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Oscillation of Forced Second Order Dynamic Equations on Time Scales

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Abstract

In this paper, by introducing a nonnegative kernel function $H(t, s)$, some oscillation criteria for a forced second-order dynamic equation on time scales are given.

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1 Introduction

This paper concerns the oscillation of solutions to the forced second order dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x^{\sigma}) = e(t), \quad t \geq t_0 \tag{1.1}$$

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and

$$x^{\Delta\Delta}(t) - p(t)f(x^\sigma) = e(t), \quad t \geq t_0, \tag{1.2}$$

where $f(x^\sigma) = f(x(\sigma(t)))$, p and e are real-valued rd-continuous functions defined on a time scales interval $[a, b]$, and $p(t) > 0$, $e(t) \not\equiv 0$, (throughout $a, b \in \mathbb{T}$ with $a < b$). Our attention is restricted to those solutions of Eq. (1.1) which exist on some half line $[t_0, \infty)$ and satisfy $\sup\{|x(t)| : t \geq t_1\} > 0$ for any $t_1 > t_0$. A solution x of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative; otherwise it is nonoscillatory. Eq. (1.1) is said to be oscillatory if all its solutions are oscillatory.

In 1988, Hilger [7] introduced the theory of time scales in order to unify continuous and discrete analysis. Bohner and Peterson [4] summarizes and organizes much of time scales calculus. From then on, there has been much research activity concerning the oscillation and nonoscillation of solutions of different dynamic equations on time scales; we refer the reader to the papers [2, 3, 5, 6, 8]. We note that in spite of the great number of investigations of dynamic equations, for forced equations, oscillation theory has not yet been elaborated unlike that for differential equations. Motivated by the ideas in [1,9], by employing the Riccati technique and the integral averaging method, we shall establish criteria for oscillation of Eq. (1.1) and Eq. (1.2).

When $e(t) = 0$, Eq. (1.1) reduces to second order dynamic equation

$$x^{\Delta\Delta}(t) + p(t)f(x(\sigma(t))) = 0, \qquad t \in [a, b],$$

which has been discussed in many papers.

2 Some Preliminaries

In this section, we present some basic definitions and elementary results concerning the calculus on time scales.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. Since we are interested in oscillatory behavior, we suppose that the time scale under consideration is not bounded above, i.e., it is a time scale interval of the form $[a, \infty)$. We define the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\emptyset$ denotes the empty set. If $\sigma(t) > t$, we say that t is right-scattered, while if $\rho(t) < t$ we say that t is left-scattered. Points that are right-scattered and left-scattered at the same time are called isolated. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right-dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left-dense. Points that are right-dense and left-dense at the same time are called dense. Finally, the graininess function $\mu(t) : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t.$$

Throughout this paper we make the blanket assumption that $a \leq b$ are points in $\mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exist a finite left limit in all left-dense points. The derivative (delta) f^{Δ} of f is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}$$

if f is continuous at t and t is right-scattered. If t is not right-scattered then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s},$$

provided this limit exists. The derivative and the forward jump operator are related by the useful formula

$$f^{\sigma} = f + \mu f^{\Delta}, \quad \text{where} \quad f^{\sigma} = f \circ \sigma.$$

We will also make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g (where $gg^{\sigma} \neq 0$) of two differentiable functions f and g:

$$(fg)^{\Delta} = f^{\Delta} g + f^{\sigma} g^{\Delta} \quad \text{and} \quad \left(\frac{f}{g}\right)^{\Delta} = \frac{f^{\Delta} g - f g^{\Delta}}{g g^{\sigma}}.$$

For $a, b \in \mathbb{T}$ and a differentiable function f, the Cauchy integral of f^{Δ} is defined by

$$\int_a^b f^{\Delta}(t) \Delta t = f(b) - f(a).$$

The integration by parts formula reads

$$\int_a^b f^{\Delta}(s) g(\sigma(t)) \Delta s = [f(t)g(t)]_a^b - \int_a^b f(s) g^{\Delta}(s) \Delta s,$$

and infinite integrals are defined as

$$\int_a^{\infty} f(s) \Delta s = \lim_{t \to \infty} \int_a^t f(s) \Delta s.$$

If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = 0$, $\mu(t) = 0$, $f^{\Delta} = f'(t)$ and Eq. (1.1) becomes the second order differential equation

$$x''(t) + p(t) f(x(t)) = e(t), \qquad t \in [t_0, \infty).$$

If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$, $\mu(t) = 1$, $y^{\Delta}(t) = \Delta y(t) = y(t+1) - y(t)$, and Eq. (1.1) becomes the second order difference equation

$$\Delta(\Delta x(t)) + p(t) f(\sigma(x(t))) = e(t).$$

3 Main Results

In this section, we establish oscillation criteria for Eq. (1.1) and Eq. (1.2). Our approach is based largely on the application of $H(t,s)$ and the following lemma.

