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Blow-up for Degenerate and Singular Nonlinear Parabolic Systems with Nonlocal Source

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Abstract

Existence of a unique classical nonnegative solution is established and sufficient conditions for the solution that exists locally or blows up in finite time are obtained for the degenerate and singular parabolic system $x^{p_1}u_t - (x^{r_1}u_x)_x = \int_0^a g(v(x,t))dx$, $x^{p_2}v_t - (x^{r_2}v_x)_x = \int_0^a f(u(x,t))dx$ in $(0,a) \times (0,T)$, where $T \leq \infty$, $a \geq 0$ are constants, f, g are given functions. Furthermore, under certain conditions it is proved that the blow-up set of the solution is the entire interval $[0,a]$. These extend a recent work of Zhou, Mu and Li, which considered the particular systems with localized sources.

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1 Introduction

In this paper, we consider degenerate and singular nonlinear reaction-diffusion equations with nonlocal source of the form

$$\begin{cases} x^{p_1}u_t - (x^{r_1}u_x)_x = \int_0^a g(v(x,t))dx, & (x,t) \in (0,a) \times (0,T), \\ x^{p_2}v_t - (x^{r_2}v_x)_x = \int_0^a f(u(x,t))dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = v(0,t) = v(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in [0,a], \end{cases} \tag{1.1}$$

where $u_0(x), v_0(x) \in C^{2+\alpha}([0,a])$ for some $\alpha \in (0,1)$ are nonnegative nontrivial functions. $u_0(0) = u_0(a) = v_0(0) = v_0(a) = 0$, u_0 and v_0 satisfy compatibility conditions, $T > 0$, $a > 0$, $r_1, r_2 \in [0,1)$, $|p_1| + r_1 \neq 0$, $|p_2| + r_2 \neq 0$.

Let $D = (0,a)$ and $\Omega_t = D \times (0,t]$. $\overline{D}$ and $\overline{\Omega}_t$ are their closures, respectively. Since $|p_1| + r_1 \neq 0$, $|p_2| + r_2 \neq 0$, the coefficients of u_t, u_x, u_{xx} and v_t, v_x, v_{xx} may tend to 0 or ∞ as x tends to 0, and thus we can regard the equations as degenerate and singular.

Floater [11] and Chan and Liu [6] investigated the blow-up properties of the problem

$$\begin{cases} x^q u_t - u_{xx} = u^p, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.2}$$

The motivation for studying problem (1.2) comes form Ockendon's model (see [15]) for the flow in a channel of a fluid whose viscosity depends on temperature

$$xu_t = u_{xx} + e^u,$$

where u represents the temperature of the fluid. Floater in [11] approximated e^u by u^p and considered equation (1.2). In [4], Chan and Chan considered the problem

$$\begin{cases} x^q u_t - u_{xx} = f(u), & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.3}$$

For $q = 0$, it is the heat equation; the problem (1.3) and (1.2) (cf. [18, p. 10]) may be used to describe the temperature $u(x,t)$ of a homogeneous and isotropic rod having a constant cross-sectional area with respect to x, and a thermal conductivity K independent of x; inside the rod, there is a nonlinear source producing heat (due to an exothermic reaction) at $Kf(u)$ per unit volume per unit time; the object has an initial distribution of temperature $u_0(x)$, and the temperature at each of its ends is kept at zero.

For $q = 1$, the problem (1.3) may be used to describe the temperature u of the channel flow of a fluid with a temperature-dependent viscosity in the boundary layer (cf. [5,15]); here, x and t denote the coordinates perpendicular and parallel to the channel wall, respectively; hence, t_b corresponds to the downstream position where u blows up at some x. In a heat conduction problem with t denoting the time, the term x^q corresponds to the reciprocal of the diffusivity (cf. [2, p. 9]); thus for $q > 0$, the amount of heat required to raise the temperature of the object approaches to zero as x tends to zero; also for a fixed $x \in D$, x^q is a decreasing function of q; physically, decreasing x or increasing q has the effect of shifting the blow-up point towards $x = 0$.

In [8], Chen and Xie discussed the degenerate and singular semilinear parabolic equation

$$\begin{cases} u_t - (x^\alpha u_x)_x = \int_0^a f(u(x,t))dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases} \tag{1.4}$$

They established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they obtained some sufficient conditions for the global existence and blow-up of a positive solution.

In [9], Chen et al. consider the following degenerate nonlinear reaction diffusion equation with nonlocal source

$$\begin{cases} x^q u_t - (x^\gamma u_x)_x = \int_0^a u^p dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,a]. \end{cases}$$

They established the local existence and uniqueness of a classical solution. Under appropriate hypotheses, they also got some sufficient conditions for the global existence and blow-up of a positive solution. Furthermore, under certain conditions, it is proved that the blow-up set of the solution is the whole domain.

Very recently, Jun Zhou et al. [18] generalized the results of [9] and investigated the blow-up properties of the following parabolic system

$$\begin{cases} x^{q_1} u_t - (x^{r_1} u_x)_x = \int_0^a (v(x,t))^{p_1} dx, & (x,t) \in (0,a) \times (0,T), \\ x^{q_2} v_t - (x^{r_2} v_x)_x = \int_0^a (u(x,t))^{p_2} dx, & (x,t) \in (0,a) \times (0,T), \\ u(0,t) = u(a,t) = v(0,t) = v(a,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \;\; v(x,0) = v_0(x), & x \in [0,a]. \end{cases} \tag{1.5}$$

Under certain conditions, Jun Zhou et al. proved that the blow-up set of the solution of (1.5) is the whole domain. The existence of a unique classical nonnegative solution is established and sufficient conditions for solution that exist globally or blows up in finite time are obtained.

In [14], J. Li et al. considered the effect of the singularity, degeneracy and localized reaction on the behavior of the solution of following problem

$$\begin{cases} x^{p_1}u_t-(x^{r_1}u_x)_x=v^{q_1}(x_0,t), & (x,t)\in(0,a)\times(0,T),\\ x^{p_2}v_t-(x^{r_2}v_x)_x=u^{q_2}(x_0,t), & (x,t)\in(0,a)\times(0,T),\\ u(0,t)=u(a,t)=v(0,t)=v(a,t)=0, & t\in(0,T),\\ u(x,0)=u_0(x),\ \ v(x,0)=v_0(x), & x\in[0,a], \end{cases} \tag{1.6}$$

and show that the blow-up set of the solution of (1.6) is the whole domain.

Motivated by the results of the papers [8, 9, 18], we slightly modify the method developed by Jun Zhou et al. [18] and Y. Chen et al. [9] and extend the results of [9,18] to a degenerate and singular parabolic system (1.1). The difficulties are the construction of the corresponding upper solution of (1.1). It is different from [6,7,11,18] that under certain conditions the blow-up set of the solution of (1.1) is the whole domain. But this is consistent with the conclusions in [1,16,17].

