Blow-up for Degenerate and Singular Nonlinear Parabolic Systems with Nonlocal Source

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Abstract

Existence of a unique classical nonnegative solution is established and sufficient conditions for the solution that exists locally or blows up in finite time are obtained for the degenerate and singular parabolic system

\[ x^{p_1} u_t - (x^{r_1} u_x)_x = \int_0^a g(v(x, t)) dx, \quad x^{p_2} v_t - (x^{r_2} v_x)_x = \int_0^a f(u(x, t)) dx \]

in \((0, a) \times (0, T)\), where \(T \leq \infty\), \(a \geq 0\) are constants, \(f, g\) are given functions. Furthermore, under certain conditions it is proved that the blow-up set of the solution is the entire interval \([0, a]\). These extend a recent work of Zhou, Mu and Li, which considered the particular systems with localized sources.

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1 Introduction

In this paper, we consider degenerate and singular nonlinear reaction-diffusion equations with nonlocal source of the form

\[
\begin{aligned}
\begin{cases}
x^p u_t - (x^r u_x)_x = \int_0^a g(v(x, t)) dx, & (x, t) \in (0, a) \times (0, T), \\
x^p v_t - (x^r v_x)_x = \int_0^a f(u(x, t)) dx, & (x, t) \in (0, a) \times (0, T), \\
u(0, t) = u(a, t) = v(0, t) = v(a, t) = 0, & t \in (0, T), \\
u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in [0, a],
\end{cases}
\end{aligned}
\]

where \( u_0(x), v_0(x) \in C^{2+\alpha}([0, a]) \) for some \( \alpha \in (0, 1) \) are nonnegative nontrivial functions. \( u_0(0) = u_0(a) = v_0(0) = v_0(a) = 0, u_0 \) and \( v_0 \) satisfy compatibility conditions, \( T > 0, a > 0, r_1, r_2 \in [0, 1], |p_1| + r_1 \neq 0, |p_2| + r_2 \neq 0. \)

Let \( D = (0, a) \) and \( \Omega_t = D \times (0, t) \). \( \overline{D} \) and \( \overline{\Omega_t} \) are their closures, respectively. Since \( |p_1| + r_1 \neq 0, |p_2| + r_2 \neq 0 \), the coefficients of \( u_t, u_x, u_{xx} \) and \( v_t, v_x, v_{xx} \) may tend to 0 or \( \infty \) as \( x \) tends to 0, and thus we can regard the equations as degenerate and singular.


\[
\begin{aligned}
\begin{cases}
x^q u_t - u_{xx} = u^p, & (x, t) \in (0, a) \times (0, T), \\
u(0, t) = u(a, t) = 0, & t \in (0, T), \\
u(x, 0) = u_0(x), & x \in [0, a].
\end{cases}
\end{aligned}
\]

The motivation for studying problem (1.2) comes form Ockendon’s model (see [15]) for the flow in a channel of a fluid whose viscosity depends on temperature

\[ xu_t = u_{xx} + e^u, \]

where \( u \) represents the temperature of the fluid. Floater in [11] approximated \( e^u \) by \( u^p \) and considered equation (1.2). In [4], Chan and Chan considered the problem

\[
\begin{aligned}
\begin{cases}
x^q u_t - u_{xx} = f(u), & (x, t) \in (0, a) \times (0, T), \\
u(0, t) = u(a, t) = 0, & t \in (0, T), \\
u(x, 0) = u_0(x), & x \in [0, a].
\end{cases}
\end{aligned}
\]

For \( q = 0 \), it is the heat equation; the problem (1.3) and (1.2) (cf. [18, p. 10]) may be used to describe the temperature \( u(x, t) \) of a homogeneous and isotropic rod having a constant cross-sectional area with respect to \( x \), and a thermal conductivity \( K \) independent of \( x \); inside the rod, there is a nonlinear source producing heat (due to an exothermic reaction) at \( K f(u) \) per unit volume per unit time; the object has an initial distribution of temperature \( u_0(x) \), and the temperature at each of its ends is kept at zero.
For $q = 1$, the problem (1.3) may be used to describe the temperature $u$ of the channel flow of a fluid with a temperature-dependent viscosity in the boundary layer (cf. [5, 15]); here, $x$ and $t$ denote the coordinates perpendicular and parallel to the channel wall, respectively; hence, $t_b$ corresponds to the downstream position where $u$ blows up at some $x$. In a heat conduction problem with $t$ denoting the time, the term $x^q$ corresponds to the reciprocal of the diffusivity (cf. [2, p. 9]); thus for $q > 0$, the amount of heat required to raise the temperature of the object approaches zero as $x$ tends to zero; also for a fixed $x \in D$, $x^q$ is a decreasing function of $q$; physically, decreasing $x$ or increasing $q$ has the effect of shifting the blow-up point towards $x = 0$.

In [8], Chen and Xie discussed the degenerate and singular semilinear parabolic equation

\[
\begin{cases}
  u_t - (x^\alpha u_x)_x = \int_0^a f(u(x, t)) dx, & (x, t) \in (0, a) \times (0, T), \\
  u(0, t) = u(a, t) = 0, & t \in (0, T), \\
  u(x, 0) = u_0(x), & x \in [0, a].
\end{cases}
\]  

(1.4)

They established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and blow-up of a positive solution.

In [9], Chen et al. consider the following degenerate nonlinear reaction diffusion equation with nonlocal source

\[
\begin{cases}
  x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p dx, & (x, t) \in (0, a) \times (0, T), \\
  u(0, t) = u(a, t) = 0, & t \in (0, T), \\
  u(x, 0) = u_0(x), & x \in [0, a].
\end{cases}
\]

They established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they also got some sufficient conditions for the global existence and blow-up of a positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

Very recently, Jun Zhou et al. [18] generalized the results of [9] and investigated the blow-up properties of the following parabolic system

\[
\begin{cases}
  x^{q_1} u_t - (x^{r_1} u_x)_x = \int_0^a (v(x, t))^{p_1} dx, & (x, t) \in (0, a) \times (0, T), \\
  x^{q_2} v_t - (x^{r_2} v_x)_x = \int_0^a (u(x, t))^{p_2} dx, & (x, t) \in (0, a) \times (0, T), \\
  u(0, t) = u(a, t) = v(0, t) = v(a, t) = 0, & t \in (0, T), \\
  u(x, 0) = u_0(x), & x \in [0, a].
\end{cases}
\]

(1.5)

Under certain conditions, Jun Zhou et al. proved that the blow-up set of the solution of (1.5) is the whole domain. The existence of a unique classical nonnegative solution is
established and sufficient conditions for solution that exist globally or blows up in finite
time are obtained.

