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Triple Positive Solutions of Two-Point BVPs for p-Laplacian Dynamic Equations on Time Scales

Tuo Li and **Xiaohong Yuan**
Hexi University
Department of Mathematics
Zhangye, Gansu 734000, People's Republic of China
lituo@hxu.edu.cn (corresponding author)

Abstract

We study the existence of positive solutions to the p-Laplacian dynamic equations $(g(u^{\Delta}(t)))^{\nabla} + a(t)f(t, u(t)) = 0$ for $t \in [0, T]_{\mathbb{T}}$ satisfying either the boundary condition $u(0) - B_0(u^{\Delta}(0)) = 0$, $u^{\Delta}(T) = 0$ or $u^{\Delta}(0) = 0$, $u^{\Delta}(T) + B_1(u(T)) = 0$, where $g(\nu) = |\nu|^{p-2}\,\nu$ with $p > 1$. By using a new five functionals fixed-point theorem due to Avery, we prove that the boundary value problems has at least three positive solutions. As an application, an example is given to illustrate our result.

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1 Introduction

The theory of dynamic equations on time scales has appeared in the Ph.D. thesis of Hilger [12]. Theoretically, this new theory cannot only unify continuous and discrete equations [13], but have also exhibited much more complicated dynamics on time scales [7]. Practically, dynamic equations on time scales have led to several important applications, e.g., in the study of insect population models, and epidemic models [1, 7].

Many of the works are concerned with the existence of positive solution for boundary value problems on time scales [6–8, 16, 17, 21]. Additionally, there is much current attention being paid to the existence of positive solution for boundary value problems with p-Laplacian differential or difference equations, see [3, 15, 18–20, 22] and the references therein. However, very few work has been done to the existence of positive

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solutions for the two-point boundary value problems for p-Laplacian dynamic equations on time scales, see [10]. In 2005, He [10] considered the two-point boundary value problem

$$(g(u^{\Delta}(t)))^{\nabla} + a(t)f(u(t)) = 0,\ t \in [0,T]_{\mathbb{T}} \tag{1.1}$$

with the boundary conditions

$$u(0) - B_0(u^{\Delta}(0)) = 0,\ u^{\Delta}(T) = 0, \tag{1.2}$$

or

$$u^{\Delta}(0) = 0,\ u^{\Delta}(T) + B_1(u^{\Delta}(T)) = 0, \tag{1.3}$$

where B_0 and B_1 satisfy

$$Bx \leq B_i(x) \leq Ax,\ x \in \mathbb{R}^+,\ i = 0, 1;$$

here B and A are nonnegative numbers, $0, T$ are points in $\mathbb{T}$, $\mathbb{T}$ (the time scale) is a nonempty closed subset of $\mathbb{R}$. Under some assumptions on f and $a(t)$, he proved that the boundary value problem (1.1) and (1.2) or (1.3) has at least *two* positive solutions by applying the double fixed point theorem due to Avery and Henderson [5].

In this paper, motivated by [10], we consider the existence of triple positive solutions to the boundary value problem

$$(g(u^{\Delta}(t)))^{\nabla} + a(t)f(t, u(t)) = 0,\ t \in [0,T]_{\mathbb{T}} \tag{1.4}$$

satisfying the boundary conditions (1.2) or (1.3). We establish the existence of at least *three* positive solutions of boundary value problem (1.4) and (1.2) or (1.3) by using the five functionals fixed point theorem in a cone [4]. Our results are new for the special cases of difference equations and differential equations as well as in the general time scale setting. As an application, an example is given to illustrate our result.

Throughout this paper, it is assumed that

(H_1) $f(t, u(t)) : [0,T]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function ($\mathbb{R}^+$ denotes the set of nonnegative real numbers);

(H_2) $a(t) : [0,T]_{\mathbb{T}} \to [0,\infty)$ is left-dense continuous (i.e., $a \in C_{\mathrm{ld}}(\mathbb{T}, [0,\infty))$) and does not vanish identically on any closed subinterval of $[0,T]_{\mathbb{T}}$ ($C_{\mathrm{ld}}(\mathbb{T}, [0,\infty))$ denotes the set of all left-dense continuous functions from $\mathbb{T}$ to $[0,\infty)$).

2 Preliminaries

In this section, we begin by presenting some basic definitions which can be found in Atici and Guseinov [2], and Bohner and Peterson [7]. Another excellent source on dynamic systems on measure chains is the book [14].