Before stating our main results, we make some assumptions on the initial data $u_0(x), v_0(x)$ and $f(s), g(s)$ as follows:

(A$_1$) $f,g\in C^2([0,\infty))$, $f(0)\geq 0$, $g(0)\geq 0$, $f'(s)>0$, $g'(s)>0$, $f''(s)\geq 0$, $g''(s)\geq 0$ for $s>0$ and $\displaystyle\int_{s_0}^{\infty}\frac{ds}{f(s)}<\infty$, $\displaystyle\int_{s_0}^{\infty}\frac{ds}{g(s)}<\infty$ for some $s_0>0$,

(A$_2$) $(u_0(x),v_0(x))\in C^{2+\beta}([0,a])\times C^{2+\beta}([0,a])$ for some $\beta\in(0,1)$ and

$$(u_0(0),v_0(0))=(u_0(a),v_0(a))=(0,0),\ (u_0(x),v_0(x))\neq(0,0) \text{ if } x\in(0,a).$$

This paper is organized as follows. In the next section, we show the existence of a unique classical solution. In Section 3, we give some criteria for the solution $(u(x,t),v(x,t))$ to blow-up in finite time and discuss the blow-up set.

2 Local Existence

In this section, we start with the definition of an upper solution of system (1.1).

Definition 2.1. A pair of nonnegative functions $(\bar{u}(x,t),\bar{v}(x,t))$ is called an upper solution of (1.1) if $(\overline{u}(x,t),\overline{v}(x,t)) \in (C([0,a]\times[0,T))^2$ is such that

$$
\begin{cases}
x^{p_1}\bar{u}_t - (x^{r_1}\bar{u}_x)_x \geq \int_0^a g(\bar{v}(x,t))dx, & (x,t)\in(0,a)\times(0,T),\\
x^{p_2}\bar{v}_t - (x^{r_2}\bar{v}_x)_x \geq \int_0^a f(\bar{u}(x,t))dx, & (x,t)\in(0,a)\times(0,T),\\
\bar{u}(0,t)\geq 0, \bar{u}(a,t)\geq 0, & t\in(0,T)\\
\bar{v}(0,t)\geq 0, \bar{v}(a,t)\geq 0, & t\in(0,T),\\
\bar{u}(x,0)\geq \bar{u}_0(x), \ \ \bar{v}(x,0)\geq \bar{v}_0(x), & x\in[0,a].
\end{cases}
\tag{2.1}
$$

Similarly, $(\underline{u}(x,t),\underline{v}(x,t)) \in (C([0,a]\times[0,T))^2$ is called a lower solution if it satisfies all the reversed inequalities in (2.1).

In order to prove the existence of a unique positive solution to (1.1), we must construct the following comparison principle.

Lemma 2.2. *Let* $b_1(x,t)$ *and* $b_2(x,t)$ *be continuous nonnegative functions defined on* $[0,a]\times[0,r]$ *for any* $r\in(0,T)$*, and let* $(u(x,t),v(x,t)) \in (C(\overline{\Omega}_r)\cap C^{2,1}(\Omega_r))^2$ *satisfy*

$$
\begin{cases}
x^{p_1}u_t - (x^{r_1}u_x)_x \geq \int_0^a b_1(x,t)v(x,t)dx, & (x,t)\in(0,a)\times(0,r],\\
x^{p_2}v_t - (x^{r_2}v_x)_x \geq \int_0^a b_2(x,t)u(x,t)dx, & (x,t)\in(0,a)\times(0,r],\\
u(0,t)\geq 0, \ \ u(a,t)\geq 0, \ \ v(0,t)\geq 0, \ \ v(a,t)\geq 0, & t\in(0,r],\\
u(x,0)\geq 0, \ \ v(x,0)\geq 0, & x\in[0,a].
\end{cases}
$$

Then $u(x,t)\geq 0$ *and* $v(x,t)\geq 0$ *on* $[0,a]\times[0,T)$*.*

Proof. Jun Zhou et al. proved this lemma in [18], so we omit it. □

Lemma 2.3. *Let* (u,v) *be the nonnegative solution of* (1.1)*. Let us assume that a pair of nonnegative functions* $(w(x,t),z(x,t)) \in (C(\overline{\Omega}_r)\cap C^{2,1}(\Omega_r))^2$ *is such that*

$$
\begin{cases}
x^{p_1}w_t - (x^{r_1}w_x)_x \geq (\leq) \int_0^a g(z(x,t))dx, & (x,t)\in(0,a)\times(0,r],\\
x^{p_2}z_t - (x^{r_2}z_x)_x \geq (\leq) \int_0^a f(w(x,t))dx, & (x,t)\in(0,a)\times(0,r],\\
w(0,t)\geq(=)0, \ \ w(a,t)\geq(=)0, & t\in(0,r],\\
z(0,t)\geq(=)0, \ \ z(a,t)\geq(=)0, & t\in(0,r],\\
w(x,0)\geq(\leq)u_0(x), \ \ z(x,0)\geq(\leq)v_0(x), & x\in[0,a].
\end{cases}
\tag{2.2}
$$

Then $(w(x,t),z(x,t))\geq(\leq)(u(x,t),v(x,t))$ *on* $[0,a]\times[0,T)$*.*

Proof. We only consider the case " $\geq$ " (as for the other case " $\leq$ " the proof is similar). Let $\varphi_1(x,t) = w(x,t) - u(x,t)$ and $\varphi_2(x,t) = z(x,t) - v(x,t)$. Subtracting (1.1) from (2.2) and using the mean value theorem, we obtain

$$\begin{cases} x^{p_1}\varphi_{1t} - (x^{r_1}\varphi_{1x})_x \geq \int_0^a g'(\eta_2)\varphi_2(x,t)dx, & (x,t) \in (0,a) \times (0,r], \\ x^{p_2}\varphi_{2t} - (x^{r_2}\varphi_{2x})_x \geq \int_0^a f'(\eta_1)\varphi_1(x,t)dx, & (x,t) \in (0,a) \times (0,r], \\ \varphi_1(0,t) \geq 0, \quad \varphi_1(a,t) \geq 0, & t \in (0,r], \\ \varphi_2(0,t) \geq 0, \quad \varphi_2(a,t) \geq 0, & t \in (0,r], \\ \varphi_1(x,0) \geq 0, \quad \varphi_1(x,0) \geq 0, & x \in [0,a], \end{cases}$$

where η_1 and η_2 are some intermediate values between (w,u) and (z,v) satisfying $f'(\eta_1), g'(\eta_2) \geq 0$. Then Lemma 2.2 ensures that $(\varphi_1(x,t), \varphi_2(x,t)) \geq (0,0)$, that is, $(w(x,t), z(x,t)) \geq (u(x,t), v(x,t))$ on $[0,a] \times [0,T)$. $\square$

Obviously, $(0,0)$ is a lower solution of (1.1), and we need to construct an upper solution. We modify the proof of Jun Zhou et al. [18, Lemma 2.2] to show the following result.

Lemma 2.4. *There exists a positive constant* $t_0 (t_0 < T)$ *such that the problem* (1.1) *has an upper solution* $(h_1(x,t), h_2(x,t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$.