In [14], J. Li et al. considered the effect of the singularity, degeneracy and localized
reaction on the behavior of the solution of following problem

\[
\begin{cases}
  x^{p_1} u_t - (x^{r_1} u_x)_x = u^{q_1}(x_0, t), & (x, t) \in (0, a) \times (0, T), \\
  x^{p_2} v_t - (x^{r_2} v_x)_x = u^{q_2}(x_0, t), & (x, t) \in (0, a) \times (0, T), \\
  u(0, t) = u(a, t) = v(0, t) = v(a, t) = 0, & t \in (0, T), \\
  u(x, 0) = u_0(x), v(x, 0) = v_0(x), & x \in [0, a],
\end{cases}
\]  

and show that the blow-up set of the solution of (1.6) is the whole domain.

Motivated by the results of the papers [8, 9, 18], we slightly modify the method
developed by Jun Zhou et al. [18] and Y. Chen et al. [9] and extend the results of [9, 18]
to a degenerate and singular parabolic system (1.1). The difficulties are the construction
of the corresponding upper solution of (1.1). It is different from [6, 7, 11, 18] that under
certain conditions the blow-up set of the solution of (1.1) is the whole domain. But this
is consistent with the conclusions in [1, 16, 17].

Before stating our main results, we make some assumptions on the initial data
\(u_0(x), v_0(x)\) and \(f(s), g(s)\) as follows:

(A1) \( f, g \in C^2([0, \infty)), f(0) \geq 0, g(0) \geq 0, f'(s) > 0, g'(s) > 0, f''(s) \geq 0, \)
\(g''(s) \geq 0 \text{ for } s > 0 \text{ and } \int_{s_0}^{\infty} \frac{ds}{f(s)} < \infty, \int_{s_0}^{\infty} \frac{ds}{g(s)} < \infty \text{ for some } s_0 > 0, \)

(A2) \((u_0(x), v_0(x)) \in C^{2+\beta}([0, a]) \times C^{2+\beta}([0, a]) \) for some \(\beta \in (0, 1)\) and
\((u_0(0), v_0(0)) = (u_0(a), v_0(a)) = (0, 0), (u_0(x), v_0(x)) \neq (0, 0) \text{ if } x \in (0, a).\)

This paper is organized as follows. In the next section, we show the existence
of a unique classical solution. In Section 3, we give some criteria for the solution
\((u(x, t), v(x, t))\) to blow-up in finite time and discuss the blow-up set.

## 2 Local Existence

In this section, we start with the definition of an upper solution of system (1.1).

**Definition 2.1.** A pair of nonnegative functions \((\bar{u}(x, t), \bar{v}(x, t))\) is called an upper so-
lemma 2.3. blow-up for nonlinear parabolic systems

of nonnegative functions

proof. jun zhou et al. proved this lemma in [18], so we omit it.

in order to prove the existence of a unique positive solution to (1.1), we must construct the following comparison principle.

lemma 2.2. let \( b_1(x, t) \) and \( b_2(x, t) \) be continuous nonnegative functions defined on \([0, a] \times [0, r]\) for any \( r \in (0, T) \), and let \( (u(x, t), v(x, t)) \in (C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r))^2 \) satisfy

\[
\begin{align*}
    x^{p_1}u_t - (x^{r_1}u_x)_x &\geq \int_0^a b_1(x, t)v(x, t)dx, & (x, t) &\in (0, a) \times (0, r), \\
    x^{p_2}v_t - (x^{r_2}v_x)_x &\geq \int_0^a b_2(x, t)u(x, t)dx, & (x, t) &\in (0, a) \times (0, r), \\
    u(0, t) &\geq 0, \quad u(a, t) &\geq 0, \\
    v(0, t) &\geq 0, \quad v(a, t) &\geq 0, \\
    u(x, 0) &\geq 0, \quad v(x, 0) &\geq 0,
\end{align*}
\]

(2.1)

then \( u(x, t) \geq 0 \) and \( v(x, t) \geq 0 \) on \([0, a] \times [0, T]\).

proof. jun zhou et al. proved this lemma in [18], so we omit it.

lemma 2.3. let \( (u, v) \) be the nonnegative solution of (1.1). let us assume that a pair of nonnegative functions \((w(x, t), z(x, t)) \in (C(\overline{\Omega}_r) \cap C^{2,1}(\Omega_r))^2\) is such that

\[
\begin{align*}
    x^{p_1}w_t - (x^{r_1}w_x)_x &\geq \int_0^a g(z(x, t))dx, & (x, t) &\in (0, a) \times (0, r), \\
    x^{p_2}z_t - (x^{r_2}z_x)_x &\geq \int_0^a f(w(x, t))dx, & (x, t) &\in (0, a) \times (0, r), \\
    w(0, t) &\geq (=)0, \quad w(a, t) &\geq (=)0, \\
    z(0, t) &\geq (=)0, \quad z(a, t) &\geq (=)0, \\
    w(x, 0) &\geq (\leq)u_0(x), \quad z(x, 0) &\geq (\leq)v_0(x),
\end{align*}
\]

(2.2)

then \((w(x, t), z(x, t)) \geq (\leq)(u(x, t), v(x, t))\) on \([0, a] \times [0, T]\).
Proof. We only consider the case "\( > \)" (as for the other case "\( \leq \)" the proof is similar). Let \( \varphi_1(x, t) = w(x, t) - u(x, t) \) and \( \varphi_2(x, t) = z(x, t) - v(x, t) \). Subtracting (1.1) from (2.2) and using the mean value theorem, we obtain

\[
\begin{cases}
x^{p_1}\varphi_{1t} - (x^{r_1}\varphi_{1x})_x \geq \int_0^a g'(\eta_2)\varphi_2(x, t)dx, & (x, t) \in (0, a) \times (0, r], \\
x^{p_2}\varphi_{2t} - (x^{r_2}\varphi_{2x})_x \geq \int_0^a f'(\eta_1)\varphi_1(x, t)dx, & (x, t) \in (0, a) \times (0, r], \\
\varphi_1(0, t) \geq 0, & \varphi_1(a, t) \geq 0, \quad t \in (0, r], \\
\varphi_2(0, t) \geq 0, & \varphi_2(a, t) \geq 0, \quad t \in (0, r], \\
\varphi_1(x, 0) \geq 0, & \varphi_1(x, 0) \geq 0, \quad x \in [0, a],
\end{cases}
\]

where \( \eta_1 \) and \( \eta_2 \) are some intermediate values between \((w, u)\) and \((z, v)\) satisfying \( f'(\eta_1), g'(\eta_2) \geq 0 \). Then Lemma 2.2 ensures that \((\varphi_1(x, t), \varphi_2(x, t)) \geq (0, 0)\), that is, \((w(x, t), z(x, t)) \geq (u(x, t), v(x, t))\) on \([0, a] \times [0, T]\).

Obviously, \((0, 0)\) is a lower solution of (1.1), and we need to construct an upper solution. We modify the proof of Jun Zhou et al. [18, Lemma 2.2] to show the following result.