Lemma 3.1 (see [6]). *If A and B are positive constants, then*

$$A^\lambda + (\lambda-1)B^\lambda - \lambda A B^{\lambda-1} \geq 0, \quad \lambda > 1.$$

Let $D = \{(t,s) \in \mathbb{T}\times\mathbb{T} : t \geq s \geq t_0\}$, $H(t,s) \in C^1(D, \mathbb{R}_+)$, $\mathbb{R}_+ = (0,\infty)$, $f \in C(\mathbb{R}\times\mathbb{R},\mathbb{R})$, $xf(x) > 0$ for all $x \neq 0$.

Theorem 3.2. *Assume that*

$$H(t,t) = 0,\ \ t \geq t_0,\ \ H(t,s) > 0,\ \ (t,s) \in D, \tag{3.1}$$

$$h_i(t,s) = -[H(t,s)]_i^{\Delta s} \geq 0 \in D,\ \ h_i(t,t) = 0,\ \ t \geq t_0,\ \ i = 1,2, \tag{3.2}$$

$$0 \leq \liminf_{t\to\infty} \frac{h_i(t,t_0)}{H(t,t_0)} < \infty. \tag{3.3}$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H^\sigma e(s)\Delta s = \infty, \tag{3.4}$$

$$\liminf_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^t H^\sigma e(s)\Delta s = -\infty. \tag{3.5}$$

Then every solution of Eq. (1.1) *oscillates.*

Proof. Let x be a nonoscillatory solution of Eq. (1.1). Suppose that $x(t) > 0$ for $t \geq t_0$. Multiplying Eq. (1.1) by $H^\sigma = H(t,\sigma(s))$ for $t \geq t_0$ and integrating from t_0 to t, we have

$$\int_{t_0}^t H^\sigma x^{\Delta\Delta}(s)\Delta s + \int_{t_0}^t H^\sigma q(s) f(x^\sigma)\Delta s = \int_{t_0}^t H^\sigma e(s)\Delta s.$$

Since

$$\int_{t_0}^t H^\sigma x^{\Delta\Delta}(s)\Delta s = -H(t,t_0)x^\Delta(t_0) + \int_{t_0}^t h_1(t,s) x^\Delta(s)\Delta s$$

and

$$\int_{t_0}^t h_1(t,s)x^\Delta(s)\Delta s = -h_1(t,t_0)x(t_0) + \int_{t_0}^t h_2(t,s) x(\sigma(s))\Delta s,$$

we have

$$\begin{aligned}\int_{t_0}^t H^\sigma e(s)\Delta s &= -H(t,t_0)x^\Delta(t_0) - h_1(t,t_0)x(t_0) + \int_{t_0}^t h_2(t,s)x^\sigma \Delta s \\ &\quad + \int_{t_0}^t H^\sigma q(s) f(x^\sigma)\Delta s.\end{aligned}$$

Dividing by $H(t,t_0)$, we arrive at

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H^{\sigma}e(s)\Delta s &= \frac{-H(t,t_0)x^{\Delta}(t_0)-h_1(t,t_0)x(t_0)}{H(t,t_0)}\\ &\quad+\frac{1}{H(t,t_0)}\int_{t_0}^{t} h_2(t,s)x^{\sigma}\Delta s\\ &\quad+\frac{1}{H(t,t_0)}\int_{t_0}^{t} H^{\sigma}q(s)f(x^{\sigma})\Delta s.\end{aligned}$$

In view of (3.3), there exists a finite number M_1 such that for $t \geq t_0$

$$-x^{\Delta}(t_0)-\frac{h_1(t,t_0)}{H(t,t_0)}x(t_0) \geq M_1.$$

So we have

$$\begin{aligned}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H^{\sigma}e(s)\Delta s &\geq M_1+\frac{1}{H(t,t_0)}\int_{t_0}^{t} h_2(t,s)x^{\sigma}\Delta s\\ &\quad+\frac{1}{H(t,t_0)}\int_{t_0}^{t} H^{\sigma}q(s)f(x^{\sigma})\Delta s.\end{aligned}$$

Taking the liminf as $t \to \infty$, we obtain a desired contradiction to (3.5). The case $x(t) < 0$ can be handled similarly by using (3.4). $\square$

Let $H(t,s)=(t-s)^{\beta}$ with $\beta > 1$. It is easy to see that $H(t,s)$ satisfies all conditions of (3.1)–(3.3).