Proof. Let

$$\psi(x) = \left(\frac{x}{a}\right)^{1-r_1}\left(1 - \frac{x}{a}\right) + \left(\frac{x}{a}\right)^{\frac{1-r_1}{2}}\left(1 - \frac{x}{a}\right)^{\frac{1}{2}},$$
$$\varphi(x) = \left(\frac{x}{a}\right)^{1-r_2}\left(1 - \frac{x}{a}\right) + \left(\frac{x}{a}\right)^{\frac{1-r_2}{2}}\left(1 - \frac{x}{a}\right)^{\frac{1}{2}},$$

and let K_0 be a positive constant such that

$$K_0\psi(x) + f(0) \geq f(u_0(x)), \quad K_0\varphi(x) + g(0) \geq g(v_0(x)).$$

Introduce the positive constants $M_1 = \sup_{x \in [0,a]} \psi(x)$, $M_2 = \sup_{x \in [0,a]} \varphi(x)$ and

$$b_{20} = \int_0^1 \left(s^{1-r_1}(1-s) + s^{\frac{1-r_1}{2}}(1-s)^{\frac{1}{2}}\right) ds,$$
$$b_{10} = \int_0^1 \left(s^{1-r_2}(1-s) + s^{\frac{1-r_2}{2}}(1-s)^{\frac{1}{2}}\right) ds,$$
$$N_1 = af'(f^{-1}(2K_0M_1 + f(0)))(2b_{10}K_0 + g(0)),$$
$$N_2 = ag'(g^{-1}(2K_0M_2 + g(0)))(2b_{20}K_0 + f(0)).$$

Let $K_{10} \in \left(0, \dfrac{1-r_1}{2-r_1}\right)$ and $K_{20} \in \left(0, \dfrac{1-r_2}{2-r_2}\right)$ be positive constants such that

$$K_{10} \le \left(\frac{2a^{2-r_1}N_1}{K_0}\right)^{-\frac{2}{1-r_1}}, \quad K_{20} \le \left(\frac{2a^{2-r_2}N_2}{K_0}\right)^{-\frac{2}{1-r_2}}.$$

Let $K_1(t), K_2(t)$ be the positive solutions of the initial value problems

$$K_1'(t) = \begin{cases} \dfrac{f'(f^{-1}(2K_0M_1 + f(0)))(b_{10}K_2(t) + g(0))}{a^{p_1-1}K_{10}^{p_1}[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{\frac{1}{2}}(1-K_{10})^{\frac{1-r_1}{2}}]}, & p_1 \ge 0, \\ \dfrac{f'(f^{-1}(2K_0M_1 + f(0)))(b_{10}K_2(t) + g(0))}{a^{p_1-1}(1-K_{10})^{p_1}[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{\frac{1}{2}}(1-K_{10})^{\frac{1-r_1}{2}}]}, & p_1 < 0, \\ K_1(0) = K_0, \end{cases} \tag{2.3}$$

$$K_2'(t) = \begin{cases} \dfrac{g'(g^{-1}(2K_0M_2 + g(0)))(b_{20}K_1(t) + f(0))}{a^{p_2-1}K_{20}^{p_2}[K_{20}(1-K_{20})^{1-r_2} + K_2^{\frac{1}{2}}(1-K_{20})^{\frac{1-r_2}{2}}]}, & p_2 \ge 0, \\ \dfrac{g'(g^{-1}(2K_0M_2 + g(0)))(b_{20}K_1(t) + f(0))}{a^{p_2-1}(1-K_{20})^{p_2}[K_{20}(1-K_{20})^{1-r_2} + K_{20}^{\frac{1}{2}}(1-K_{20})^{\frac{1-r_2}{2}}]}, & p_2 < 0, \\ K_2(0) = K_0. \end{cases} \tag{2.4}$$

Since $K_1(t)$ and $K_2(t)$ are increasing functions, we can choose $t_0 > 0$ such that

$$K_1(t) \le 2K_0, \quad K_2(t) \le 2K_0 \quad \text{for all} \quad t \in [0, t_0].$$

Set

$$h_1(x,t) = f^{-1}(K_1(t)\psi(x) + f(0)), \qquad h_2(x,t) = g^{-1}(K_2(t)\varphi(x) + g(0)).$$

Then $h_1(x,t) \ge 0$ and $h_2(x,t) \ge 0$ on $\overline{\Omega}_{t_0}$. We show that $(h_1(x,t), h_2(x,t))$ is an upper solution of (1.1) in Ω_{t_0}. To do this, let us construct two functions J_1 and J_2 by

$$J_1 = f'(h_1)\left[x^{p_1}h_{1t} - (x^{r_1}h_{1x})_x - \int_0^a g(h_2)dx\right],$$

and

$$J_2 = g'(h_2)\left[x^{p_2}h_{2t} - (x^{r_2}h_{2x})_x - \int_0^a f(h_1)dx\right].$$

Then

$$\begin{aligned}
J_1 &= f'(h_1)\left[x^{p_1}h_{1t}-(x^{r_1}h_{1x})_x-\int_0^a g(h_2)dx\right]\\
&= x^{p_1}(f(h_1))_t-r_1x^{r_1-1}(f(h_1))_x-x^{r_1}(f(h_1))_{xx}\\
&\quad -f'(h_1)\int_0^a g(h_2)dx+x^{r_1}f''(h_1)h_{1x}^2\\
&\geq x^{p_1}K_1'(t)\psi(x)-K_1(t)(r_1x^{r_1-1}\psi_x+x^{r_1}\psi_{xx})\\
&\quad -f'(h_1)\int_0^a (K_2(t)\varphi(x)+g(0))dx\\
&= x^{p_1}K_1'(t)\psi(x)+K_1(t)\left[\frac{2-r_1}{a^{2-r_1}}+\left(\frac{(1-r_1)^2}{4}x^{(r_1-3)/2}(a-x)^{1/2}\right.\right.\\
&\quad\left.\left.+\frac{1}{2}x^{(r_1-1)/2}(a-x)^{-1/2}+\frac{1}{4}x^{(r_1+1)/2}(a-x)^{-3/2}\right)\frac{1}{a^{1-r_1/2}}\right]\\
&\quad -af'(f^{-1}(K_1(t)\psi(x)+f(0)))(b_{10}K_2(t)+g(0))\\
&\geq x^{p_1}K_1'(t)\psi(x)+x^{(r_1-1)/2}(a-x)^{-1/2}\frac{K_1(t)}{2a^{1-r_1/2}}\\
&\quad -af'(f^{-1}(K_1(t)M_1+f(0)))(b_{10}K_2(t)+g(0)),\\
J_2 &= g'(h_2)\left[x^{p_2}h_{2t}-(x^{r_2}h_{2x})_x-\int_0^a f(h_1)dx\right]\\
&\geq x^{p_2}K_2'(t)\varphi(x)+x^{(r_2-1)/2}(a-x)^{-1/2}\frac{K_2(t)}{2a^{1-r_2/2}}\\
&\quad -ag'(g^{-1}(K_2(t)M_2+g(0)))(b_{20}K_1(t)+f(0)).
\end{aligned}$$

For $(x,t)\in(0,aK_{10})\times(0,t_0]\cup(a(1-K_{10}),a)\times(0,t_0]$, we have

$$\begin{aligned}
J_1 &\geq x^{(r_1-1)/2}(a-x)^{-1/2}\frac{K_1(t)}{2a^{1-r_1/2}}-af'(f^{-1}(K_1(t)M_1+f(0)))(b_{10}K_2(t)+g(0))\\
&\geq \left[\frac{K_{10}^{\frac{r_1-1}{2}}}{2a^{2-r_1}}\right]K_0-af'(f^{-1}(2K_0M_1+f(0)))(2b_{10}K_0+g(0))\\
&= \left[\frac{K_{10}^{\frac{r_1-1}{2}}}{2a^{2-r_1}}\right]K_0-N_1\geq 0.
\end{aligned}$$

