(s)ds, u'(t)\right), \quad t \in (0,1),$$

with one of the following boundary conditions

$$au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = \mu_1 u(\xi),$$
$$au(0) - bp(0)u'(0) = \mu_2 u(\xi), \quad cu(1) + dp(1)u'(1) = 0,$$
$$au(0) - bp(0)u'(0) = \mu_1 u(\xi), \quad cu(1) + dp(1)u'(1) = \mu_2 u(\xi).$$

Under the appropriate nonlinear restriction of nonlinearity, solvability criteria for generalized Sturm–Liouville boundary value problems at resonance are established by means of coincidence degree theory of Mawhin type.

AMS Subject Classifications: 34B15.
Keywords: Sturm–Liouville problem, Fredholm mapping, coincidence degree.

1 Introduction

Consider the second order nonlinear equation

$$(p(t)u'(t))' - q(t)u(t) = f\left(t, \int_0^t u(s)ds, u'(t)\right), \quad t \in (0,1), \tag{1.1}$$

Received October 22, 2007; Accepted February 25, 2008
Communicated by Gerhard Freiling

subject to one of the following boundary conditions

$$au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i), \tag{1.2}$$

$$au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + dp(1)u'(1) = 0, \tag{1.3}$$

$$au(0) - bp(0)u'(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \quad cu(1) + dp(1)u'(1) = \sum_{i=1}^{m-2} b_i u(\xi_i). \tag{1.4}$$

It is known [8] that the solutions of (1.1) with the m-point boundary conditions (1.2), (1.3) and (1.4) can be obtained via existence subject to the respective three-point boundary conditions

$$au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = \mu_1 u(\xi), \tag{1.5}$$

$$au(0) - bp(0)u'(0) = \mu_2 u(\xi), \quad cu(1) + dp(1)u'(1) = 0, \tag{1.6}$$

$$au(0) - bp(0)u'(0) = \mu_1 u(\xi), \quad cu(1) + dp(1)u'(1) = \mu_2 u(\xi). \tag{1.7}$$

In [6], Gupta studied some existence results for solutions of the boundary value problem

$$x''(t) = f(t, x(t), x'(t)) + e(t), \quad t \in (0, 1),$$

$$x(0) = 0, \quad x'(1) = \sum_{i=1}^{m-2} a_i x'(\xi_i)$$

with $\sum_{i=1}^{m-2} a_i = 1$, where $f(t, x, y)$ satisfies Carathéodory's conditions and $e(t)$ is a function in $L^1[0, 1]$. Feng and Webb [4] considered the solvability of second order differential equations

$$u''(t) = f(t, u(t), u'(t)) + e(t), \quad t \in (0, 1) \tag{1.8}$$

with the three-point boundary conditions

$$u'(0) = 0, \quad u(1) = \alpha u(\eta), \tag{1.9}$$

$$u(0) = 0, \quad u(1) = \alpha u(\eta) \tag{1.10}$$

when $\alpha = 1$ for (1.8)/(1.9) and $\alpha = \dfrac{1}{\eta}$ for (1.8)/(1.10) are at resonance.

There has been increasing interest in questions of the solvability of boundary value problems for ordinary differential equations at resonance. There were many excellent results on the existence of solutions for two-point or multipoint boundary value problems, for which the nonlinearity is only dependent of the first-order derivative. The main techniques used are the Leray–Schauder continuation theorem and the coincidence degree theory, see [3–5, 7, 9, 10, 13, 14] and references therein. The second-order nonlinear equation, for which the nonlinearity is involved with integration and first-order derivative, is a special case of an integro-differential equation. It is known that integro-differential equations arise from many fields of science, for example in applied areas which include engineering, mechanics, financial mathematics, etc. [1, 2, 11].

For the equation (1.1), with the generalized Sturm–Liouville boundary conditions, nothing is known regarding the solvability of this class of boundary value problems. So, in this paper, using coincidence degree theory of Mawhin type [12], we establish some solvability criteria for boundary value problem at resonance (1.1)/(1.5), (1.1)/(1.6) and (1.1)/(1.7), respectively. The problem (1.1)/(1.5) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$(p(t)u'(t))' = 0, \quad t \in (0,1),$$
$$au(0) - bp(0)u'(0) = 0, \quad cu(1) + dp(1)u'(1) = \mu_1 u(\xi)$$

has $u(t) = a\int_0^t \frac{1}{p(\tau)}d\tau + b$ as a nontrivial solution, while we assume that

$$\mu_1\left(a\int_0^\xi \frac{1}{p(\tau)}d\tau + b\right) = ad + bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau, \quad \xi \in (0,1).$$

This result implies that $q(t)u(t) + f\left(t, \int_0^t u(s)ds, u'(t)\right) \in L^1[0,1]$ and

$$c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau - \mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau + d\int_0^1 z(s)ds = 0.$$

If $\mu_1\left(a\int_0^\xi \frac{1}{p(\tau)}d\tau + b\right) \neq ad+bc+ac\int_0^1 \frac{1}{p(\tau)}d\tau$, then this problem has $u(t) \equiv 0$ as its only solution. So we say that the boundary value problem (1.1)/(1.5) is at resonance when

$$\mu_1\left(a\int_0^\xi \frac{1}{p(\tau)}d\tau + b\right) = ad + bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau.$$

The cases such that the linear mapping $Lu = (p(t)u'(t))'$ is noninvertible are called resonance cases. Otherwise, they are called nonresonance cases.