Corollary 3.3. *If*

$$\limsup_{t\to\infty}\frac{1}{(t-t_0)^{\beta}}\int_{t_0}^{t}(t-\sigma(s))^{\beta}e(s)\Delta s=\infty$$

and

$$\liminf_{t\to\infty}\frac{1}{(t-t_0)^{\beta}}\int_{t_0}^{t}(t-\sigma(s))^{\beta}e(s)\Delta s=-\infty,$$

then every solution of Eq. (1.1) *is oscillatory.*

Next, we consider oscillation criteria for Eq. (1.2).

Theorem 3.4. *Let the function H be as in Theorem 3.2 such that* (3.1)–(3.3) *hold, and suppose there exist two positive constants λ and c such that for all x either*

$$|f(x)| \geq c|x|^{\lambda} > 0, \quad \textit{for} \quad \lambda > 1, \tag{3.6}$$

or

$$|f(x)| \leq c|x|^{\lambda} > 0, \quad \textit{for} \quad 0 < \lambda < 1. \tag{3.7}$$

If

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H^\sigma e(s)-(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\left(\frac{h_2^\lambda(t,s)}{cq(s)H^\sigma}\right)^{\frac{1}{\lambda-1}}\right]\Delta s=\infty \tag{3.8}$$

and

$$\liminf_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H^\sigma e(s)-(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\left(\frac{h_2^\lambda(t,s)}{cq(s)H^\sigma}\right)^{\frac{1}{\lambda-1}}\right]\Delta s=-\infty, \tag{3.9}$$

then every solution of Eq. (1.2) *oscillates.*

Proof. Proceeding as in the proof of Theorem 3.2, we assume that Eq. (1.2) has a nonoscillatory solution, say $x(t)>0$ for $t\geq t_0$. Multiplying Eq. (1.2) by $H^\sigma=H(t,\sigma(s))$ for $t\geq t_0$ and integrating from t_0 to t, we have

$$\begin{aligned}\int_{t_0}^{t}H^\sigma e(s)\Delta s &= -H(t,t_0)x^\Delta(t_0)-h_1(t,t_0)x(t_0)+\int_{t_0}^{t}h_2(t,s)x^\sigma\Delta s\\ &\quad -\int_{t_0}^{t}H^\sigma q(s)f(x^\sigma)\Delta s\\ &= M(t,t_0)+\int_{t_0}^{t}h_2(t,s)x^\sigma\Delta s-\int_{t_0}^{t}H^\sigma q(s)f(x^\sigma)\Delta s.\end{aligned}$$

Case I: For $|f(x)|\geq c|x|^\lambda$, $\lambda>1$, we have

$$\begin{aligned}\int_{t_0}^{t}H^\sigma e(s)\Delta s &= M(t,t_0)+\int_{t_0}^{t}h_2(t,s)x^\sigma\Delta s-\int_{t_0}^{t}H^\sigma q(s)f(x^\sigma)\Delta s\\ &\leq M(t,t_0)+\int_{t_0}^{t}h_2(t,s)x^\sigma\Delta s-\int_{t_0}^{t}cH^\sigma q(s)x^\lambda(\sigma(s))\Delta s\\ &= M(t,t_0)-\int_{t_0}^{t}[cH^\sigma q(s)x^\lambda(\sigma(s))-h_2(t,s)x^\sigma]\Delta s.\end{aligned}$$

Set

$$A=[cH^\sigma q(s)]^{\frac{1}{\lambda}}x^\sigma,\quad B=\left(\frac{1}{\lambda}h_2(t,s)[cH^\sigma q(s)]^{-\frac{1}{\lambda}}\right)^{\frac{1}{\lambda-1}}$$

and apply Lemma 3.1 to obtain

$$\begin{aligned}&\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H^\sigma e(s)-(\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}}\left(\frac{h_2^\lambda(t,s)}{cq(s)H^\sigma}\right)^{\frac{1}{\lambda-1}}\right]\Delta s\\ &\qquad\qquad\leq -x^\Delta(t_0)-\frac{h_1(t,t_0)}{H(t,t_0)}x(t_0).\end{aligned}$$