For $(x,t)\in(0,aK_{20})\times(0,t_0]\cup(a(1-K_{20}),a)\times(0,t_0]$, we have

$$\begin{aligned}
J_2 &\geq x^{(r_2-1)/2}(a-x)^{-1/2}\frac{K_2(t)}{2a^{1-r_2/2}}-ag'(g^{-1}(K_2(t)M_2+g(0)))(b_{20}K_1(t)+f(0))\\
&\geq \left[\frac{K_{20}^{\frac{r_2-1}{2}}}{2a^{2-r_2}}\right]K_0-N_2\geq 0.
\end{aligned}$$

For $(x,t) \in [aK_{10}, a(1-K_{10})] \times (0,t_0]$ by (2.3), we have

$$
\begin{aligned}
J_1 &\geq x^{p_1} K_1'(t)\psi(x) - af'(f^{-1}(K_1(t)M_1 + f(0)))(b_{10}K_2(t) + g(0)) \\
&\geq \begin{cases} a^{p_1} K_{10}^{p_1} K_1'(t)[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{\frac{1}{2}}(1-K_{10})^{\frac{1-r_1}{2}}] \\ \qquad -af'(f^{-1}(K_1(t)M_1 + f(0)))(b_{10}K_2(t) + g(0)), & p_1 \geq 0, \\ a^{p_1}(1-K_{10})^{p_1} K_1'(t)[K_{10}(1-K_{10})^{1-r_1} + K_{10}^{\frac{1}{2}}(1-K_{10})^{\frac{1-r_1}{2}}] \\ \qquad -af'(f^{-1}(K_1(t)M_1 + f(0)))(b_{10}K_2(t) + g(0)), & p_1 < 0, \end{cases} \\
&\geq 0.
\end{aligned}
$$

For $(x,t) \in [aK_{20}, a(1-K_{20})] \times (0,t_0]$ by (2.4), we can get $J_2 \geq 0$ with the same argument as that for J_1. Thus, $J_1(x,t) \geq 0, J_2(x,t) \geq 0$ in Ω_{t_0}. Since $f'(s) > 0$ and $g'(s) > 0$ in Ω_{t_0}, we have

$$
x^{p_1}h_{1t} - (x^{r_1}h_{1x})_x - \int_0^a g(h_2)dx \geq 0, \quad x^{p_2}h_{2t} - (x^{r_2}h_{2x})_x - \int_0^a f(h_1)dx \geq 0 \ \text{ in } \ \Omega_{t_0},
$$

and

$$
\begin{aligned}
h_1(0,t) = h_1(a,t) = f^{-1}(f(0)) = 0, &\quad 0 < t < t_0, \\
h_2(0,t) = h_2(a,t) = g^{-1}(g(0)) = 0, &\quad 0 < t < t_0, \\
h_1(x,0) = f^{-1}(K_0\psi(x) + f(0)) \geq f^{-1}(f(u_0(x))) = u_0(x), \\
h_2(x,0) = g^{-1}(K_0\varphi(x) + g(0)) \geq g^{-1}(g(v_0(x))) = v_0(x).
\end{aligned}
$$

So $(h_1(x,t), h_2(x,t))$ is an upper solution of (1.1). The proof is complete. □

To show the existence of the classical solution $(u(x,t), v(x,t))$ of (1.1), let us introduce a cutoff function $\rho(x)$. By Dunford and Schwartz [10, p. 1640], there exists a nondecreasing $\rho(x) \in C^3(R)$ such that $\rho(x) = 0$ if $x \leq 0$ and $\rho(x) = 1$ if $x \geq 1$. Let $0 < \delta < \min\left\{\frac{1-r_1}{2-r_1}a, \frac{1-r_2}{2-r_2}a\right\}$,

$$
\rho_\delta(x) = \begin{cases} 0, & x \leq \delta, \\ \rho\left(\frac{x}{\delta} - 1\right), & \delta < x < 2\delta, \\ 1, & x \geq 2\delta, \end{cases}
$$

and $u_{0\delta}(x) = \rho_\delta(x)u_0(x)$, $v_{0\delta}(x) = \rho_\delta(x)v_0(x)$. We note that

$$
\frac{\partial u_{0\delta}(x)}{\partial \delta} = \begin{cases} 0, & x \leq \delta, \\ -\frac{x}{\delta^2}\rho'\left(\frac{x}{\delta} - 1\right)u_0(x), & \delta < x < 2\delta, \\ 0, & x \geq 2\delta, \end{cases}
$$

$$\frac{\partial v_{0\delta}(x)}{\partial \delta} = \begin{cases} 0, & x \le \delta, \\ -\dfrac{x}{\delta^2}\rho'\left(\dfrac{x}{\delta}-1\right)v_0(x), & \delta < x < 2\delta, \\ 0, & x \ge 2\delta. \end{cases}$$

Since ρ is nondecreasing, we have $\dfrac{\partial u_{0\delta}(x)}{\partial \delta} \le 0$ and $\dfrac{\partial v_{0\delta}(x)}{\partial \delta} \le 0$. From $0 \le \rho(x) \le 1$, we have $u_0(x) \ge u_{0\delta}(x)$, $v_0(x) \ge v_{0\delta}(x)$ and $\lim_{\delta\to 0} u_{0\delta}(x) = u_0(x)$, $\lim_{\delta\to 0} v_{0\delta}(x) = v_0(x)$.

Let $D_\delta = (\delta, a)$, let $w_\delta = D_\delta \times (0, t_0]$, let $\overline{D}_\delta$ and $\overline{w}_\delta$ be their respective closures, and let $S_\delta = \{0, a\} \times (0, t_0]$. We consider the following regularized problem

$$\begin{cases} x^{p_1}u_{\delta t} - (x^{r_1}u_{\delta x})_x = \displaystyle\int_0^a g(v_\delta(x,t))dx, & (x,t) \in w_\delta, \\ x^{p_2}v_{\delta t} - (x^{r_2}v_{\delta x})_x = \displaystyle\int_0^a f(u_\delta(x,t))dx, & (x,t) \in w_\delta, \\ u_\delta(0,t) = u_\delta(a,t) = v_\delta(0,t) = v_\delta(a,t) = 0, & t \in (0,t_0), \\ u_\delta(x,0) = u_{0\delta}(x), \quad v_\delta(x,0) = v_{0\delta}(x), & x \in \overline{D}_\delta. \end{cases} \tag{2.5}$$

By using Schauder's fixed point theorem,we have the following theorem.