Lemma 2.4. There exists a positive constant \( t_0 \) \((t_0 < T)\) such that the problem (1.1) has an upper solution \((h_1(x, t), h_2(x, t)) \in (C(\Omega_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2\).

Proof. Let

\[
\varphi(x) = \left(\frac{x}{a}\right)^{1-r_2} \left(1 - \frac{x}{a}\right) + \left(\frac{x}{a}\right)^{1-r_2} \left(1 - \frac{x}{a}\right)^{\frac{1}{2}},
\]

and let \( K_0 \) be a positive constant such that

\[
K_0\varphi(x) + f(0) \geq f(u_0(x)), \quad K_0\varphi(x) + g(0) \geq g(v_0(x)).
\]

Introduce the positive constants \( M_1 = \sup_{x \in [0, a]} \psi(x) \), \( M_2 = \sup_{x \in [0, a]} \varphi(x) \) and

\[
\begin{align*}
b_{20} &= \int_0^1 \left(s^{1-r_1}(1 - s) + s^{\frac{1}{2}}(1 - s)^{\frac{1}{2}}\right) ds, \\
b_{10} &= \int_0^1 \left(s^{1-r_2}(1 - s) + s^{\frac{1}{2}}(1 - s)^{\frac{1}{2}}\right) ds, \quad N_1 = a f^{-1}(2K_0M_1 + f(0))(2b_{10}K_0 + g(0)), \\
N_2 &= a g^{-1}(2K_0M_2 + g(0))(2b_{20}K_0 + f(0)).
\end{align*}
\]
Let $K_{10} \in \left(0, \frac{1 - r_1}{2 - r_1}\right)$ and $K_{20} \in \left(0, \frac{1 - r_2}{2 - r_2}\right)$ be positive constants such that

$$K_{10} \leq \left(\frac{2a^{2-r_1}N_1}{K_0}\right)^{-\frac{2}{1-r_1}}, \quad K_{20} \leq \left(\frac{2a^{2-r_2}N_2}{K_0}\right)^{-\frac{2}{1-r_2}}.$$ 

Let $K_1(t), K_2(t)$ be the positive solutions of the initial value problems

$$K_1'(t) = \begin{cases} 
\frac{f'(f^{-1}(2K_0M_1 + f(0))) (b_{10}K_2(t) + g(0))}{a^{p_1-1}K_{10}^{-\frac{1}{2}}[K_{10}(1 - K_{10})^{1 - r_1} + K_{10}^{\frac{1}{2}}(1 - K_{10})^{\frac{1-r_1}{2}}]}, & p_1 \geq 0, \\
\frac{f'(f^{-1}(2K_0M_1 + f(0))) (b_{10}K_2(t) + g(0))}{a^{p_1-1}(1 - K_{10})^{p_1}[K_{10}(1 - K_{10})^{1 - r_1} + K_{10}^{\frac{1}{2}}(1 - K_{10})^{\frac{1-r_1}{2}}]}, & p_1 < 0,
\end{cases}$$

(2.3)

$$K_2'(t) = \begin{cases} 
g'(g^{-1}(2K_0M_2 + g(0))) (b_{20}K_1(t) + f(0))}{a^{p_2-1}K_{20}^{-\frac{1}{2}}[K_{20}(1 - K_{20})^{1 - r_2} + K_{20}^{\frac{1}{2}}(1 - K_{20})^{\frac{1-r_2}{2}}]}, & p_2 \geq 0, \\
a^{p_2-1}(1 - K_{20})^{p_2}[K_{20}(1 - K_{20})^{1 - r_2} + K_{20}^{\frac{1}{2}}(1 - K_{20})^{\frac{1-r_2}{2}}]}, & p_2 < 0,
\end{cases}$$

(2.4)

Since $K_1(t)$ and $K_2(t)$ are increasing functions, we can choose $t_0 > 0$ such that

$$K_1(t) \leq 2K_0, \quad K_2(t) \leq 2K_0 \quad \text{for all} \quad t \in [0, t_0].$$

Set

$$h_1(x, t) = f^{-1}(K_1(t)\psi(x) + f(0)), \quad h_2(x, t) = g^{-1}(K_2(t)\varphi(x) + g(0)).$$

Then $h_1(x, t) \geq 0$ and $h_2(x, t) \geq 0$ on $\Omega_{t_0}$. We show that $(h_1(x, t), h_2(x, t))$ is an upper solution of (1.1) in $\Omega_{t_0}$. To do this, let us construct two functions $J_1$ and $J_2$ by

$$J_1 = f'(h_1) \left[ x^{p_1}h_{1t} - (x^{r_1}h_{1x})_x - \int_0^a g(h_2)dx \right],$$

and

$$J_2 = g'(h_2) \left[ x^{p_2}h_{2t} - (x^{r_2}h_{2x})_x - \int_0^a f(h_1)dx \right].$$
Then

\[
J_1 = f'(h_1) \left[ x^{p_1} h_{1t} - (x^{r_1} h_{1x})_x - \int_0^a g(h_2) \, dx \right]
\]

\[
= x^{p_1} (f(h_1))_t - r_1 x^{r_1 - 1} (f(h_1))_x - x^{r_1} (f(h_1))_{xx}
- f'(h_1) \int_0^a g(h_2) \, dx + x^{r_1} f''(h_1) h_1^2
\]

\[
\geq x^{p_1} K'_1(t) \psi(x) - K_1(t) (r_1 x^{r_1 - 1} \psi_x + x^{r_1} \psi_{xx})
- f'(h_1) \int_0^a (K_2(t) \varphi(x) + g(0)) \, dx
\]

\[
= x^{p_1} K'_1(t) \psi(x) + K_1(t) \left[ \frac{2 - r_1}{a^2 - r_1} + \left( \frac{1 - r_1}{4} x^{r_1 - 3/2} (a - x)^{1/2}
+ \frac{1}{2} x^{r_1 - 1/2} (a - x)^{-1/2} + \frac{1}{4} x^{r_1 - 1/2} (a - x)^{-3/2} \right) \frac{1}{a^{1 - r_1/2}} \right]
- a f'\left( f^{-1}(K_1(t) \psi(x) + f(0)) \right) (b_{10} K_2(t) + g(0))
\]

\[
\geq x^{p_1} K'_1(t) \psi(x) + x^{(r_1 - 1)/2} (a - x)^{-1/2} K_1(t)
- a f'\left( f^{-1}(K_1(t) M_1 + f(0)) \right) (b_{10} K_2(t) + g(0)),
\]

\[
J_2 = g'(h_2) \left[ x^{p_2} h_{2t} - (x^{r_2} h_{2x})_x - \int_0^a f(h_1) \, dx \right]
\]

\[
\geq x^{p_2} K'_2(t) \varphi(x) + x^{(r_2 - 1)/2} (a - x)^{-1/2} K_2(t)
- a g'\left( g^{-1}(K_2(t) M_2 + g(0)) \right) (b_{20} K_1(t) + f(0)).
\]