A time scale $\mathbb{T}$ is a nonempty closed subset of $\mathbb{R}$. It follows that the jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$

$$\sigma(t) = \inf\{\tau \in \mathbb{T} : \tau > t\} \text{ and } \rho(t) = \sup\{\tau \in \mathbb{T} : \tau < t\}$$

(supplemented by $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$) are well defined. The point $t \in \mathbb{T}$ is called left-dense, left-scattered, right-dense, right-scattered if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, $\sigma(t) > t$, respectively. If $\mathbb{T}$ has a right-scattered minimum m, define $\mathbb{T}_\kappa = \mathbb{T} - \{m\}$; otherwise, set $\mathbb{T}_\kappa = \mathbb{T}$. If $\mathbb{T}$ has a left-scattered maximum M, define $\mathbb{T}^\kappa = \mathbb{T} - \{M\}$; otherwise, set $\mathbb{T}^\kappa = \mathbb{T}$. The forward graininess is $\mu(t) := \sigma(t) - t$. Similarly, the backward graininess is $\upsilon(t) := t - \rho(t)$.

We make the blanket assumption that $0, T$ are points in $\mathbb{T}$. By an interval $(0, T)_\mathbb{T}$ we always mean $(0, T) \cap \mathbb{T}$. Other types of intervals are defined similarly. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa$, the delta derivative [7] of f at t, denoted by $f^\Delta(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood $U \subset \mathbb{T}$ of t such that

$$\left| f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s] \right| \le \epsilon \left| \sigma(t) - s \right|$$

for all $s \in U$. For $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$, the nabla derivative [2] of f at t, denoted by $f^\nabla(t)$, is the number (provided it exists) with the property that given any $\epsilon > 0$, there is a neighborhood U of t such that

$$\left| f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s] \right| \le \epsilon \left| \rho(t) - s \right|$$

for all $s \in U$. In the case $\mathbb{T} = \mathbb{R}$, $f^\Delta(t) = f'(t) = f^\nabla(t)$. When $\mathbb{T} = \mathbb{Z}$, $f^\Delta(t) = f(t+1) - f(t)$ and $f^\nabla(t) = f(t) - f(t-1)$.

A function $f : \mathbb{T} \to \mathbb{R}$ is called ld-continuous provided it is continuous at left-dense points in $\mathbb{T}$ and its right sided limit exists (finite) at right-dense points in $\mathbb{T}$. If $\mathbb{T} = \mathbb{R}$, then f is ld-continuous if and only if f is continuous. If $\mathbb{T} = \mathbb{Z}$, then any function is ld-continuous. It is known from [2] that if f is ld-continuous, then there is a function F such that $F^\nabla(t) = f(t)$. In this case, we define

$$\int_a^b f(\tau) \nabla \tau = F(b) - F(a).$$

Now, we provide some background materials on the theory of cones in Banach spaces [9], which will be used in the rest of the paper.

Definition 2.1. Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is said to be a cone provided the following conditions are satisfied:

(i) If $x \in P$ and $\lambda \ge 0$, then $\lambda x \in P$;

(ii) if $x \in P$ and $-x \in P$, then $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2.2. Given a cone P in a real Banach space E, a functional $\psi : P \to \mathbb{R}$ is said to be increasing on P, provided $\psi(x) \leq \psi(y)$, for all $x, y \in P$ with $x \leq y$.

Definition 2.3. A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0, 1]$.

Let γ, β, θ be nonnegative continuous convex functionals on P and α, φ be nonnegative continuous concave functionals on P. For nonnegative real numbers h, a, b, d and c we define the following convex sets:

$$\begin{aligned} P(\gamma, c) &= \{x \in P : \gamma(x) < c\}, \\ P(\gamma, \alpha, a, c) &= \{x \in P : a \leq \alpha(x), \gamma(x) \leq c\}, \\ Q(\gamma, \beta, d, c) &= \{x \in P : \beta(x) \leq d, \gamma(x) \leq c\}, \\ P(\gamma, \theta, \alpha, a, b, c) &= \{x \in P : a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\}, \end{aligned}$$

and

$$Q(\gamma, \beta, \varphi, h, d, c) = \{x \in P : h \leq \varphi(x), \beta(x) \leq d, \gamma(x) \leq c\}.$$

To prove our main results, we need the following theorem, which is the five functionals fixed point theorem [4].