Similarly, we can obtain that the problem (1.1)/(1.6) is at resonance when

$$\mu_2\left(-c\int_0^\xi \frac{1}{p(\tau)}d\tau + c\int_0^1 \frac{1}{p(\tau)}d\tau + d\right) = ad + bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau,$$

and the problem (1.1)/(1.7) is at resonance when

$$\mu_1\left(c\int_0^1 \frac{1}{p(\tau)}d\tau - c\int_0^\xi \frac{1}{p(\tau)}d\tau + d\right) + \mu_2\left(a\int_0^\xi \frac{1}{p(\tau)}d\tau + b\right)$$
$$= ad + bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau.$$

Throughout this paper, we use the following hypotheses:

(H_1) $a, b, c, d \in \mathbb{R} \setminus \{0\}$, $\mu_i \neq 0$ ($i = 1, 2$), and $\xi \in (0, 1)$ are given constants;

(H_2) $p \in C^1([0,1], (0,\infty))$, $q \in L^1([0,1], \mathbb{R})$, $f \in C([0,1] \times \mathbb{R}^2, \mathbb{R})$;

(H_3) $1 - (2\zeta + |b/a|) \|q\|_1 > 0$, $1 - (2\zeta + |b/a|) \|q\|_1 - (2\zeta + |b/a|) \|\alpha\|_1 > 0$, where $\zeta = \sup_{t\in[0,1]} \dfrac{1}{p(t)} \in (0, \infty)$.

The rest of the paper is organized as follows: Section 2 gathers together the definitions of Fredholm mapping of index zero and L-completely continuity, which will be useful in proving the main results. Using coincidence degree theory of Mawhin type, solvability criteria for the generalized Sturm–Liouville boundary value problems at resonance (1.1)/(1.5), (1.1)/(1.6) and (1.1)/(1.7) are established in Section 3 and Section 4, respectively.

2 Preliminaries

Definition 2.1. Let X and Z be normed spaces. A linear mapping $L : \mathrm{Dom}L \subset X \to Z$ is called a Fredholm mapping if the following two conditions hold:

(i) $\ker L$ has a finite dimension,

(ii) $\mathrm{Im}L$ is closed and has a finite codimension.

If L is a Fredholm mapping, its Fredholm index is the integral $\mathrm{Ind}L = \dim\ker L - \mathrm{codimIm}L$. In this paper, we are interested in a Fredholm mapping of index zero, that is, $\dim\ker L = \mathrm{codimIm}L$. From Definition 2.1, it follows that there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\mathrm{Im}P = \ker L$, $\ker Q = \mathrm{Im}L$, $X = \ker L \oplus \ker P$, $Z = \mathrm{Im}L \oplus \mathrm{Im}Q$, and that the mapping

$$L_{\mathrm{Dom}L\cap\ker P} : \mathrm{Dom}L \cap \ker P \to \mathrm{Im}L$$

is invertible. We denote the inverse of $L\,|_{\mathrm{Dom}L\cap\ker P}$ by $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Z \to \mathrm{Dom}L \cap \ker P$ is defined by $K_{P,Q} : K_P(I - Q)$.

Definition 2.2. Let $L : \mathrm{Dom}L \subset X \to Z$ be a Fredholm mapping, let E be a metric space, and $N : E \to Z$ be a mapping. N is called L-compact on E if $QN : E \to Z$ and $K_{P,Q}N : E \to X$ are compact on E. In addition, we say that N is L-completely continuous if it is L-compact on every bounded $E \subset X$.

Theorem 2.3 (see [12]). *Let $\Omega \subset X$ be open and bounded, L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(*i*) $Lu \neq \lambda Nu$ *for every* $(u, \lambda) \in ((\mathrm{Dom}L \setminus \ker L) \cap \partial\Omega) \times (0, 1)$*;*

(*ii*) $Nu \notin \mathrm{Im}L$ *for every* $u \in \ker L \cap \partial\Omega$*;*

(*iii*) $\deg(QN \mid_{\ker L \cap \partial\Omega}, \Omega \cap \ker L, 0) \neq 0$ *with* $Q : Z \to Z$ *a continuous projection such that* $\ker Q = \mathrm{Im}L$*.*

Then the equation $Lu = Nu$ *has at least one solution in* $\mathrm{Dom}L \cap \overline{\Omega}$*.*

3 Existence Results for (1.1)/(1.5)

Let $X = C^1[0, 1]$ with the norm

$$\|u\| = \max\left\{\|u\|_\infty\,,\ \|u'\|_\infty\right\}, \quad \text{where} \quad \|u\|_\infty = \sup_{t\in[0,1]} |u(t)|\,,$$

and $Z = L^1[0, 1]$ with the norm $\|u\|_1 = \displaystyle\int_0^1 |u(t)|\,dt$. We use the Sobolev space

$$W^{2,1}(0,1) = \left\{u : [0,1] \to \mathbb{R} :\ u, u' \text{ are absolutely continuous on } [0,1],\ u'' \in L^1[0,1]\right\}.$$

Define L to be the linear mapping from $\mathrm{Dom}L \subset X$ to Z with

$$\begin{aligned}\mathrm{Dom}L \;=\; &\left\{u \in W^{2,1}(0,1) :\ au(0) - bp(0)u'(0) = 0,\right.\\ &\left. cu(1) + dp(1)u'(1) = \mu_1 u(\xi) \quad \xi \in (0,1)\right\}\end{aligned}$$

by

$$Lu(t) = (p(t)u'(t))', \quad u \in \mathrm{Dom}L,$$

and define the mapping $N : X \to Z$ by

$$Nu(t) = f\left(t, \int_0^t u(s)ds, u'(t)\right) + q(t)u(t), \quad t \in [0,1].$$