Taking limsup as $t \to \infty$ in the above inequality, we obtain a contradiction to (3.8). The case $x(t) < 0$ can be handled similarly.
Case II: For $|f(x)| \le c|x|^\lambda > 0$, $0 < \lambda < 1$, we have

$$\begin{aligned}\int_{t_0}^{t} H^\sigma e(s)\Delta s &= M(t,t_0) + \int_{t_0}^{t} h_2(t,s)x^\sigma \Delta s - \int_{t_0}^{t} H^\sigma q(s) f(x^\sigma)\Delta s \\ &\ge M(t,t_0) + \int_{t_0}^{t} h_2(t,s)x^\sigma \Delta s - \int_{t_0}^{t} cH^\sigma q(s)x^\lambda(\sigma(s))\Delta s.\end{aligned}$$

For given t and s,

$$F(x) = h_2(t,s)x^\sigma - cH^\sigma q(s)x^\lambda(\sigma(s)), \quad x > 0, \ \ 0 < \lambda < 1$$

attains its minimum at $x^\sigma = \left(\dfrac{h_2(t,s)}{c\lambda H^\sigma q(s)}\right)^{\frac{1}{\lambda-1}}$, and

$$F_{\min} = (\lambda - 1)\lambda^{\frac{\lambda}{1-\lambda}} \left(\frac{h_2^\lambda(t,s)}{cq(s)H^\sigma}\right)^{\frac{1}{\lambda-1}}.$$

Thus, we obtain

$$\frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H^\sigma e(s) - (\lambda-1)\lambda^{\frac{\lambda}{1-\lambda}} \left(\frac{h_2^\lambda(t,s)}{cq(s)H^\sigma}\right)^{\frac{1}{\lambda-1}} \right] \Delta s \ge -x^\Delta(t_0) - \frac{h_1(t,t_0)}{H(t,t_0)} x(t_0).$$

Taking liminf as $t \to \infty$, we obtain a desired contradiction to (3.9). The case $x(t) < 0$ can be handled similarly. $\square$

Corollary 3.5. *Let the function H be as in Theorem 3.2 such that* (3.1)–(3.5) *hold, and suppose there exist two positive constants λ and c such that* (3.6) *and* (3.7) *hold. If*

$$\lim_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left(\frac{h_2^\lambda(t,s)}{cq(s)H^\sigma}\right)^{\frac{1}{\lambda-1}} \Delta s < \infty,$$

then every solution of Eq. (1.2) *is oscillatory.*

If $\lambda = 1$,

$$|f(x)| \ge |x| \quad \text{or} \quad |f(x)| \le |x|,$$

then the inequality in the proof of Case I reduces to

$$\int_{t_0}^{t} H^\sigma e(s)\Delta s \le M(t,t_0) - \int_{t_0}^{t} [cH^\sigma q(s) - h_2(t,s)]x^\sigma \Delta s.$$

We then have the following result.

Theorem 3.6. *Let the function H be as in Theorem 3.2 such that* (3.1)–(3.5) *hold, and* $|f(x)| \geq |x|$ *holds with*

$$h_2(t,s) - cH^\sigma q(s) \leq 0, \quad t \geq s \geq t_0$$

or $|f(x)| \leq |x|$ *holds with*

$$h_2(t,s) - cH^\sigma q(s) \geq 0, \quad t \geq s \geq t_0.$$

Then Eq. (1.2) *with $\lambda = 1$ is oscillatory.*

Example 3.7. Consider the differential equation ($\mathbb{T} = \mathbb{R}$)

$$x^{\Delta\Delta}(t) + x^\sigma(t) = e^t(2\cos t + \sin t), \tag{3.10}$$

where $q(t) = 1$, $e(t) = e^t(2\cos t + \sin t)$. Let $t_0 = 1$ and $H(t,s) = (t-s)^3$. Obviously,

$$\limsup_{t\to\infty} \frac{1}{(t-1)^3} \int_1^t (t - \sigma(s))^3 e^s \Delta s = \infty$$

and

$$\liminf_{t\to\infty} \frac{1}{(t-1)^3} \int_1^t (t - \sigma(s))^3 e^s \Delta s = -\infty.$$

By Corollary 3.3, we can see that every solution of Eq. (3.10) is oscillatory. In fact, $x(t) = e^t \sin t$ is a such solution for Eq. (3.10).

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