For \((x, t) \in (0, a K_{10}) \times (0, t_0] \cup (a(1 - K_{10}), a) \times (0, t_0)\), we have

\[
J_1 \geq x^{(r_1 - 1)/2} (a - x)^{-1/2} K_1(t)
- a f'\left( f^{-1}(K_1(t) M_1 + f(0)) \right) (b_{10} K_2(t) + g(0))
\]

\[
\geq \left[ \frac{K_{10}^{r_1 - 1}}{2 a^{2 - r_1}} \right] K_0 - a f'\left( f^{-1}(2K_0 M_1 + f(0)) \right) (2b_{10} K_0 + g(0))
\]

\[
= \left[ \frac{K_{10}^{r_1 - 1}}{2 a^{2 - r_1}} \right] K_0 - N_1 \geq 0.
\]

For \((x, t) \in (0, a K_{20}) \times (0, t_0] \cup (a(1 - K_{20}), a) \times (0, t_0)\), we have

\[
J_2 \geq x^{(r_2 - 1)/2} (a - x)^{-1/2} K_2(t)
- a g'\left( g^{-1}(K_2(t) M_2 + g(0)) \right) (b_{20} K_1(t) + f(0))
\]

\[
\geq \left[ \frac{K_{20}^{r_2 - 1}}{2 a^{2 - r_2}} \right] K_0 - N_2 \geq 0.
\]
For \((x, t) \in [a K_1, a (1 - K_10)] \times (0, t_0]\) by (2.3), we have
\[
J_1 \geq \frac{d}{dt} \int_0^a (\frac{a \psi^p}{f(K_1)}(f^{-1}(K_1 t) M_1 + f(0))\big(b_{10} K_2(t) + g(0)\big)) u(0) dt + \int_0^t (\frac{a \psi^p}{f(K_1)}(f^{-1}(K_1 t) M_1 + f(0))\big(b_{10} K_2(t) + g(0)\big)) v(0) dt.
\]
For \((x, t) \in [a K_2, a (1 - K_20)] \times (0, t_0]\) by (2.4), we can get \(J_2 \geq 0\) with the same argument as that for \(J_1\). Thus, \(J_1(x, t) \geq 0, J_2(x, t) \geq 0\) in \(\Omega_{t_0}\). Since \(f' > 0\) and \(g'(s) > 0\) in \(\Omega_{t_0}\), we have
\[
x^{p_1} h_{11t} - (x^{r_1} h_{11x})_x - \int_0^a g(h_2) dx \geq 0, \quad x^{p_2} h_{21t} - (x^{r_2} h_{22x})_x - \int_0^a f(h_1) dx \geq 0 \quad \text{in} \quad \Omega_{t_0},
\]
and
\[
\begin{aligned}
    h_1(0, t) &= h_1(a, t) = f^{-1}(f(0)) = 0, \quad 0 < t < t_0, \\
    h_2(0, t) &= h_2(a, t) = g^{-1}(g(0)) = 0, \quad 0 < t < t_0, \\
    h_1(x, 0) &= f^{-1}(K_0 \psi(x) + f(0)) \geq f^{-1}(f(u_0(x))) = u_0(x), \\
    h_2(x, 0) &= g^{-1}(K_0 \varphi(x) + g(0)) \geq g^{-1}(g(v_0(x))) = v_0(x).
\end{aligned}
\]
So \((h_1(x, t), h_2(x, t))\) is an upper solution of (1.1). The proof is complete. \(\square\)

To show the existence of the classical solution \((u(x, t), v(x, t))\) of (1.1), let us introduce a cutoff function \(\rho(x)\). By Dunford and Schwartz [10, p. 1640], there exists a nondecreasing \(\rho(x) \in C^3(R)\) such that \(\rho(x) = 0\) if \(x \leq 0\) and \(\rho(x) = 1\) if \(x \geq 1\). Let
\[
0 < \delta < \min \left\{ \frac{1 - r_1}{a}, \frac{1 - r_2}{2 - r_2} a \right\}, \quad \rho_\delta(x) = \begin{cases} 
    0, & x \leq \delta, \\
    \rho \left( \frac{x}{\delta} - 1 \right), & \delta < x < 2\delta, \\
    1, & x \geq 2\delta.
\end{cases}
\]
and \(u_{0\delta}(x) = \rho_\delta(x) u_0(x), v_{0\delta}(x) = \rho_\delta(x) v_0(x)\). We note that
\[
\frac{\partial u_{0\delta}(x)}{\partial \delta} = \begin{cases} 
    0, & x \leq \delta, \\
    -\frac{x}{\delta^2} \rho'(\frac{x}{\delta} - 1) u_0(x), & \delta < x < 2\delta, \\
    0, & x \geq 2\delta.
\end{cases}
\]
By using Lemma 2.3, there exists at most one nonnegative solution
\( u_0(x) \geq u_{0\delta}(x), v_0(x) \geq v_{0\delta}(x) \) and \( \lim_{\delta \to 0} u_{0\delta}(x) = u_0(x), \lim_{\delta \to 0} v_{0\delta}(x) = v_0(x) \).

Let \( D_\delta = (\delta, a) \), let \( w_\delta = D_\delta \times (0, t_0) \), let \( \overline{D}_\delta \) and \( \overline{w}_\delta \) be their respective closures, and let \( S_\delta = \{0, a\} \times (0, t_0) \). We consider the following regularized problem

\[
\begin{align*}
\left\{ \begin{array}{l}
x^{p_1}u_{\delta t} - (x^{r_1}u_{\delta x})_x = \int_0^x g(v_\delta(x, t))dx, \\
x^{p_2}v_{\delta t} - (x^{r_2}v_{\delta x})_x = \int_0^x f(u_\delta(x, t))dx,
\end{array} \right. \\
u_\delta(0, t) = u_\delta(a, t) = v_\delta(0, t) = v_\delta(a, t) = 0, \\
u_\delta(x, 0) = u_{0\delta}(x), v_\delta(x, 0) = v_{0\delta}(x), \quad x \in \overline{D}_\delta.
\end{align*}
\]

By using Schauder’s fixed point theorem, we have the following theorem.

**Theorem 2.5.** The problem (2.5) admits a unique nonnegative solution
\( (u_\delta, v_\delta) \in (C^{2+\alpha, 1/2}(\overline{w}_\delta))^2 \).

Moreover, \( 0 \leq u_\delta \leq h_1(x, t), 0 \leq v_\delta \leq h_2(x, t), (x, t) \in \overline{w}_\delta \), where \( h_1(x, t) \) and \( h_2(x, t) \) are given by Lemma 2.4.