For convenience, we set

$$\Lambda_1 = c\int_0^1 \frac{\tau}{p(\tau)}d\tau - \mu_1\int_0^\xi \frac{\tau}{p(\tau)}d\tau + d \neq 0.$$

Lemma 3.1. *The mapping* $L : \mathrm{Dom}L \subset X \to Z$ *is a Fredholm mapping of index zero when*

$$\mu_1\left(a\int_0^\xi \frac{1}{p(\tau)}d\tau + b\right) = ad + bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau.$$

Furthermore, the linear continuous projection $Q : Z \to Z$ *can be defined by*

$$Qz = \frac{1}{\Lambda_1}\left(c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau - \mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau + d\int_0^1 z(s)ds\right),$$

the mapping $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \ker P$ *can be written as*

$$K_P z(t) = \int_0^t \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau,$$

and

$$\|K_P z\|_\infty \le \zeta\,\|z\|_1 \text{ for } z \in \mathrm{Im}L.$$

Proof. It is clear that $\ker L = \mathbb{R}$. Let $u \in \mathrm{Dom}L$, and $z \in Z$ and consider the linear equation

$$(p(t)u'(t))' = z(t), \quad t \in (0,1),$$

Taking the Cauchy integral from 0 to t, we obtain

$$u'(t) = \frac{p(0)u'(0)}{p(t)} + \frac{1}{p(t)}\int_0^t z(\tau)d\tau$$

Again taking the Cauchy integral from 0 to t, we get

$$u(t) = u(0) + p(0)u'(0)\int_0^t \frac{1}{p(\tau)}d\tau + \int_0^t \frac{1}{p(s)}\int_0^s z(\tau)d\tau ds, \tag{3.1}$$

which satisfies (1.5) if and only if

$$c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau - \mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau + d\int_0^1 z(s)ds = 0. \tag{3.2}$$

On the other hand, if (3.2) holds for some $z \in Z$, then we take $u \in \mathrm{Dom}L$ as given by (3.1), $(p(t)u'(t))' = z(t)$ for $t \in (0,1)$, and (1.5) is satisfied. So

$$\mathrm{Im}L = \left\{z \in L^1[0,1] :\ c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau - \mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau + d\int_0^1 z(s)ds = 0\right\}.$$

Further, we define the mapping $Q : Z \to Z$ by

$$Qz = \frac{1}{\Lambda_1}\left(c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau - \mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau + d\int_0^1 z(s)ds\right)$$

for $z \in Z$, and it is easy to check that $Q : Z \to Z$ is a linear continuous projection. Furthermore, $\mathrm{Im}L = \ker Q$. Let $z = (z - Qz) + Qz$. Then $z - Qz \in \ker Q = \mathrm{Im}L$

and $Qz \in \mathrm{Im}Q$, so $Z = \mathrm{Im}L + \mathrm{Im}Q$. If $z \in \mathrm{Im}L \cap \mathrm{Im}Q$, then $z(t) = 0$. Hence $Z = \mathrm{Im}L \oplus \mathrm{Im}Q$. From $\ker L = \mathbb{R}$, we obtain that

$$\mathrm{Ind}L = \dim\ker L - \mathrm{codim}\mathrm{Im}L = \dim\ker L - \dim\mathrm{Im}Q = 0.$$

Thus L is a Fredholm mapping of index zero. Take $P : X \to X$ as

$$(Pu)(t) = u(0) + p(0)u'(0)\int_0^t \frac{1}{p(\tau)}d\tau, \quad t \in (0,1)$$

and let $u \in X$ be in the form

$$u(t) = \left(u(0) + p(0)u'(0)\int_0^t \frac{1}{p(\tau)}d\tau\right) + \left(u(t) - u(0) - p(0)u'(0)\int_0^t \frac{1}{p(\tau)}d\tau\right).$$

Obviously, $\mathrm{Im}P = \ker L$ and $X = \ker L \oplus \ker P$. Then the generalized inverse $K_P : \mathrm{Im}L \to \mathrm{Dom}L \cap \ker P$ is given by

$$K_P z(t) = \int_0^t \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau.$$

It follows that

$$(K_P z(t))' = \frac{1}{p(t)}\int_0^t z(\tau)d\tau.$$

We have

$$\begin{aligned} \|K_P z\|_\infty &= \sup_{t\in[0,1]}\left|\int_0^t \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau\right| \le \sup_{t\in[0,1]}\int_0^t \frac{1}{p(\tau)}\left|\int_0^\tau z(s)ds\right|d\tau \\ &\le \|z\|_1 \sup_{t\in[0,1]}\int_0^t \frac{1}{p(\tau)}d\tau \le \zeta\|z\|_1 \end{aligned}$$

and

$$\left\|(K_P z)'\right\|_\infty = \sup_{t\in[0,1]}\frac{1}{p(t)}\left|\int_0^t z(\tau)d\tau\right| \le \zeta\|z\|_1.$$

Thus

$$\|K_P z\| \le \zeta\|z\|_1. \tag{3.3}$$

In fact, for $z \in \mathrm{Im}L$, we know

$$(LK_P)z(t) = \left(p(t)\left(\int_0^t \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau\right)'\right)' = z(t), \tag{3.4}$$

and for $u \in \mathrm{Dom}L \cap \ker P$, we also know

$$\begin{aligned} (K_P L)u(t) &= \int_0^t \frac{1}{p(\tau)}\int_0^\tau (p(s)u'(s))'dsd\tau = \int_0^t \frac{1}{p(\tau)}(p(\tau)u'(\tau) - p(0)u'(0))d\tau \\ &= u(t) - u(0) - p(0)u'(0)\int_0^t \frac{1}{p(\tau)}d\tau. \end{aligned}$$