Theorem 2.5. *The problem* (2.5) *admits a unique nonnegative solution*

$$(u_\delta, v_\delta) \in (C^{2+\alpha, 1+\alpha/2}(\overline{w}_\delta))^2.$$

Moreover, $0 \le u_\delta \le h_1(x,t)$, $0 \le v_\delta \le h_2(x,t)$, $(x,t) \in \overline{w}_\delta$, *where* $h_1(x,t)$ *and* $h_2(x,t)$ *are given by Lemma 2.4.*

Proof. By using Lemma 2.3, there exists at most one nonnegative solution (u_δ, v_δ). To prove existence, we use the Schauder fixed point theorem. Let

$$X_1 = \{v_1 \in C^{\alpha,\alpha/2}(\overline{w}_\delta) : 0 \le v_1(x,t) \le h_2(x,t), (x,t) \in \overline{w}_\delta\},$$
$$X_2 = \{u_1 \in C^{\alpha,\alpha/2}(\overline{w}_\delta) : 0 \le u_1(x,t) \le h_1(x,t), (x,t) \in \overline{w}_\delta\}.$$

We note that X_1 and X_2 are closed convex subsets of the Banach space $C^{\alpha,\alpha/2}(\overline{w}_\delta)$. In order to obtain the conclusion, we define another set $X = X_1 \times X_2$. Obviously $(C^{\alpha,\alpha/2}(\overline{w}_\delta))^2$ is a Banach space with the norm

$$\| (v_1, u_1) \|_{\alpha,\alpha/2} = \| v_1 \|_{\alpha,\alpha/2} + \| u_1 \|_{\alpha,\alpha/2}, \quad \text{for any } (v_1,u_1) \in (C^{\alpha,\alpha/2}(\overline{w}_\delta))^2$$

and X is a closed convex subset of the Banach space $(C^{\alpha,\alpha/2}(\overline{w}_\delta))^2$. For any $(v_1, u_1) \in (X_1 \times X_2)$, let us consider the following linearized uniformly parabolic problem

$$\begin{cases} x^{p_1}W_{\delta t} - (x^{r_1}W_{\delta x})_x = \displaystyle\int_\delta^a g(v_1)dx, & (x,t) \in w_\delta, \\ x^{p_2}Z_{\delta t} - (x^{r_2}Z_{\delta x})_x = \displaystyle\int_\delta^a f(u_1)dx, & (x,t) \in w_\delta, \\ W_\delta(\delta,t) = W_\delta(a,t) = Z_\delta(\delta,t) = Z_\delta(a,t) = 0, & t \in (0,t_0], \\ W_\delta(x,0) = u_{0\delta}(x), \quad Z_\delta(x,0) = v_{0\delta}(x), & x \in [\delta, a]. \end{cases} \tag{2.6}$$

By construction, $(0,0)$ and $(h_1(x,t), h_2(x,t))$ are lower and upper solutions of problem (2.6). We also note that

$$x^{-p_1+r_1}, x^{-p_1-1+r_1}, x^{-p_1}, x^{-p_2+r_2}, x^{-p_2-1+r_2}, x^{-p_2} \in C^{\alpha,\alpha/2}(\overline{w}_\delta)$$

and

$$x^{-p_1}\int_\delta^a g(v_1)dx, x^{-p_2}\int_\delta^a f(u_1)dx \in C^{\alpha,\alpha/2}(\overline{w}_\delta), \quad u_{0\delta}, v_{0\delta} \in C^{2+\alpha}(\overline{D}_\delta).$$

It follows from Ladde et al. [12, Theorem 4.2.2 on p. 143] that problem (2.6) has a unique solution $(W_\delta(x,t;v_1,u_1).Z_\delta(x,t;v_1,u_1)) \in (C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta))^2$ such that

$$0 \le W_\delta(x,t;v_1,u_1) \le h_1(x,t), \quad 0 \le Z_\delta(x,t;v_1,u_1) \le h_2(x,t).$$

Thus, we can define a mapping T from X into $(C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta))^2$ such that

$$T(v_1(x,t), u_1(x,t)) = (W_\delta(x,t;v_1,u_1), Z_\delta(x,t;v_1,u_1)),$$

where $(W_\delta(x,t;v_1,u_1), Z_\delta(x,t;v_1,u_1))$ denotes the unique solution of (2.6) corresponding to $(v_1(x,t), u_1(x,t)) \in X$. To use the Schauder fixed point theorem, we need to verify that T maps X into itself and that T is continuous and compact. In fact, $TX \subset X$, and the embedding operator from the Banach space $(C^{2+\alpha,1+\alpha/2}(\overline{w}_\delta))^2$ to the Banach space $(C^{\alpha,\alpha/2}(\overline{w}_\delta))^2$ is compact. Therefore, T is compact. To show that T is continuous, let us consider sequence $v_{1n}(x,t)$ which converges to $v_1(x,t)$ uniformly and $u_{1n}(x,t)$ which converges to $u_1(x,t)$ uniformly in the norm $\|\cdot\|_{\alpha,\alpha/2}$. We know that $v_1(x,t) \in X_1$ and $u_1(x,t) \in X_2$. So we get a sequence $\{(v_{1n}(x,t), u_{1n}(x,t))\} \in X$, which converges to $(v_1(x,t), u_1(x,t))$ uniformly in the norm $\|\cdot,\cdot\|_{\alpha,\alpha/2}$. Let $(W_{\delta n}(x,t), Z_{\delta n}(x,t))$ and $(W_\delta(x,t), Z_\delta(x,t))$ be the solutions of (2.6) corresponding to $(v_{1n}(x,t), u_{1n}(x,t))$ and $(v_1(x,t), u_1(x,t))$, respectively. Without loss of generality, let us assume that

$$\|v_{1n}(x,t)\|_{\alpha,\alpha/2} \le \|v_1(x,t)\|_{\alpha,\alpha/2} + 1, \qquad \text{for any } n \ge 1,$$
$$\|u_{1n}(x,t)\|_{\alpha,\alpha/2} \le \|u_1(x,t)\|_{\alpha,\alpha/2} + 1, \qquad \text{for any } n \ge 1.$$

Let $W(x,t) = W_{\delta n}(x,t) - W_\delta(x,t), Z(x,t) = Z_{\delta n}(x,t) - Z_\delta(x,t)$. Then we have

$$\begin{cases} x^{p_1}W_t - (x^{r_1}W_x)_x = \displaystyle\int_\delta^a (g(v_{1n}) - g(v_1))dx, & (x,t) \in w_\delta, \\ x^{p_2}Z_t - (x^{r_2}Z_x)_x = \displaystyle\int_\delta^a (f(u_{1n}) - f(u_1))dx, & (x,t) \in w_\delta, \\ W(\delta,t) = W(a,t) = Z(\delta,t) = Z(a,t) = 0, & t \in (0,t_0], \\ W(x,0) = 0, \quad Z(x,0) = 0, & x \in \overline{D}_\delta. \end{cases}$$