**Proof.** By using Lemma 2.3, there exists at most one nonnegative solution \( (u_\delta, v_\delta) \). To prove existence, we use the Schauder fixed point theorem. Let

\[
\begin{align*}
X_1 &= \{v_1 \in C^{\alpha, \alpha/2}(\overline{w}_\delta) : 0 \leq v_1(x, t) \leq h_2(x, t), (x, t) \in \overline{w}_\delta \}, \\
X_2 &= \{u_1 \in C^{\alpha, \alpha/2}(\overline{w}_\delta) : 0 \leq u_1(x, t) \leq h_1(x, t), (x, t) \in \overline{w}_\delta \}.
\end{align*}
\]

We note that \( X_1 \) and \( X_2 \) are closed convex subsets of the Banach space \( C^{\alpha, \alpha/2}(\overline{w}_\delta) \).

In order to obtain the conclusion, we define another set \( X = X_1 \times X_2 \). Obviously \((C^{\alpha, \alpha/2}(\overline{w}_\delta))^2\) is a Banach space with the norm

\[
\| (v_1, u_1) \|_{\alpha, \alpha/2} = \| v_1 \|_{\alpha, \alpha/2} + \| u_1 \|_{\alpha, \alpha/2}, \quad \text{for any} \quad (v_1, u_1) \in (C^{\alpha, \alpha/2}(\overline{w}_\delta))^2
\]

and \( X \) is a closed convex subset of the Banach space \((C^{\alpha, \alpha/2}(\overline{w}_\delta))^2\). For any \( (v_1, u_1) \in (X_1 \times X_2) \), let us consider the following linearized uniformly parabolic problem

\[
\begin{align*}
\left\{ \begin{array}{l}
x^{p_1}W_{\delta t} - (x^{r_1}W_{\delta x})_x = \int_0^x g(v_1)dx, \\
x^{p_2}Z_{\delta t} - (x^{r_2}Z_{\delta x})_x = \int_0^x f(u_1)dx,
\end{array} \right. \\
W_\delta(\delta, t) = W_\delta(a, t) = Z_\delta(\delta, t) = Z_\delta(a, t) = 0, \\
W_\delta(x, 0) = u_{0\delta}(x), Z_\delta(x, 0) = v_{0\delta}(x), \quad x \in [\delta, a].
\end{align*}
\]
By construction, \((0, 0)\) and \((h_1(x, t), h_2(x, t))\) are lower and upper solutions of problem (2.6). We also note that
\[
x^{-p_1 + r_1}, x^{-p_1 - 1 + r_1}, x^{-p_1}, x^{-p_2 + r_2}, x^{-p_2 - 1 + r_2}, x^{-p_2} \in C^{\alpha, \alpha/2}(\bar{\omega}_\delta)
\]
and
\[
x^{-p_1} \int_\delta^a g(v_1) dx, x^{-p_2} \int_\delta^a f(u_1) dx \in C^{\alpha, \alpha/2}(\bar{\omega}_\delta), \quad u_{0\delta}, v_{0\delta} \in C^{2+\alpha}(\bar{D}_\delta).
\]
It follows from Ladde et al. [12, Theorem 4.2.2 on p. 143] that problem (2.6) has a unique solution \((W_\delta(x, t; v_1, u_1), Z_\delta(x, t; v_1, u_1)) \in (C^{2+\alpha, 1+\alpha/2}(\bar{\omega}_\delta))^2\) such that
\[
0 \leq W_\delta(x, t; v_1, u_1) \leq h_1(x, t), \quad 0 \leq Z_\delta(x, t; v_1, u_1) \leq h_2(x, t).
\]
Thus, we can define a mapping \(T\) from \(X\) into \((C^{2+\alpha, 1+\alpha/2}(\bar{\omega}_\delta))^2\) such that
\[
T(v_1(x, t), u_1(x, t)) = (W_\delta(x, t; v_1, u_1), Z_\delta(x, t; v_1, u_1)),
\]
where \((W_\delta(x, t; v_1, u_1), Z_\delta(x, t; v_1, u_1))\) denotes the unique solution of (2.6) corresponding to \((v_1(x, t), u_1(x, t)) \in X\). To use the Schauder fixed point theorem, we need to verify that \(T\) maps \(X\) into itself and that \(T\) is continuous and compact. In fact, \(TX \subset X\), and the embedding operator from the Banach space \((C^{2+\alpha, 1+\alpha/2}(\bar{\omega}_\delta))^2\) to the Banach space \((C^{\alpha, \alpha/2}(\bar{\omega}_\delta))^2\) is compact. Therefore, \(T\) is compact. To show that \(T\) is continuous, let us consider sequence \(v_{1n}(x, t)\) which converges to \(v_1(x, t)\) uniformly and \(u_{1n}(x, t)\) which converges to \(u_1(x, t)\) uniformly in the norm \(\|\cdot\|_{\alpha, \alpha/2}\). We know that \(v_{1n}(x, t) \in X_1\) and \(u_{1n}(x, t) \in X_2\). So we get a sequence \(\{v_{1n}(x, t), u_{1n}(x, t)\}\) \(\in X\), which converges to \((v_1(x, t), u_1(x, t))\) uniformly in the norm \(\|\cdot\|_{\alpha, \alpha/2}\). Let \((W_{\delta n}(x, t), Z_{\delta n}(x, t))\) and \((W_\delta(x, t), Z_\delta(x, t))\) be the solutions of (2.6) corresponding to \((v_{1n}(x, t), u_{1n}(x, t))\) and \((v_1(x, t), u_1(x, t))\), respectively. Without loss of generality, let us assume that
\[
\|v_{1n}(x, t)\|_{\alpha, \alpha/2} \leq \|v_1(x, t)\|_{\alpha, \alpha/2} + 1, \quad \text{for any} \ n \geq 1,
\]
\[
\|u_{1n}(x, t)\|_{\alpha, \alpha/2} \leq \|u_1(x, t)\|_{\alpha, \alpha/2} + 1, \quad \text{for any} \ n \geq 1.
\]
Let \(W(x, t) = W_{\delta n}(x, t) - W_\delta(x, t), Z(x, t) = Z_{\delta n}(x, t) - Z_\delta(x, t)\). Then we have
\[
\begin{align*}
\begin{cases}
x^{p_1}W_t - (x^{r_1}W)_x = \int_\delta^a (g(v_{1n}) - g(v_1)) dx, & (x, t) \in \bar{w}_\delta; \\
x^{p_2}Z_t - (x^{r_2}Z)_x = \int_\delta^a (f(u_{1n}) - f(u_1)) dx, & (x, t) \in \bar{w}_\delta; \\
W(\delta, t) = W(a, t) = Z(\delta, t) = Z(a, t) = 0, & t \in (0, t_0], \\
W(x, 0) = 0, \quad Z(x, 0) = 0, & x \in \bar{D}_\delta.
\end{cases}
\end{align*}
\]
From Ladyženskaja et al. [13, Theorem 4.5.2 on p. 320], there exist positive constants \(C_1\) (independent of \(g, v_{1n}\) and \(v_1\)) and \(C_2\) (independent of \(f, u_{1n}\) and \(u_1\)) such that
\[
\|W\|_{2+\alpha, 1+\alpha/2} \leq C_1 \left| \int_\delta^a (g(v_{1n}) - g(v_1)) \right|_{\alpha, \alpha/2} \leq C_1 a \|g'(v_1 + \tau(v_{1n} - v_1))\|_{\alpha, \alpha/2} \|v_{1n} - v_1\|_{\alpha, \alpha/2},
\]
and we note that
\[
\|g'(v_1 + \tau(v_{1n} - v_1))\|_{a,a/2} \leq \|g'(h_2)\|_{\infty} \\
+ \sup_{\frac{t}{\tau} \leq a} \left| \frac{g'(v_1 + \tau(v_{1n} - v_1))(x, t) - g'(v_1 + \tau(v_{1n} - v_1))(\tilde{x}, \tilde{t})}{|x - \tilde{x}|^a} \right| \\
+ \sup_{0 \leq t \leq t_0} \left| \frac{g'(v_1 + \tau(v_{1n} - v_1))(x, t) - g'(v_1 + \tau(v_{1n} - v_1))(x, \hat{t})}{|t - \hat{t}|^{a/2}} \right|
\]
\[
\leq \|g'(h_2)\|_{\infty} + \max_{0 \leq s \leq h(x,t)} |g''(s)|(|v_1|_{a,a/2} + |v_{1n}|_{a,a/2}) \\
\leq \|g'(h_2)\|_{\infty} + \max_{0 \leq s \leq h(x,t)} |g''(s)| (2|v_1|_{a,a/2} + 1),
\]
where \( \tau \in (0, 1) \). Therefore,
\[
\|W\|_{2+a,1+a/2} \leq C_1 a \|g'(h_2)\|_{\infty} \\
+ \max_{0 \leq s \leq h(x,t)} |g''(s)| (2|v_1|_{a,a/2} + 1) \|v_{1n} - v_1\|_{a,a/2} \\
\leq C_1' \|v_{1n} - v_1\|_{a,a/2}
\]
and
\[
\|Z\|_{2+a,1+a/2} \leq C_2 a \|f'(h_1)\|_{\infty} \\
+ \max_{0 \leq s \leq h(x,t)} |f''(s)| (2|u_1|_{a,a/2} + 1) \|u_{1n} - u_1\|_{a,a/2} \\
\leq C_2' \|u_{1n} - u_1\|_{a,a/2}.
\]
It follows that
\[
\|(W, Z)\|_{2+a,1+a/2} = \|W\|_{2+a,1+a/2} + \|Z\|_{2+a,1+a/2} \leq C\|(v_{1n} - v_1, u_{1n} - u_1)\|_{a,a/2}.
\]
This shows that the mapping \( T \) is continuous. By the Schauder fixed point theorem, the proof is complete. \( \square \)