Then, in view of $u \in \mathrm{Dom}L \cap \ker P$, $Pu = u(0) + p(0)u'(0)\displaystyle\int_0^t \frac{1}{p(\tau)}d\tau = 0$, and thus

$$(K_P L)u(t) = u(t). \tag{3.5}$$

By (3.4) and (3.5), we obtain

$$K_P = (L\mid_{\mathrm{Dom}L\cap\ker P})^{-1}.$$

The proof is complete. □

Note that

$$\begin{aligned}(QN)u(t) \quad = \quad & \frac{1}{\Lambda_1}\left(c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) dsd\tau\right. \\ & -\mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) dsd\tau \\ & \left. +d\int_0^1 \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) ds\right)\end{aligned}$$

and

$$(K_{P,Q}N)u(t) = \int_0^t \frac{1}{p(\tau)}\int_0^\tau (p(s)(Nu(s) - QNu(s))')'\, dsd\tau.$$

Theorem 3.2. *Assume* (H_1)–(H_3). *Suppose that*

(A_1) *There exists a constant* $M > 0$ *such that for* $u \in \mathrm{Dom}L$, *if* $|u'(t)| > M/p(t)$ *for all* $t \in [0,1]$, *then*

$$\begin{aligned} 0 \quad \neq \quad & c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) dsd\tau \\ & -\mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) dsd\tau \\ & +d\int_0^1 \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) ds.\end{aligned}$$

(A_2) *There exist functions* $\alpha, \beta, \gamma, \theta \in L^1[0,1]$ *and a constant* $\epsilon \in [0,1)$ *such that for all* $(x, y) \in \mathbb{R}^2$ *and* $t \in [0,1]$ *either*

$$|f(t,x,y)| \le \alpha(t)\,|x| + \beta(t)\,|y| + \gamma(t)\,|y|^\epsilon + \theta(t) \tag{3.6}$$

or

$$|f(t,x,y)| \le \alpha(t)\,|x| + \beta(t)\,|y| + \gamma(t)\,|x|^\epsilon + \theta(t). \tag{3.7}$$

(A_3) *There exists a constant* $\overline{M} > 0$ *such that for any* $\omega \in \mathbb{R}$*, if* $|\omega| > \overline{M}$*, then either*

$$0 > \omega \left(c \int_0^1 \frac{1}{p(\tau)} \int_0^\tau (f(s, \omega s, 0) + \omega q(s)) ds d\tau - \mu_1 \int_0^\xi \frac{1}{p(\tau)} \int_0^\tau (f(s, \omega s, 0) + \omega q(s)) ds d\tau + d \int_0^1 (f(s, \omega s, 0) + \omega q(s)) ds \right) \quad (3.8)$$

or

$$0 < \omega \left(c \int_0^1 \frac{1}{p(\tau)} \int_0^\tau (f(s, \omega s, 0) + \omega q(s)) ds d\tau - \mu_1 \int_0^\xi \frac{1}{p(\tau)} \int_0^\tau (f(s, \omega s, 0) + \omega q(s)) ds d\tau + d \int_0^1 (f(s, \omega s, 0) + \omega q(s)) ds \right). \quad (3.9)$$

Then for each $q \in L^1[0,1]$*, the boundary value problem* (1.1)/(1.5) *when*

$$\mu_1 \left(a \int_0^\xi \frac{1}{p(\tau)} d\tau + b \right) = ad + bc + ac \int_0^1 \frac{1}{p(\tau)} d\tau$$

has at least one solution in $C^1[0,1]$ *provided that*

$$\|\alpha\|_1 + \|\beta\|_1 < \frac{1 - (2\zeta + |b/a|) \|q\|_1}{2\zeta + |b/a|}.$$

Proof. Let

$$\Omega_1 = \{ u \in \mathrm{Dom} L \setminus \ker L : \ Lu = \lambda N u \text{ for some } \lambda \in (0,1) \}.$$

For $u \in \Omega_1$, $u \notin \ker L$, and $Nu \in \mathrm{Im} L = \ker Q$. Thus

$$\begin{aligned} 0 \ = \ & c \int_0^1 \frac{1}{p(\tau)} \int_0^\tau \left(f \left(s, \int_0^s u(\eta) d\eta, u'(s) \right) + q(s) u(s) \right) ds d\tau \\ & - \mu_1 \int_0^\xi \frac{1}{p(\tau)} \int_0^\tau \left(f \left(s, \int_0^s u(\eta) d\eta, u'(s) \right) + q(s) u(s) \right) ds d\tau \\ & + d \int_0^1 \left(f \left(s, \int_0^s u(\eta) d\eta, u'(s) \right) + q(s) u(s) \right) ds \end{aligned}$$

since $QNu = 0$. It follows from (A_1) that there exists $t_0 \in [0,1]$ with $|p(t_0)u'(t_0)| \leq M$. From $\int_0^{t_0} (p(s)u'(s))' ds = p(t_0)u'(t_0) - p(0)u'(0)$, we get

$$\begin{aligned} |p(0)u'(0)| &\leq |p(t_0)u'(t_0)| + \left| \int_0^{t_0} (p(s)u'(s))' ds \right| \\ &\leq M + \|(pu')'\|_1 = M + \|Lu\|_1 \leq M + \|Nu\|_1 . \end{aligned} \tag{3.10}$$