Theorem 2.4. *Let P be a cone in a real Banach space E. Suppose that there exist positive numbers c and M, nonnegative continuous concave functionals α and φ on P, and nonnegative continuous convex functionals γ, β and θ on P, with*

$$\alpha(x) \leq \beta(x) \text{ and } \|x\| \leq M\gamma(x) \text{ for all } x \in \overline{P(\gamma, c)}.$$

Suppose $\Phi : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}$ is completely continuous and there exist nonnegative numbers h, a, k, b, with $0 < a < b$ such that

(*i*) *for $x \in P(\gamma, \theta, \alpha, b, k, c)$,*

$$\{x \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(x) > b\} \neq \emptyset \quad \text{and} \quad \alpha(\Phi(x)) > b;$$

(*ii*) *for* $x \in Q(\gamma, \beta, \varphi, h, a, c)$,

$$\{x \in Q(\gamma, \beta, \varphi, h, a, c) : \beta(x) < a\} \neq \emptyset \quad \textit{and} \quad \beta(\Phi(x)) < a;$$

(*iii*) $\alpha(\Phi(x)) > b$ *for* $x \in P(\gamma, \alpha, b, c)$ *with* $\theta(\Phi(x)) > k$*;*

(*iv*) $\beta(\Phi(x)) < a$ *for* $x \in Q(\gamma, \beta, a, c)$ *with* $\varphi(\Phi(x)) < h$*.*

Then Φ *has at least three fixed points* $x_1, x_2, x_3 \in \overline{P(\gamma, c)}$ *such that*

$$\beta(x_1) < a,\ b < \alpha(x_2) \textit{ and } a < \beta(x_3) \textit{ with } \alpha(x_3) < b.$$

3 Existence Results

In this section, by using the five functionals fixed point theorem, we shall establish the existence of at least *three* positive solutions of the boundary value problem (1.4) and (1.2) or (1.3).

We note that, from the nonnegativity of $a(t)$ and $f(t, u(t))$, a solution of the problem (1.4) and (1.2) or (1.3) is nonnegative and concave on $[0, T]_{\mathbb{T}}$, see [11].

Assume that $\eta \in (0, T)_{\mathbb{T}}$ is a constant and let $E = C_{\mathrm{ld}}([0, T]_{\mathbb{T}}, \mathbb{R})$. Then E is a Banach space with norm $\|u\| = \sup_{t\in[0,T]_{\mathbb{T}}} |u(t)|$. Define the cone $P \subset E$ by

$$P = \left\{u \in E \,\middle|\, u^{\Delta}(T) = 0,\ u \text{ is concave, nonnegative on } [0, \sigma(T)]_{\mathbb{T}} \right\}.$$

Fix $l \in \mathbb{T}$ such that $0 < \eta < l < T$. Then we have the following.

Lemma 3.1. *If* $u \in P$*, then*

(*i*) $u(t) \geq \dfrac{t}{T} \|u\|$ *for* $t \in [0, T]_{\mathbb{T}}$*;*

(*ii*) $\eta u(l) \leq l u(\eta)$.

Proof. From [10] we obtain that (i) is true. In view of the concavity of $u(t) \in P$, by letting $t = \dfrac{\eta}{l}$, $x = l$ and $y = 0$ in Definition 2.3, (ii) is satisfied. □

First we define the nonnegative continuous concave functionals α and φ and nonnegative continuous convex functionals γ, β and θ on P respectively by

$$\gamma(u) := \theta(u) := \max_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta),$$
$$\alpha(u) := \min_{t\in[l,T]_{\mathbb{T}}} u(t) = u(l),$$
$$\beta(u) := \max_{t\in[0,l]_{\mathbb{T}}} u(t) = u(l),$$

and

$$\varphi(u) := \min_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta).$$

It is clear that $\alpha(u) = \beta(u)$ for all $u \in P$. For notational convenience, we denote

$$\begin{aligned} M &= AG\left(\int_0^T a(r)\nabla r\right) + \eta G\left(\int_0^T a(r)\nabla r\right), \\ m &= (B+l)G\left(\int_l^T a(r)\nabla r\right), \\ \lambda_l &= (A+l)G\left(\int_0^T a(r)\nabla r\right), \end{aligned}$$

where $G(w) = |w|^{\frac{1}{p-1}}\operatorname{sgn}(w)$ is the inverse of g.

Now we state and prove our main result.