Now we can prove the following local existence result.

**Theorem 2.6.** There exists some \( t_0 < T \) such that problem (1.1) has a unique nonnegative solution \((u(x, t), v(x, t)) \in (C(\bar{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2\).

**Proof.** By Theorem 2.5, the problem (2.5) has a unique nonnegative solution \((u_{\delta}, v_{\delta}) \in (C^{2+a,1+a/2}(\bar{\Omega}_{\delta}))^2\). It follows from Lemma 2.3 that \((u_{\delta_1}, v_{\delta_1}) \leq (u_{\delta_2}, v_{\delta_2})\) if \( \delta_1 > \delta_2 \). Therefore, \( \lim_{\delta \to 0} (u_{\delta}(x, t), v_{\delta}(x, t)) \) exists for all \((x, t) \in (0, a] \times [0, t_0]\). Let
\[
(u(x, t), v(x, t)) = \lim_{\delta \to 0} (u_{\delta}(x, t), v_{\delta}(x, t)), \quad (x, t) \in (0, a] \times [0, t_0]
\]
and define \((u(0,t), v(0,t)) = (0,0), t \in [0, t_0]\). We show that \((u(x,t), v(x,t))\) is a classical solution of (1.1) in \(Q_0\). For any \((x_1, t_1) \in \Omega_{t_0}\), there exist three domains \(Q' = (a_1', a_2') \times (t_2', t_3']\), \(Q'' = (a_1'', a_2'') \times (t_2', t_3']\), \(Q''' = (a_1''', a_2''') \times (t_2', t_3']\)

such that \((x_1, t_1) \in Q' \subset Q'' \subset Q''' \subset (0, a) \times (0, t_0)\) with

\[
0 < a_1'' < a_1' < a_1'' < a_1' < a_2'' < a_2' < a_2'' < a
\]

and

\[
0 < t_2'' < t_2' < t_2 < t_3' < t_3 < t_3'' < t_0.
\]

By the conditions of \(f\) and \(g\), we know that \(h_1(x,t)\) and \(h_2(x,t)\) are finite on \(Q'''\). For any constant \(q > 1\) and some positive constants \(K_3\) and \(K_4\), we have

\[
\|u_\delta\|_{L^q(Q')} \leq \|h_1\|_{L^q(Q')} \leq K_3, \quad \|v_\delta\|_{L^q(Q')} \leq \|h_2\|_{L^q(Q')} \leq K_3,
\]

\[
\|x^{-p_1} \int_\delta^a g(v_\delta)dx\|_{H^{\alpha, \alpha/2}(Q''')} \leq (a_1')^{-p_1} \|\int_\delta^a g(h_2)dx\|_{\infty}
\]

\[
+ \sup_{(x,t) \in Q''} \left\| x^{-p_1} \int_\delta^a g(v_\delta)dx \cdot (x^{-p_1} - \tilde{x}^{-p_1}) \right\|_{\infty}
\]

\[
+ \sup_{(x,t) \in Q''} \left\| x^{-p_1} \int_\delta^a g(v_\delta(x,t) + \tau(v_\delta(x,t) - v_\delta(x,\tilde{t}))(v_\delta(x,t) - v_\delta(x,\tilde{t}))dx \right\|_{\infty}
\]

\[
\leq (a_1')^{-p_1} \|\int_\delta^a g(h_2)dx\|_{\infty}
\]

\[
+ \|\int_0^a g(h_2)dx\|_{\infty} \cdot \|x^{-p_1} \|_{H^{\alpha, \alpha/2}(Q''')} + (a_1')^{-p_1} \|\int_\delta^a g(h_2)dx\|_{\infty} \cdot \|v_\delta\|_{H^{\alpha, \alpha/2}(Q''')}
\]