Also, for $u \in \Omega_1$, observe that $(I-P)u \in \text{Im} K_P = \text{Dom} L \cap \ker P$. Then using (3.3), we obtain

$$\|(I-P)u\| = \|K_P L(I-P)u\| \leq \zeta \|L(I-P)u\|_1 = \zeta \|Lu\|_1 \leq \zeta \|Nu\|_1 . \tag{3.11}$$

By (3.10) and (3.11)

$$\begin{aligned} \|u\| &\leq \|Pu\| + \|(I-P)u\| \leq |u(0)| + \zeta |p(0)u'(0)| + \zeta \|Nu\|_1 \\ &= (\zeta + |b/a|) |p(0)u'(0)| + \zeta \|Nu\|_1 \\ &\leq (2\zeta + |b/a|) \|Nu\|_1 + M(\zeta + |b/a|) \\ &\leq (2\zeta + |b/a|) \left(\left| f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) \right| + \|q\|_1 \|u\|_1 \right) + M(\zeta + |b/a|) \\ &\leq (2\zeta + |b/a|) \left(\left| f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) \right| + \|q\|_1 \|u\| \right) + M(\zeta + |b/a|) \end{aligned}$$

since $\|u\|_1 \leq \|u\|_\infty \leq \|u\|$. Hence

$$\|u\| \leq \frac{2\zeta + |b/a|}{1 - (2\zeta + |b/a|)\|q\|_1} \left| f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) \right| + \frac{M(\zeta + |b/a|)}{1 - (2\zeta + |b/a|)\|q\|_1}.$$

If (3.6) holds, then

$$\begin{aligned} \|u\| \leq\ & \frac{2\zeta + |b/a|}{1 - (2\zeta + |b/a|)\|q\|_1} \times \\ & \times \left(\|\alpha\|_1 \left| \int_0^t u(s)ds \right| + \|\beta\|_1 \|u'\|_\infty + \|\gamma\|_1 \|u'\|_\infty^\epsilon + \|\theta\|_1 \right) \\ & + \frac{M(\zeta + |b/a|)}{1 - (2\zeta + |b/a|)\|q\|_1}, \end{aligned} \tag{3.12}$$

that is,

$$\begin{aligned} \|u\| \leq\ & \frac{(2\zeta + |b/a|)\|\alpha\|_1}{1 - (2\zeta + |b/a|)\|q\|_1} \left| \int_0^t u(s)ds \right| + \frac{(2\zeta + |b/a|)\|\beta\|_1}{1 - (2\zeta + |b/a|)\|q\|_1} \|u'\|_\infty \\ & + \frac{(2\zeta + |b/a|)\|\gamma\|_1}{1 - (2\zeta + |b/a|)\|q\|_1} \|u'\|_\infty^\epsilon + \frac{2\zeta + |b/a|}{1 - (2\zeta + |b/a|)\|q\|_1} \|\theta\|_1 \\ & + \frac{M(\zeta + |b/a|)}{1 - (2\zeta + |b/a|)\|q\|_1}. \end{aligned}$$

It is easy to check that $\left|\int_0^t u(s)ds\right|, \|u'\|_\infty \le \|u\|$. So, we have

$$\begin{aligned}\left|\int_0^t u(s)ds\right| &\le \|u\| \\ &\le \frac{(2\zeta+|b/a|)\|\alpha\|_1}{1-(2\zeta+|b/a|)\|q\|_1}\left|\int_0^t u(s)ds\right| + \frac{(2\zeta+|b/a|)\|\beta\|_1}{1-(2\zeta+|b/a|)\|q\|_1}\|u'\|_\infty \\ &\quad + \frac{(2\zeta+|b/a|)\|\gamma\|_1}{1-(2\zeta+|b/a|)\|q\|_1}\|u'\|_\infty^\epsilon + \frac{2\zeta+|b/a|}{1-(2\zeta+|b/a|)\|q\|_1}\|\theta\|_1 \\ &\quad + \frac{M(\zeta+|b/a|)}{1-(2\zeta+|b/a|)\|q\|_1},\end{aligned}$$

that is,

$$\begin{aligned}\left|\int_0^t u(s)ds\right| &\le \frac{(2\zeta+|b/a|)\|\beta\|_1}{1-(2\zeta+|b/a|)\|q\|_1-(2\zeta+|b/a|)\|\alpha\|_1}\|u'\|_\infty \\ &\quad + \frac{(2\zeta+|b/a|)\|\gamma\|_1}{1-(2\zeta+|b/a|)\|q\|_1-(2\zeta+|b/a|)\|\alpha\|_1}\|u'\|_\infty^\epsilon \\ &\quad + \frac{2\zeta+|b/a|}{1-(2\zeta+|b/a|)\|q\|_1-(2\zeta+|b/a|)\|\alpha\|_1}\|\theta\|_1 \\ &\quad + \frac{M(\zeta+|b/a|)}{1-(2\zeta+|b/a|)\|q\|_1-(2\zeta+|b/a|)\|\alpha\|_1}.\end{aligned} \tag{3.13}$$

Also, by (3.12) and (3.13), we obtain

$$\begin{aligned}\|u'\|_\infty &\le \frac{(2\zeta+|b/a|)\|\gamma\|_1}{1-(2\zeta+|b/a|)\|q\|_1-(2\zeta+|b/a|)(\|\alpha\|_1+\|\beta\|_1)}\|u'\|_\infty^\epsilon \\ &\quad + \frac{2\zeta+|b/a|}{1-(2\zeta+|b/a|)\|q\|_1-(2\zeta+|b/a|)(\|\alpha\|_1+\|\beta\|_1)}\|\theta\|_1 \\ &\quad + \frac{M(\zeta+|b/a|)}{1-(2\zeta+|b/a|)\|q\|_1-(2\zeta+|b/a|)(\|\alpha\|_1+\|\beta\|_1)}.\end{aligned}$$