Theorem 3.2. *Assume that* (H_1) *and* (H_2) *hold and that there exist nonnegative real numbers* a*,* b*,* c *with* $0 < a < lb/T < l\eta c/T^2$*,* $Mb < mc$*, and* $f(t,u(t))$ *such that the following conditions hold:*

(F_1) $f(t,x) < g\left(\dfrac{c}{M}\right)$ *for all* $(t,x) \in [0,T]_{\mathbb{T}} \times \left[0, \dfrac{Tc}{\eta}\right]$*;*

(F_2) $f(t,x) > g\left(\dfrac{b}{m}\right)$ *for all* $(t,x) \in [l,T]_{\mathbb{T}} \times \left[b, \dfrac{T^2 b}{\eta^2}\right]$*;*

(F_3) $f(t,x) < g\left(\dfrac{a}{\lambda_l}\right)$ *for all* $(t,x) \in [0,T]_{\mathbb{T}} \times \left[0, \dfrac{Ta}{l}\right]$.

Then the boundary value problem (1.4) *and* (1.2) *or* (1.3) *has at least three positive solutions* u_1*,* u_2 *and* u_3 *such that*

$$\max_{t\in[0,l]_{\mathbb{T}}} u_1(t) < a,\ b < \min_{t\in[l,T]_{\mathbb{T}}} u_2(t) \text{ and } a < \max_{t\in[0,l]_{\mathbb{T}}} u_3(t) \text{ with } \min_{t\in[l,T]_{\mathbb{T}}} u_3(t) < b.$$

Proof. Define a completely continuous integral operator $\Phi : P \to E$ by

$$(\Phi u)(t) = B_0\left(G\left(\int_0^T a(r)f(r,u(r))\nabla r\right)\right) + \int_0^t G\left(\int_s^T a(r)f(r,u(r))\nabla r\right)\Delta s,$$

$t \in [0,T]_{\mathbb{T}}$. We first note that for $u \in P$ we have $\Phi u(t) \geq 0$, $\Phi u^{\Delta}(T) = 0$, and, applying the fundamental theorem of calculus, we deduce that $\Phi u(t)$ is concave. Consequently, $\Phi u(t) \in P$. Since

$$(\Phi(u^{\Delta}(t)))^{\nabla} = -a(t)f(t,u(t)) \leq 0 \text{ for } t \in (0,T)_{\mathbb{T}},$$

all fixed points of Φ are solutions of the boundary value problem (1.4) and (1.2). Assume that $u \in \overline{P(\gamma, c)}$. Then $\gamma(u) = \max_{t \in [0,\eta]_{\mathbb{T}}} u(t) = u(\eta) \leq c$. Consequently, $0 \leq u(t) \leq c$ for $t \in [0, \eta]_{\mathbb{T}}$. It follows from Lemma 3.1 (i) that $\|u\| \leq \frac{Tu(\eta)}{\eta} \leq \frac{Tc}{\eta}$. This implies $0 \leq u(t) \leq \frac{Tc}{\eta}$ for $t \in [0, T]_{\mathbb{T}}$. Furthermore

$$\begin{aligned}
&\gamma(\Phi(u)) = (\Phi u)(\eta) \\
&\quad = B_0\left(G\left(\int_0^T a(r) f(r, u(r)) \nabla r\right)\right) + \int_0^\eta G\left(\int_s^T a(r) f(r, u(r)) \nabla r\right) \Delta s \\
&\quad \leq AG\left(\int_0^T a(r) f(r, u(r)) \nabla r\right) + \eta G\left(\int_0^T a(r) f(r, u(r)) \nabla r\right) \\
&\quad \leq \frac{c}{M}\left[AG\left(\int_0^T a(r)) \nabla r\right) + \eta G\left(\int_0^T a(r) \nabla r\right)\right] \\
&\quad = c.
\end{aligned}$$

Therefore $\Phi(u) \in \overline{P(\gamma, c)}$. From Lemma 3.1 (i), we have $\gamma(u) = u(\eta) \geq \frac{\eta}{T}\|u\|$. Hence $\|u\| \leq \frac{T}{\eta} u(\eta) = \frac{T}{\eta}\gamma(u)$ for all $u \in P$.