and

\[
\left\| x^{-p_2} \int_\delta^a f(u_\delta)dx \right\|_{H^{\alpha, \alpha/2}(Q''')} \leq K_6
\]
for some positive $K_6$, which is independent of $\delta$, where $\tau \in (0, 1)$. By Ladyženskaja et al. [16, pp. 351–352], we have

$$\|u_\delta\|_{H^{2+\alpha, 1+\alpha/2}(Q')} \leq K_7, \quad \|v_\delta\|_{H^{2+\alpha, 1+\alpha/2}(Q')} \leq K_7$$

for some positive constant $K_7$ independent of $\delta$. This implies that $u_\delta, v_\delta, u_\delta t, u_\delta x, u_\delta xx$ and $v_\delta, v_\delta t, v_\delta x, v_\delta xx$ are equicontinuous in $Q'$. By the Ascoli–Arzela theorem, we know that

$$\|u\|_{H^{2+\alpha', 1+\alpha'/2}(Q')} \leq K_8, \quad \|v\|_{H^{2+\alpha', 1+\alpha'/2}(Q')} \leq K_8$$

for some $\alpha' \in (0, \alpha)$ and some positive constant $K_8$ independent of $\delta$, and that the derivatives of $u$ and $v$ are uniform limits of the corresponding partial derivatives of $u_\delta$ and $v_\delta$, respectively. Hence $(u(x, t), v(x, t))$ satisfy (1.1), and

$$\lim_{t \to 0}(u(x, t), v(x, t)) = \lim_{\delta \to 0}\lim_{t \to 0}(u_\delta(x, t), v_\delta(x, t)) = \lim_{\delta \to 0}(u_{0\delta}(x, t), v_{0\delta}(x, t)) = (u_0(x), v_0(x)).$$

It follows from $0 \leq u(x, t) \leq h_1(x, t), 0 \leq v(x, t) \leq h_2(x, t)$ and $h_1(x, t) \to 0, h_2(x, t) \to 0$ as $x \to 0$ or $x \to a$ that

$$\lim_{x \to 0}(u(x, t), v(x, t)) = \lim_{x \to a}(u(x, t), v(x, t)) = (0, 0).$$

Thus $(u, v) \in C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$ is the solution of (1.1) in $\Omega_{t_0}$. This completes the proof. $\square$

**Theorem 2.7.** Let $T$ be the supremum over $t_0$ for which there is a unique nonnegative solution $(u(x, t), v(x, t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$ of (1.1). Then (1.1) has a unique nonnegative solution $(u(x, t), v(x, t)) \in (C([0, a] \times [0, T]) \cap C^{2,1}([0, a] \times (0, T)))^2$. If $T < \infty$, then $\limsup_{t \to T} \max_{x \in [0, a]} |u(x, t)| + |v(x, t)| = \infty$.

**Proof.** The proof of this theorem is similar to the proof of [11, Theorem 2.5], so we omit it. $\square$

## 3 Blow-up of Solutions

In this section, we give some global blow-up result of the solution of (1.1). In order to obtain the blow-up result, we assume that $p_1 \geq r_1 - 1, p_2 \geq r_2 - 1$ and $f(s), g(s)$ satisfy

$$f(s) + g(t) \geq \eta \min\{f(s + t), g(s + t)\} \equiv h(s + t),$$

for some positive constant $\eta$.

**Remark 3.1.** Since $(s + t)^p \leq 2^{p-1}(s^p + t^p)$, power functions satisfy the property (3.1).
Now, we consider the eigenvalue problem
\[-(x^r \varphi_1'(x))' = \lambda_1 x^{p_1} \varphi_1(x), \quad x \in (0, a), \quad \varphi_1(0) = \varphi_1(a) = 0. \tag{3.2}\]
Using the argument by Jun Zhou et al. [18], we can make \( \varphi_1(x) \) satisfy
\[
\max_{x \in [0,a]} x^{p_1} \varphi_1(x) = 1. \tag{3.3}
\]
Analogously, we consider the eigenvalue problem
\[-(x^r \varphi_2'(x))' = \lambda_2 x^{p_2} \varphi_2(x), \quad x \in (0, a), \quad \varphi_2(0) = \varphi_2(a) = 0. \tag{3.4}\]
As above, we can obtain
\[
\max_{x \in [0,a]} x^{p_2} \varphi_2(x) = 1. \tag{3.5}
\]
Let
\[
C_1 = \int_0^a \varphi_1(x)dx, \quad C_2 = \int_0^a \varphi_2(x)dx \quad \text{and} \quad \lambda = \max\{\lambda_1, \lambda_2\}, \quad C = \min\{C_1, C_2\}.
\]
Then we have the following result.

**Theorem 3.2.** Let \((u(x, t), v(x, t))\) be the solution of problem (1.1). Then the solution of (1.1) blows up in finite time.

**Proof.** We set
\[
U(t) = \int_0^a x^{p_1} \varphi_1(x)u(x, t)dx, \quad V(t) = \int_0^a x^{p_2} \varphi_2(x)v(x, t)dx.
\]
By (1.1), (3.2) and (3.4), we have
\[
U'(t) = \int_0^a x^{p_1} \varphi_1(x)u_t(x, t)dx = \int_0^a \left( (x^{r_1} u_x)_x + \int_0^a g(v)dx \right) \varphi_1(x)dx \geq -\lambda_1 U(t) + C_1 \int_0^a g(v)dx
\]
and
\[
V'(t) \geq -\lambda V(t) + C \int_0^a f(u)dx.
\]
Using Jensen’s inequality and (3.3), (3.5), we get
\[
U'(t) \geq -\lambda U(t) + C \int_0^a g(v)dx \geq -\lambda U(t) + C ag \left( \frac{1}{a} \int_0^a vdx \right) \geq -\lambda U(t) + C ag \left( \frac{V(t)}{a} \right)
\]
\geq -\lambda U(t) + C \left( \frac{V(t)}{a} \right) \tag{3.6}
\]
and

\[ V'(t) \geq -\lambda V(t) + Ca \left( \frac{U(t)}{a} \right). \tag{3.7} \]

Since \( \int_{s_0}^{\infty} \frac{ds}{f(s)} < \infty \), \( \int_{s_0}^{\infty} \frac{ds}{g(s)} < \infty \), and \( h(s + t) = h(y) \), we have \( \int_{s_0}^{\infty} \frac{dy}{h(y)} < \infty \).

Then we can obtain

\[ \lim_{y \to \infty} \frac{h(y)}{y} = \infty. \]

In fact, we have \( \lim_{y \to \infty} h(y) = \infty \) for \( \int_{s_0}^{\infty} \frac{ds}{h(y)} < \infty \). By \( h'(y) \geq 0 \) (by \( f''(s) \geq 0 \), \( g''(s) \geq 0 \)), we have that \( h'(y) \) is nondecreasing if \( y > 0 \).