Since $\epsilon \in [0,1)$ and $\|\alpha\|_1 + \|\beta\|_1 < \dfrac{1-(2\zeta+|b/a|)\|q\|_1}{2\zeta+|b/a|}$, there exists $M_1 > 0$ such that $\|u'\|_\infty \le M_1$ for all $u \in \Omega_1$, and so there exists $M_2 > 0$ such that $\displaystyle\int_0^1 |u(s)|\,ds \le M_2$ for all $u \in \Omega_1$. Therefore, Ω_1 is bounded. If (3.7) holds, then similar to the above arguments, we can derive the same conclusion. Now let

$$\Omega_2 = \{u \in \ker L : \ Nu \in \operatorname{Im} L\}.$$

If $u \in \Omega_2$, then $u(t) \equiv e \in \mathbb{R}$, $t \in [0,1]$, $Nu \in \mathrm{Im}L = \ker Q$, that is,

$$\begin{aligned} 0 &= \frac{1}{\Lambda_1}\Bigg(c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau (f(s,es,0)+eq(s))dsd\tau \\ &\quad -\mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau (f(s,es,0)+eq(s))dsd\tau + d\int_0^1 (f(s,es,0)+eq(s))ds\Bigg) \end{aligned}$$

since $QNu = 0$. From (A_1), we know $\|u\| = |e| \le \overline{M}$. Thus Ω_2 is bounded. Define

$$\Omega_3 = \{u \in \ker L : \ -\lambda Ju + (1-\lambda)QNu = 0, \ \lambda \in [0,1]\},$$

where $J : \ker L \to \mathrm{Im}Q$ is a linear isomorphism given by $J(k) = k$ for all $k \in \mathbb{R}$. If $u(t) = k$, then

$$\begin{aligned} \lambda k &= (1-\lambda)QNk \\ &= (1-\lambda)\frac{1}{\Lambda_1}\Bigg(c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau (f(s,ks,0)+kq(s))dsd\tau \\ &\quad -\mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau (f(s,ks,0)+kq(s))dsd\tau + d\int_0^1 (f(s,ks,0)+kq(s))ds\Bigg). \end{aligned}$$

If $\lambda = 1$, then $k = 0$ and in the case $\lambda \in [0,1)$, if $|k| > \overline{M}$, in view of (3.8), we have

$$\begin{aligned} \lambda k^2 &= k(1-\lambda)\frac{1}{\Lambda_1}\Bigg(c\int_0^1 \frac{1}{p(\tau)}\int_0^\tau (f(s,ks,0)+kq(s))dsd\tau \\ &\quad -\mu_1\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau (f(s,ks,0)+kq(s))dsd\tau + d\int_0^1 (f(s,ks,0)+kq(s))ds\Bigg) < 0, \end{aligned}$$

which is a contradiction. If (3.9) holds, then we take

$$\Omega_3 = \{u \in \ker L : \ \lambda Ju + (1-\lambda)QNu = 0, \ \lambda \in [0,1]\},$$

where J is as above, similar to the above argument. Thus, in either case

$$\|u\| = |k| \le \overline{M} \quad \text{for all} \quad u \in \Omega_3,$$

that is, Ω_3 is bounded.

Let Ω be a bounded open subset of X such that $\cup_{i=1}^3 \Omega_i \subset \Omega$. By using the Ascoli–Arzela theorem, we can prove that $K_P(I-Q)N : \overline{\Omega} \to X$ is compact. Thus N is L-compact on $\overline{\Omega}$. Finally, it only remains to verify that the condition (iii) of Theorem 2.3 is fulfilled. We define a homotopy

$$H(u,\lambda) = \pm\lambda Ju + (1-\lambda)QNu.$$

According to the above argument, we have

$$H(u,\lambda) \neq 0 \quad \text{for } u \in \partial\Omega \cap \ker L.$$

Thus, by the degree property of homotopy invariance, we obtain

$$\begin{aligned}\deg(QN_{\ker L}, \Omega \cap \ker L, 0) &= \deg\left(H(\cdot,0), \Omega \cap \ker L, 0\right)\\ &= \deg\left(H(\cdot,1), \Omega \cap \ker L, 0\right)\\ &= \deg(\pm J, \Omega \cap \ker L, 0) \neq 0.\end{aligned}$$

Then by Theorem 2.3, $Lu = Nu$ has at least one solution in $\mathrm{Dom}L \cap \overline{\Omega}$. Therefore, the boundary value problem (1.1)/(1.5) has a solution in $C^1[0,1]$. □

4 Existence Results for (1.1)/(1.6), (1.1)/(1.7)

In this section, we discuss existence of nontrivial solutions for (1.1)/(1.6) and (1.1)/(1.7), respectively. By using the same arguments as in Section 3, we easily show the following lemmas and theorems, and thus we omit their proofs. For the sake of convenience, we set

$$\Lambda_2 = c\mu_2 \int_0^\xi \frac{\tau}{p(\tau)}d\tau + c(a-\mu_2)\int_0^1 \frac{\tau}{p(\tau)}d\tau + d(a-\mu_2) \neq 0,$$
$$\Lambda_3 = (c\mu_1 - a\mu_2)\int_0^\xi \frac{\tau}{p(\tau)}d\tau + c(a-\mu_1)\int_0^1 \frac{\tau}{p(\tau)}d\tau + d(a-\mu_1) \neq 0.$$