From Ladyženskaja et al. [13, Theorem 4.5.2 on p. 320], there exist positive constants C_1 (independent of g, v_{1n} and v_1) and C_2 (independent of f, u_{1n} and u_1) such that

$$\begin{aligned} \|W\|_{2+\alpha,1+\alpha/2} &\le C_1 \left\| \int_\delta^a (g(v_{1n}) - g(v_1)) \right\|_{\alpha,\alpha/2} \\ &\le C_1 a \|g'(v_1 + \tau(v_{1n} - v_1))\|_{\alpha,\alpha/2} \|v_{1n} - v_1\|_{\alpha,\alpha/2}, \end{aligned}$$

and we note that

$$\begin{aligned}
&\|g'(v_1+\tau(v_{1n}-v_1))\|_{\alpha,\alpha/2} \le \|g'(h_2)\|_\infty \\
&\qquad + \sup_{\substack{\delta\le x\\ \tilde{x}\le a}} \frac{|g'(v_1+\tau(v_{1n}-v_1))(x,t)-g'(v_1+\tau(v_{1n}-v_1))(\tilde{x},t)|}{|x-\tilde{x}|^\alpha} \\
&\qquad + \sup_{\substack{0\le t\\ \tilde{t}\le t_0}} \frac{|g'(v_1+\tau(v_{1n}-v_1))(x,t)-g'(v_1+\tau(v_{1n}-v_1))(x,\tilde{t})|}{|t-\tilde{t}|^{\alpha/2}} \\
&\le \|g'(h_2)\|_\infty + \max_{0\le s\le h_2(x,t)} |g''(s)|(\|v_1\|_{\alpha,\alpha/2}+\|v_{1n}\|_{\alpha,\alpha/2}) \\
&\le \|g'(h_2)\|_\infty + \max_{0\le s\le h_2(x,t)} |g''(s)|(2\|v_1\|_{\alpha,\alpha/2}+1),
\end{aligned}$$

where $\tau\in(0,1)$. Therefore,

$$\begin{aligned}
\|W\|_{2+\alpha,1+\alpha/2} &\le C_1 a(\|g'(h_2)\|_\infty \\
&\qquad + \max_{0\le s\le h_2(x,t)} |g''(s)|(2\|v_1\|_{\alpha,\alpha/2}+1))\|v_{1n}-v_1\|_{\alpha,\alpha/2} \\
&\le C_1'\|v_{1n}-v_1\|_{\alpha,\alpha/2}
\end{aligned}$$

and

$$\begin{aligned}
\|Z\|_{2+\alpha,1+\alpha/2} &\le C_2 a(\|f'(h_1)\|_\infty \\
&\qquad + \max_{0\le s\le h_1(x,t)} |f''(s)|(2\|u_1\|_{\alpha,\alpha/2}+1))\|u_{1n}-u_1\|_{\alpha,\alpha/2} \\
&\le C_2'\|u_{1n}-u_1\|_{\alpha,\alpha/2}.
\end{aligned}$$

It follows that

$$\|(W,Z)\|_{2+\alpha,1+\alpha/2} = \|W\|_{2+\alpha,1+\alpha/2} + \|Z\|_{2+\alpha,1+\alpha/2} \le C\|(v_{1n}-v_1, u_{1n}-u_1)\|_{\alpha,\alpha/2}.$$

This shows that the mapping T is continuous. By the Schauder fixed point theorem, the proof is complete. □

Now we can prove the following local existence result.

Theorem 2.6. *There exists some $t_0 < T$ such that problem* (1.1) *has a unique nonnegative solution* $(u(x,t),v(x,t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$.

Proof. By Theorem 2.5, the problem (2.5) has a unique nonnegative solution $(u_\delta, v_\delta) \in (C^{2+\alpha,1+\alpha/2}(\bar{w}_\delta))^2$. It follows from Lemma 2.3 that $(u_{\delta 1}, v_{\delta 1}) \le (u_{\delta 2}, v_{\delta 2})$ if $\delta 1 > \delta 2$. Therefore, $\lim_{\delta\to 0}(u_\delta(x,t), v_\delta(x,t))$ exists for all $(x,t)\in(0,a]\times[0,t_0]$. Let

$$(u(x,t),v(x,t)) = \lim_{\delta\to 0}(u_\delta(x,t),v_\delta(x,t)), \quad (x,t)\in(0,a]\times[0,t_0]$$

and define $(u(0,t),v(0,t)) = (0,0), t \in [0,t_0]$. We show that $(u(x,t),v(x,t))$ is a classical solution of (1.1) in Ω_{t_0}. For any $(x_1,t_1) \in \Omega_{t_0}$, there exist three domains

$$Q' = (a_1', a_2') \times (t_2', t_3'],\ Q'' = (a_1'', a_2'') \times (t_2'', t_3''],\ Q''' = (a_1''', a_2''') \times (t_2''', t_3''']$$

such that $(x_1,t_1) \in Q' \subset Q'' \subset Q''' \subset (0,a) \times (0,t_0]$ with

$$0 < a_1''' < a_1'' < a_1' < x_1 < a_2' < a_2'' < a_2''' < a$$

and

$$0 < t_2''' < t_2'' < t_2' < t_1 < t_3' < t_3'' < t_3''' < t_0.$$

By the conditions of f and g, we know that $h_1(x,t)$ and $h_2(x,t)$ are finite on $\bar{Q}'''$. For any constant $q > 1$ and some positive constants K_3 and K_4, we have

$$\begin{aligned}
&\|u_\delta\|_{L^q(Q''')} \le \|h_1\|_{L^q(Q''')} \le K_3, \quad \|v_\delta\|_{L^q(Q''')} \le \|h_2\|_{L^q(Q''')} \le K_3,\\
&\|x^{-p_1} \int_\delta^a g(v_\delta)dx\|_{L^q(Q''')} \le (a_1^*)^{-p_1} \| \int_0^a g(h_2)dx\|_{L^q(Q''')} \le K_4,\\
&\|x^{-p_2} \int_\delta^a f(u_\delta)dx\|_{L^q(Q''')} \le (a_2^*)^{-p_2} \| \int_0^a f(h_1)dx\|_{L^q(Q''')} \le K_4,
\end{aligned}$$

where $a_1^* = a_1'''$ if $q_1 \ge 0$, $a_1^* = a_2'''$ if $q_1 < 0$, and $a_2^* = a_1'''$ if $q_2 \ge 0$, $a_2^* = a_2'''$ if $q_2 < 0$. By the local L^p-estimate of Ladyženskaja et al. [13, pp. 341–342, 352], $(u_\delta, v_\delta) \in (W_q^{2,1}(Q''))^2$. By the embedding theorem in [15, pp. 61, 80], $W_q^{2,1}(Q'') \hookrightarrow H^{\alpha,\alpha/2}(Q'')$ if we choose $q > 2/(1-\alpha)$. Then, $\|u_\delta\|_{H^{\alpha,\alpha/2}(Q'')} \le K_5$ and $\|v_\delta\|_{H^{\alpha,\alpha/2}(Q'')} \le K_5$, for some positive constant K_5, and we have

$$\begin{aligned}
\left\| x^{-p_1} \int_\delta^a g(v_\delta)dx \right\|_{H^{\alpha,\alpha/2}(Q'')} &\le (a_1^*)^{-p_1} \left\| \int_\delta^a g(h_2)dx \right\|_\infty \\
&\quad + \sup_{\substack{(x,t)\in Q''\\(\tilde{x},t)\in Q''}} \frac{|\int_\delta^a g(v_\delta)dx| \cdot |x^{-p_1} - \tilde{x}^{-p_1}|}{|x-\tilde{x}|^\alpha} \\
&\quad + \sup_{\substack{(x,t)\in Q''\\(x,\tilde{t})\in Q''}} \frac{|x^{-p_1}| |\int_\delta^a g'(v_\delta(x,\tilde{t}) + \tau(v_\delta(x,t) - v_\delta(x,\tilde{t})))(v_\delta(x,t) - v_\delta(x,\tilde{t}))dx|}{|t-\tilde{t}|^{\alpha/2}} \\
&\le (a_1^*)^{-p_1} \| \int_0^a g(h_2)dx\|_\infty \\
&\quad + \left\| \int_0^a g(h_2)dx \right\|_\infty \cdot \|x^{-p_1}\|_{H^\alpha(a_1'',a_2'')} + (a_1^*)^{-p_1} \left\| \int_0^a g(h_2)dx \right\|_\infty \cdot \|v_\delta\|_{H^{\alpha,\alpha/2}(Q'')} \\
&\le K_6
\end{aligned}$$