Using L’Hospital’s principle, we obtain

\[ \lim_{y \to \infty} \frac{h(y)}{y} = \lim_{y \to \infty} h'(y). \]

Assume by contradiction that \( \lim_{y \to \infty} h'(y) = N < \infty \). Then there exists \( y_0 \geq s_0 \) such that \( h(y) \leq \frac{3}{2} Ny \), and we have

\[ \int_{s_0}^{\infty} \frac{dy}{h(y)} \geq \frac{2}{3N} \int_{s_0}^{\infty} \frac{dy}{y} = \infty. \]

Now, by \( \lambda > 0 \), by \( \lim_{y \to \infty} \frac{h(y)}{y} = \infty \), we know that there exists \( s_1 \geq s_0 \) such that \( \frac{h(y)}{y} \geq \frac{2\lambda}{C} \) if \( y \geq s_1 \). Let \( (u_0(x), v_0(x)) \) be sufficiently larger such that

\[ \int_{a}^{0} x^{p_1} u_0(x) \varphi_1(x) \geq \frac{as_1}{2}, \quad \int_{0}^{a} x^{p_2} v_0(x) \varphi_2(x) \geq \frac{as_1}{2}. \]

Now, by (3.6), (3.7), we have

\[ U'(t) + V'(t) = Ca \left( g \left( \frac{V(t)}{a} \right) + f \left( \frac{U(t)}{a} \right) \right) - \lambda(U(t) + V(t)) \]

\[ \geq Ca \left( \frac{U(t) + V(t)}{a} \right) - \lambda \frac{U(t) + V(t)}{a}, \]

and integrating this inequality over \( t \) from 0 to \( T \), we have

\[ T \leq a \int_{0}^{T} \frac{dU(t)+V(t)}{Ca h(U(t)+V(t)) - \lambda a(U(t)+V(t))} \]

\[ = \frac{1}{C} \int_{s_0}^{s_1} \frac{dy}{h(y)} - \frac{\lambda y}{C} \leq \frac{2}{C} \int_{s_0}^{\infty} \frac{dy}{h(y)} < \infty. \]

This completes the proof. \( \square \)

Now, we discuss the global blow-up under the following hypothesis.
Case 1: $p_1 > 0$, $r_1 = 0$ or $p_2 > 0$, $r_2 = 0$.

Chan et al. [3,7] showed that Green’s function $G(x, \xi, t - \tau)$ associated with the operator $L = x^p(\partial / \partial t) - \partial^2 / \partial^2 x$ with the first boundary condition exists. For ease of reference, we state their results in the following lemma.

**Lemma 3.3.** (i) For $t > \tau$, $G(x, \xi, t - \tau)$ is continuous for $(x, t, \xi, \tau) \in ([0, a] \times (0, T]) \times ([0, a] \times [0, T])$.

(ii) For each fixed $(\xi, \tau) \in (0, a] \times [0, T)$, $G_t(x, \xi, t - \tau) \in C([0, a] \times (\tau, T])$.

(iii) In $\{(x, t, \xi, \tau): x$ and $\xi$ are in $(0, a)$, $t \geq \tau \geq 0\}$, $G(x, \xi, t - \tau)$ is positive.

**Lemma 3.4.** For fixed $x_0 \in (0, a)$, given any $x \in (0, a)$ and any finite time $T$, there exist positive constants $C_1$ (depending on $x$ and $T$) and $C_2$ (depending on $T$) such that

$$\int_0^a G(x, \xi, t) d\xi > C_1, \quad \int_0^a G(x_0, \xi, t) d\xi < C_2.$$ 

Now we give the global blow-up result.

**Theorem 3.5.** Under the assumption of Case 1, if the solution of (1.1) blows up at the point $x_0 \in (0, a)$, then the blow-up set of the solution of (1.1) is $[0, a]$.

**Proof.** Obviously, the system (1.1) is completely coupled. Therefore, $u$ and $v$ blow up simultaneously if the solution $(u, v)$ blows up in finite time. Without loss of generality, we assume $p_1 > 0$, $r_1 = 0$, and $u(x, t)$ blows up in finite time $T$. By Green’s second identity we have

$$u(x, t) = \int_0^a \xi^{p_1} G(x, \xi, t) u_0(\xi) d\xi + \int_0^t \int_0^a G(x, \xi, t - \tau) \int_0^a g(v(y, \tau)) dy d\xi d\tau \quad (3.8)$$

for any $(x, t) \in (0, a) \times (0, T)$. Since $u(x, t)$ blows up at $x = x_0$, we have $\lim_{t \to T} u(x_0, t) = \infty$. By (3.8) and Lemma 3.4, we have

$$u(x_0, t) = \int_0^a \xi^{p_1} G(x_0, \xi, t) u_0(\xi) d\xi + \int_0^t \int_0^a G(x_0, \xi, \tau) \int_0^a g(v(y, t - \tau)) dy d\xi d\tau \leq C_2 a^{p_1} \max_{x \in [0, a]} u_0(x) + C_2 \int_0^t \int_0^a g(v(y, t - \tau)) dy d\tau,$$

and thus

$$\lim_{t \to T} \int_0^t \int_0^a g(v(y, t - \tau)) dy d\tau = \infty. \quad (3.9)$$
On the other hand, for any \( x \in (0, a) \), we have

\[
\begin{align*}
  u(x, t) &\geq \int_0^a \xi^{p_1} G(x, \xi, t) u_0(\xi) d\xi + C_1 \int_0^t \int_0^a g(v(y, t - \tau)) dy d\tau \\
  &\geq C_1 \int_0^t \int_0^a g(v(y, t - \tau)) dy d\tau, \quad t \in (0, T).
\end{align*}
\]

It follows from the above inequality and (3.9) that \( \lim_{t \to T} u(x, t) = \infty \). For any \( \tilde{x} \in \{0, a\} \), we can always find a sequence \( \{(x_n, t_n)\} \) such that \( (x_n, t_n) \to (\tilde{x}, T) \) \( (n \to \infty) \) and \( \lim_{t \to T} u(x_n, t_n) = \infty \). Thus, the blow-up set is \([0, a]\), and this completes the proof.

**Case 2:** \( p_1 = 0, \ 0 \leq r_1 < 1 \) or \( p_2 = 0, \ 0 \leq r_2 < 1 \).

We assert that the blow-up set is the whole domain under certain assumptions.

**Theorem 3.6.** Under the assumption of Case 2 and if there exists \( M \in (0, +\infty) \) such that

\[
(x^{r_1} u_0(x))_x \leq M \quad \text{or} \quad (x^{r_2} v_0(x))_x \leq M \quad \text{in} \quad (0, a),
\]

if the solution of (1.1) blows up at the point \( x_0 \in (0, a) \), then the blow-up set of the solution of (1.1) is \([0, a]\).

**Proof.** The proof is similar to the proof presented in [8, 18], so we omit it.

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**References**