Problem (1.1)/(1.6)

The mappings L and N are the same as in Section 3, and we set

$$\begin{aligned}\mathrm{Dom}L &= \big\{u \in W^{2,1}(0,1) : au(0) - bp(0)u'(0) = \mu_2 u(\xi),\\ &\qquad cu(1) + dp(1)u'(\sigma(1)) = 0, \quad \xi \in (0,1)\big\},\\ \mathrm{Im}L &= \Bigg\{z \in L^1[0,1] : c\mu_2 \int_0^\xi \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau\\ &\qquad + c(a-\mu_2)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau + d(a-\mu_2)\int_0^1 z(s)ds = 0\Bigg\}.\end{aligned}$$

Lemma 4.1. *The mapping $L : \mathrm{Dom}L \subset X \to Z$ is a Fredholm mapping of index zero when*

$$\mu_2\left(-c\int_0^\xi \frac{1}{p(\tau)}d\tau + c\int_0^1 \frac{1}{p(\tau)}d\tau + d\right) = ad + bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau.$$

Furthermore, the continuous linear projection mapping $Q : Z \to Z$ can be defined by

$$
\begin{aligned}
Qz &= \frac{1}{\Lambda_2}\left(c\mu_2\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau + c(a-\mu_2)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau z(s)dsd\tau \right.\\
&\quad \left. +d(a-\mu_2)\int_0^1 z(s)ds\right).
\end{aligned}
$$

The mapping K_P is the same as in Section 3.

Theorem 4.2. *Assume (H_1)–(H_3). Suppose that the condition (A_2) of Theorem 3.2 is satisfied and also assume*

(A_4) *There exists a constant $M > 0$ such that for $u \in \mathrm{Dom}L$, if $|u'(t)| > M/p(t)$ for all $t \in [0,1]$, then*

$$
\begin{aligned}
0 &\neq c\mu_2\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s,\int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right)dsd\tau\\
&\quad +c(a-\mu_2)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s,\int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right)dsd\tau\\
&\quad +d(a-\mu_2)\int_0^1 \left(f\left(s,\int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right)ds.
\end{aligned}
$$

(A_5) *There exists a constant $\overline{M} > 0$ such that for any $\omega \in \mathbb{R}$, if $|\omega| > \overline{M}$, then we have either*

$$
\begin{aligned}
0 &> \omega\left(c\mu_2\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0)+\omega q(s))dsd\tau\right.\\
&\quad +c(a-\mu_2)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0)+\omega q(s))dsd\tau\\
&\quad \left. +d(a-\mu_2)\int_0^1 (f(s,\omega s,0)+\omega q(s))ds\right)
\end{aligned}
$$

or else

$$
\begin{aligned}
0 &< \omega\left(c\mu_2\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0)+\omega q(s))dsd\tau\right.\\
&\quad +c(a-\mu_2)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0)+\omega q(s))dsd\tau\\
&\quad \left. +d(a-\mu_2)\int_0^1 (f(s,\omega s,0)+\omega q(s))ds\right).
\end{aligned}
$$

Then for each $q \in L^1[0,1]$*, the boundary value problem* (1.1)/(1.6) *when*

$$\mu_2\left(-c\int_0^{\xi}\frac{1}{p(\tau)}d\tau + c\int_0^1\frac{1}{p(\tau)}d\tau + d\right) = ad + bc + ac\int_0^1\frac{1}{p(\tau)}d\tau$$

has at least one solution in $C^1[0,1]$ *provided that*

$$\|\alpha\|_1 + \|\beta\|_1 < \frac{1-(2\zeta+|b/a|)\,\|q\|_1}{2\zeta+|b/a|}.$$

Problem (1.1)/(1.7)

The mappings L and N are the same as in Section 3, and we also set

$$\begin{aligned}
\mathrm{Dom}L &= \big\{u\in W^{2,1}(0,1) : au(0)-bp(0)u'(0)=\mu_1 u(\xi),\\
&\quad cu(1)+dp(1)u'(1)=\mu_2 u(\xi),\quad \xi\in(0,1)\big\},\\
\mathrm{Im}L &= \Big\{z\in L^1[0,1] : (c\mu_1-a\mu_2)\int_0^{\xi}\frac{1}{p(\tau)}\int_0^{\tau}z(s)dsd\tau\\
&\quad +c(a-\mu_1)\int_0^1\frac{1}{p(\tau)}\int_0^{\tau}z(s)dsd\tau + d(a-\mu_1)\int_0^1 z(s)ds = 0\Big\}.
\end{aligned}$$

Lemma 4.3. *The mapping* $L:\mathrm{Dom}L\subset X\to Z$ *is a Fredholm mapping of index zero when*

$$\mu_1\left(c\int_0^1\frac{1}{p(\tau)}d\tau - c\int_0^{\xi}\frac{1}{p(\tau)}d\tau + d\right) + \mu_2\left(a\int_0^{\xi}\frac{1}{p(\tau)}d\tau + b\right) = ad + bc + ac\int_0^1\frac{1}{p(\tau)}d\tau.$$

Furthermore, the continuous linear projection mapping $Q : Z \to Z$ *can be defined by*

$$\begin{aligned}
Qz &= \frac{1}{\Lambda_3}\Bigg((c\mu_1-a\mu_2)\int_0^{\xi}\frac{1}{p(\tau)}\int_0^{\tau}z(s)dsd\tau + c(a-\mu_1)\int_0^1\frac{1}{p(\tau)}\int_0^{\tau}z(s)dsd\tau\\
&\quad + d(a-\mu_1)\int_0^1 z(s)ds\Bigg).
\end{aligned}$$