and

$$\left\| x^{-p_2} \int_\delta^a f(u_\delta)dx \right\|_{H^{\alpha,\alpha/2}(Q'')} \le K_6$$

for some positive K_6, which is independent of δ, where $\tau \in (0,1)$. By Ladyženskaja et al. [16, pp. 351–352], we have

$$\|u_\delta\|_{H^{2+\alpha,1+\alpha/2}(Q')} \le K_7, \quad \|v_\delta\|_{H^{2+\alpha,1+\alpha/2}(Q')} \le K_7$$

for some positive constant K_7 independent of δ. This implies that $u_\delta, u_{\delta t}, u_{\delta x}, u_{\delta xx}$ and $v_\delta, v_{\delta t}, v_{\delta x}, v_{\delta xx}$ are equicontinuous in Q'. By the Ascoli–Arzela theorem, we know that

$$\|u\|_{H^{2+\alpha',1+\alpha'/2}(Q')} \le K_8, \quad \|v\|_{H^{2+\alpha',1+\alpha'/2}(Q')} \le K_8$$

for some $\alpha' \in (0,\alpha)$ and some positive constant K_8 independent of δ, and that the derivatives of u and v are uniform limits of the corresponding partial derivatives of u_δ and v_δ, respectively. Hence $(u(x,t), v(x,t))$ satisfy (1.1), and

$$\begin{aligned} \lim_{t\to 0}(u(x,t), v(x,t)) &= \lim_{t\to 0}\lim_{\delta\to 0}(u_\delta(x,t), v_\delta(x,t)) = \lim_{\delta\to 0}(u_{0\delta}(x,t), v_{0\delta}(x,t)) \\ &= (u_0(x), v_0(x)). \end{aligned}$$

It follows from $0 \le u(x,t) \le h_1(x,t)$, $0 \le v(x,t) \le h_2(x,t)$ and $h_1(x,t) \to 0, h_2(x,t) \to 0$ as $x \to 0$ or $x \to a$ that

$$\lim_{x\to 0}(u(x,t), v(x,t)) = \lim_{x\to a}(u(x,t), v(x,t)) = (0,0).$$

Thus $(u,v) \in C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0})$ is the solution of (1.1) in Ω_{t_0}. This completes the proof. □

Theorem 2.7. *Let T be the supremum over t_0 for which there is a unique nonnegative solution $(u(x,t), v(x,t)) \in (C(\overline{\Omega}_{t_0}) \cap C^{2,1}(\Omega_{t_0}))^2$ of* (1.1)*. Then* (1.1) *has a unique nonnegative solution $(u(x,t), v(x,t)) \in (C([0,a]\times[0,T)) \cap C^{2,1}((0,a)\times(0,T)))^2$. If $T < \infty$, then $\limsup_{t\to T} \max_{x\in[0,a]}(|u(x,t)| + |v(x,t)|) = \infty$.*

Proof. The proof of this theorem is similar to the proof of [11, Theorem 2.5], so we omit it. □

3 Blow-up of Solutions

In this section, we give some global blow-up result of the solution of (1.1). In order to obtain the blow-up result, we assume that $p_1 \ge r_1 - 1$, $p_2 \ge r_2 - 1$ and $f(s), g(s)$ satisfy

$$f(s) + g(t) \ge \eta \min\{f(s+t), g(s+t)\} \equiv h(s+t), \tag{3.1}$$

for some positive constant η.

Remark 3.1. Since $(s+t)^p \le 2^{p-1}(s^p + t^p)$, power functions satisfy the property (3.1).

Now, we consider the eigenvalue problem

$$-(x^{r_1}\varphi_1'(x))' = \lambda_1 x^{p_1}\varphi_1(x), \quad x \in (0,a), \quad \varphi_1(0) = \varphi_1(a) = 0. \tag{3.2}$$

Using the argument by Jun Zhou et al. [18], we can make $\varphi_1(x)$ satisfy

$$\max_{x\in[0,a]} x^{p_1}\varphi_1(x) = 1. \tag{3.3}$$

Analogously, we consider the eigenvalue problem

$$-(x^{r_2}\varphi_2'(x))' = \lambda_2 x^{p_2}\varphi_2(x), \quad x \in (0,a), \quad \varphi_2(0) = \varphi_2(a) = 0. \tag{3.4}$$

As above, we can obtain

$$\max_{x\in[0,a]} x^{p_2}\varphi_2(x) = 1. \tag{3.5}$$

Let $C_1 = \int_0^a \varphi_1(x)dx$, $C_2 = \int_0^a \varphi_2(x)dx$ and $\lambda = \max\{\lambda_1, \lambda_2\}$, $C = \min\{C_1, C_2\}$. Then we have the following result.

Theorem 3.2. *Let $(u(x,t), v(x,t))$ be the solution of problem* (1.1). *Then the solution of* (1.1) *blows up in finite time.*

Proof. We set

$$U(t) = \int_0^a x^{p_1}\varphi_1(x)u(x,t)dx, \quad V(t) = \int_0^a x^{p_2}\varphi_2(x)v(x,t)dx.$$

By (1.1), (3.2) and (3.4), we have

$$\begin{aligned} U'(t) &= \int_0^a x^{p_1}\varphi_1(x)u_t(x,t)dx = \int_0^a \left((x^{r_1}u_x)_x + \int_0^a g(v)dx\right)\varphi_1(x)dx \\ &= -\lambda_1 U(t) + C_1\int_0^a g(v)dx \ \geq\ -\lambda U(t) + C\int_0^a g(v)dx \end{aligned}$$

and

$$V'(t) \geq -\lambda V(t) + C\int_0^a f(u)dx.$$

Using Jensen's inequality and (3.3), (3.5), we get

$$\begin{aligned} U'(t) &\geq -\lambda U(t) + C\int_0^a g(v)dx \ \geq\ -\lambda U(t) + Cag\left(\frac{1}{a}\int_0^a v dx\right) \\ &\geq -\lambda U(t) + Cag\left(\frac{1}{a}\int_0^a x^{p_2}\varphi_2(x)v(x,t)dx\right) \\ &\geq -\lambda U(t) + Cag\left(\frac{V(t)}{a}\right) \end{aligned} \tag{3.6}$$

and

$$V'(t) \geq -\lambda V(t) + Caf\left(\frac{U(t)}{a}\right). \tag{3.7}$$

Since $\int_{s_0}^{\infty} \frac{ds}{f(s)} < \infty$, $\int_{s_0}^{\infty} \frac{ds}{g(s)} < \infty$, and $h(s+t) = h(y)$, we have $\int_{s_0}^{\infty} \frac{dy}{h(y)} < \infty$. Then we can obtain

$$\lim_{y\to\infty} \frac{h(y)}{y} = \infty.$$

In fact, we have $\lim_{y\to\infty} h(y) = \infty$ for $\int_{s_0}^{\infty} \frac{ds}{h(y)} < \infty$. By $h''(y) \geq 0$ (by $f''(s) \geq 0, g''(s) \geq 0$), we have that $h'(y)$ is nondecreasing if $y > 0$. Using L'Hospital's principle, we obtain $\lim_{y\to\infty} \frac{h(y)}{y} = \lim_{y\to\infty} h'(y)$. Assume by contradiction that $\lim_{y\to\infty} h'(y) = N < \infty$. Then there exists $y_0 \geq s_0$ such that $h(y) \leq \frac{3}{2}Ny$, and we have

$$\int_{s_0}^{\infty} \frac{dy}{h(y)} \geq \frac{2}{3N}\int_{s_0}^{\infty} \frac{dy}{y} = \infty.$$