The mapping K_P *is the same as in Section 3.*

Theorem 4.4. *Assume* (H_1)–(H_3)*. Suppose that the condition* (A_2) *of Theorem 3.2 is satisfied and also assume*

(A_6) *There exists a constant* $M > 0$ *such that for* $u \in \mathrm{Dom}L$*, if* $|u'(t)| > M/p(t)$ *for all* $t \in [0,1]$*, then*

$$\begin{aligned} 0 \neq\ & (c\mu_1 - a\mu_2)\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) dsd\tau \\ & + c(a-\mu_1)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) dsd\tau \\ & + d(a-\mu_1)\int_0^1 \left(f\left(s, \int_0^s u(\eta)d\eta, u'(s)\right) + q(s)u(s)\right) ds. \end{aligned}$$

(A_7) *There exists a constant* $\overline{M} > 0$ *such that for any* $\omega \in \mathbb{R}$*, if* $|\omega| > \overline{M}$*, then we have either*

$$\begin{aligned} 0 > \omega \Bigg(& (c\mu_1 - a\mu_2)\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0) + \omega q(s))dsd\tau \\ & + c(a-\mu_1)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0) + \omega q(s))dsd\tau \\ & + d(a-\mu_1)\int_0^1 (f(s,\omega s,0) + \omega q(s))ds \Bigg) \end{aligned}$$

or else

$$\begin{aligned} 0 < \omega \Bigg(& (c\mu_1 - a\mu_2)\int_0^\xi \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0) + \omega q(s))dsd\tau \\ & + c(a-\mu_1)\int_0^1 \frac{1}{p(\tau)}\int_0^\tau (f(s,\omega s,0) + \omega q(s))dsd\tau \\ & + d(a-\mu_1)\int_0^1 (f(s,\omega s,0) + \omega q(s))ds \Bigg). \end{aligned}$$

Then for each $q \in L^1[0,1]$*, the boundary value problem* (1.1)/(1.7) *when*

$$\mu_1\left(c\int_0^1 \frac{1}{p(\tau)}d\tau - c\int_0^\xi \frac{1}{p(\tau)}d\tau + d\right) + \mu_2\left(a\int_0^\xi \frac{1}{p(\tau)}d\tau + b\right) = ad + bc + ac\int_0^1 \frac{1}{p(\tau)}d\tau$$

has at least a solution in $C^1[0,1]$ *provided that*

$$\|\alpha\|_1 + \|\beta\|_1 < \frac{1 - (2\zeta + |b/a|)\,\|q\|_1}{2\zeta + |b/a|}.$$

Acknowledgment

The author is grateful to the referees for their very valuable comments and suggestions.

References

[1] Maya Briani, Claudia La Chioma, and Roberto Natalini. Convergence of numerical schemes for viscosity solutions to integro-differential degenerate parabolic problems arising in financial theory. *Numer. Math.*, 98(4):607–646, 2004.

[2] Rama Cont, Peter Tankov, and Ekaterina Voltchkova. Hedging with options in models with jumps. In *Stochastic analysis and applications*, volume 2 of *Abel Symp.*, pages 197–217. Springer, Berlin, 2007.

[3] W. Feng and J. R. L. Webb. Solvability of m-point boundary value problems with nonlinear growth. *J. Math. Anal. Appl.*, 212(2):467–480, 1997.

[4] W. Feng and J. R. L. Webb. Solvability of three point boundary value problems at resonance. In *Proceedings of the Second World Congress of Nonlinear Analysts, Part 6 (Athens, 1996)*, volume 30, pages 3227–3238, 1997.

[5] Chaitan P. Gupta. Existence theorems for a second order m-point boundary value problem at resonance. *Internat. J. Math. Math. Sci.*, 18(4):705–710, 1995.

[6] Chaitan P. Gupta. A second order m-point boundary value problem at resonance. *Nonlinear Anal.*, 24(10):1483–1489, 1995.

[7] Chaitan P. Gupta. Solvability of a multi-point boundary value problem at resonance. *Results Math.*, 28(3-4):270–276, 1995.

[8] V. A. Il′in and E. I. Moiseev. A nonlocal boundary value problem of the second kind for the Sturm-Liouville operator. *Differentsial′nye Uravneniya*, 23(8):1422–1431, 1471, 1987.

[9] Bing Liu. Solvability of multi-point boundary value problem at resonance. II. *Appl. Math. Comput.*, 136(2-3):353–377, 2003.

[10] Bing Liu and Jianshe Yu. Solvability of multi-point boundary value problem at resonance. III. *Appl. Math. Comput.*, 129(1):119–143, 2002.

[11] Weijiu Liu. The exponential stabilization of the higher-dimensional linear system of thermoviscoelasticity. *J. Math. Pures Appl. (9)*, 77(4):355–386, 1998.

[12] J. Mawhin. Topological degree and boundary value problems for nonlinear differential equations. In *Topological methods for ordinary differential equations (Montecatini Terme, 1991)*, volume 1537 of *Lecture Notes in Math.*, pages 74–142. Springer, Berlin, 1993.

[13] Donal O'Regan. *Theory of singular boundary value problems*. World Scientific Publishing Co. Inc., River Edge, NJ, 1994.

[14] Donal O'Regan. *Existence theory for nonlinear ordinary differential equations*, volume 398 of *Mathematics and its Applications*. Kluwer Academic Publishers Group, Dordrecht, 1997.