Since $\lambda > 0$, by $\lim_{y\to\infty} \frac{h(y)}{y} = \infty$, we know that there exists $s_1 \geq s_0$ such that $\frac{h(y)}{y} \geq \frac{2\lambda}{C}$ if $y \geq s_1$. Let $(u_0(x), v_0(x))$ be sufficiently larger such that

$$\int_0^a x^{p_1} u_0(x)\varphi_1(x) \geq \frac{as_1}{2}, \quad \int_0^a x^{p_2} v_0(x)\varphi_2(x) \geq \frac{as_1}{2}.$$

Now, by (3.6), (3.7), we have

$$\begin{aligned} U'(t) + V'(t) &= Ca\left(g\left(\frac{V(t)}{a}\right) + f\left(\frac{U(t)}{a}\right)\right) - \lambda(U(t)+V(t)) \\ &\geq Cah\left(\frac{U(t)+V(t)}{a}\right) - \lambda a\left(\frac{U(t)+V(t)}{a}\right), \end{aligned}$$

and integrating this inequality over t from 0 to T, we have

$$\begin{aligned} T &\leq a\int_0^T \frac{d\frac{U(t)+V(t)}{a}}{Cah(\frac{U(t)+V(t)}{a}) - \lambda a(\frac{U(t)+V(t)}{a})} = a\int_{\frac{U(0)+V(0)}{a}}^{\frac{U(t)+V(t)}{a}} \frac{dy}{Cah(y) - \lambda ay} \\ &= \frac{1}{C}\int_{\frac{U(0)+V(0)}{a}}^{\frac{U(t)+V(t)}{a}} \frac{dy}{h(y) - \frac{\lambda y}{C}} \leq \frac{2}{C}\int_{s_0}^{\infty} \frac{dy}{h(y)} < \infty. \end{aligned}$$

This completes the proof. $\square$

Now, we discuss the global blow-up under the following hypothesis.

Case 1: $p_1 > 0$, $r_1 = 0$ or $p_2 > 0$, $r_2 = 0$.

Chan et al. [3,7] showed that Green's function $G(x, \xi, t-\tau)$ associated with the operator $L = x^p(\partial/\partial t) - \partial^2/\partial^2 x$ with the first boundary condition exists. For ease of reference, we state their results in the following lemma.

Lemma 3.3. (*i*) *For* $t > \tau$, $G(x, \xi, t-\tau)$ *is continuous for* $(x, t, \xi, \tau) \in ([0, a] \times (0, T]) \times ((0, a] \times [0, T))$.

(*ii*) *For each fixed* $(\xi, \tau) \in (0, a] \times [0, T)$, $G_t(x, \xi, t-\tau) \in C([0, a] \times (\tau, T])$.

(*iii*) *In* $\{(x, t, \xi, \tau) : x \ \text{ and } \ \xi \ \text{ are in } \ (0, a), T \geq t > \tau \geq 0\}$, $G(x, \xi, t-\tau)$ *is positive.*

Lemma 3.4. *For fixed* $x_0 \in (0, a]$, *given any* $x \in (0, a)$ *and any finite time* T, *there exist positive constants* C_1 *(depending on* x *and* T*) and* C_2 *(depending on* T*) such that*

$$\int_0^a G(x, \xi, t) d\xi > C_1, \quad \int_0^a G(x_0, \xi, t) d\xi < C_2.$$

Now we give the global blow-up result.

Theorem 3.5. *Under the assumption of Case 1, if the solution of* (1.1) *blows up at the point* $x_0 \in (0, a)$, *then the blow-up set of the solution of* (1.1) *is* $[0, a]$.

Proof. Obviously, the system (1.1) is completely coupled. Therefore, u and v blow up simultaneously if the solution (u, v) blows up in finite time. Without loss of generality, we assume $p_1 > 0$, $r_1 = 0$, and $u(x, t)$ blows up in finite time T. By Green's second identity we have

$$u(x, t) = \int_0^a \xi^{p_1} G(x, \xi, t) u_0(\xi) d\xi + \int_0^t \int_0^a G(x, \xi, t-\tau) \int_0^a g(v(y, \tau)) dy d\xi d\tau \quad (3.8)$$

for any $(x, t) \in (0, a) \times (0, T)$. Since $u(x, t)$ blows up at $x = x_0$, we have $\lim_{t \to T} u(x_0, t) = \infty$. By (3.8) and Lemma 3.4, we have

$$\begin{aligned} u(x_0, t) &= \int_0^a \xi^{p_1} G(x_0, \xi, t) u_0(\xi) d\xi + \int_0^t \int_0^a G(x_0, \xi, \tau) \int_0^a g(v(y, t-\tau)) dy d\xi d\tau \\ &\leq C_2 a^{p_1} \max_{x \in [0, a]} u_0(x) + C_2 \int_0^t \int_0^a g(v(y, t-\tau)) dy d\tau, \end{aligned}$$

and thus

$$\lim_{t \to T} \int_0^t \int_0^a g(v(y, t-\tau)) dy d\tau = \infty. \quad (3.9)$$

On the other hand, for any $x \in (0,a)$, we have

$$\begin{aligned} u(x,t) &\geq \int_0^a \xi^{p_1} G(x,\xi,t) u_0(\xi) d\xi + C_1 \int_0^t \int_0^a g(v(y,t-\tau)) dy d\tau \\ &\geq C_1 \int_0^t \int_0^a g(v(y,t-\tau)) dy d\tau, \quad t \in (0,T). \end{aligned}$$

It follows from the above inequality and (3.9) that $\lim\limits_{t\to T} u(x,t) = \infty$. For any $\tilde{x} \in \{0,a\}$, we can always find a sequence $\{(x_n,t_n)\}$ such that $(x_n,t_n) \to (\tilde{x},T)(n \to \infty)$ and $\lim\limits_{t\to T} u(x_n,t_n) = \infty$. Thus, the blow-up set is $[0,a]$, and this completes the proof. $\square$

Case 2: $p_1 = 0$, $0 \leq r_1 < 1$ or $p_2 = 0$, $0 \leq r_2 < 1$.

We assert that the blow-up set is the whole domain under certain assumptions.

Theorem 3.6. *Under the assumption of Case 2 and if there exists $M \in (0,+\infty)$ such that*

$$(x^{r_1} u_{0x}(x))_x \leq M \quad \text{or} \quad (x^{r_2} v_{0x}(x))_x \leq M \quad \text{in} \quad (0,a),$$

if the solution of (1.1) *blows up at the point $x_0 \in (0,a)$, then the blow-up set of the solution of* (1.1) *is* $[0,a]$.

Proof. The proof is similar to the proof presented in [8, 18], so we omit it. $\square$

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