Now, we show that (i)–(iv) of Theorem 2.4 are satisfied. If we let $u \equiv \frac{Tb}{\eta}$ and $k = \frac{Tb}{\eta}$, then

$$\alpha(u) = u(l) = \frac{Tb}{\eta} > b, \ \theta(u) = u(\eta) = \frac{Tb}{\eta} = k \text{ and } \gamma(u) = \frac{Tb}{\eta} < c,$$

which imply that

$$\{u \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(u) > b\} \neq \emptyset.$$

For $u \in P\left(\gamma, \theta, \alpha, b, \frac{Tb}{\eta}, c\right)$, we have

$$\theta(u) = \max_{t \in [0,\eta]_{\mathbb{T}}} u(t) = u(\eta) \leq \frac{Tb}{\eta}.$$

Consequently,

$$0 \leq u(t) \leq \frac{Tb}{\eta} \text{ for all } t \in [0, \eta]_{\mathbb{T}}.$$

By Lemma 3.1 (i), we have

$$\|u\| \leq \frac{Tu(\eta)}{\eta} = \frac{T^2 b}{\eta^2}.$$

This means

$$b \le u(t) \le \frac{T^2 b}{\eta^2} \text{ for all } t \in [l,T]_{\mathbb{T}}$$

and

$$\begin{aligned}
\alpha(\Phi(u)) &= (\Phi u)(l) \\
&= B_0\left(G\left(\int_0^T a(r)f(r,u(r))\nabla r\right)\right) + \int_0^l G\left(\int_s^T a(r)f(r,u(r))\nabla r\right)\Delta s \\
&\ge BG\left(\int_l^T a(r)f(r,u(r))\nabla r\right) + lG\left(\int_l^T a(r)f(r,u(r))\nabla r\right) \\
&\ge (B+l)G\left(\int_l^T a(r)\nabla r\right)\frac{b}{m} = b.
\end{aligned}$$

Therefore, condition (i) of Theorem 2.4 is true. Next, if we take $u = \dfrac{\eta a}{T}$ and $h = \dfrac{\eta a}{T}$, then

$$\gamma(u) = u(\eta) = \frac{\eta a}{T} < c,\ \varphi(u) = u(\eta) = \frac{\eta a}{T} = h,\ \beta(u) = u(l) = \frac{\eta a}{T} < a,$$

which means

$$\{u \in Q\,(\gamma,\beta,\varphi,h,a,c) : \beta(u) < a\} \ne \emptyset.$$

For

$$u \in Q\left(\gamma,\beta,\varphi,\frac{\eta a}{T},a,c\right),$$

we have

$$\beta(u) := \max_{t\in[0,l]_{\mathbb{T}}} u(t) = u(l) \le a.$$

Consequently

$$0 \le u(t) \le a \text{ for } t \in [0,l]_{\mathbb{T}}.$$

In view of Lemma 3.1 (i)

$$\|u\| \le \frac{Tu(l)}{l} \le \frac{Ta}{l} \text{ for } t \in [0,T]_{\mathbb{T}},$$

and thus

$$0 \le u(t) \le \frac{Ta}{l} \text{ for } t \in [0,T]_{\mathbb{T}}.$$

Therefore,

$$\begin{aligned}\beta(\Phi(u)) &= (\Phi u)(l)\\ &= B_0\left(G\left(\int_0^T a(r)f(r,u(r))\nabla r\right)\right)+\int_0^l G\left(\int_s^T a(r)f(r,u(r))\nabla r\right)\Delta s\\ &\le AG\left(\int_0^T a(r)f(r,u(r))\nabla r\right)+lG\left(\int_0^T a(r)f(r,u(r))\nabla r\right)\\ &\le (A+l)G\left(\int_0^T a(r)\nabla r\right)\frac{a}{\lambda_l}\\ &= a.\end{aligned}$$

This implies that condition (ii) of Theorem 2.4 holds. Next, if $u \in P(\gamma,\alpha,b,c)$ and

$$\theta(\Phi(u)) = \Phi(u(\eta)) > k = \frac{Tb}{\eta},$$

then

$$\alpha(\Phi(u)) = (\Phi u)(l) \ge \frac{l}{T}\Phi(u(l)) \ge \frac{l}{T}\Phi(u(\eta)) > \frac{l}{T}\times\frac{Tb}{\eta} = \frac{lb}{\eta} > b.$$

Therefore, condition (iii) of Theorem 2.4 is satisfied. Finally, if $u \in Q(\gamma,\beta,a,c)$ and

$$\varphi(\Phi(u)) = \Phi(u(\eta)) < h = \frac{a\eta}{T},$$

then by Lemma 3.1 (ii) we obtain

$$\beta(\Phi(u)) = (\Phi u)(l) \le \frac{T}{l}(\Phi u(l)) \le \frac{T}{\eta}\Phi(u(\eta)) < \frac{T}{\eta}\times\frac{a\eta}{T} = a,$$

which shows that the condition (iv) of Theorem 2.4 is fulfilled. Thus, all the conditions in Theorem 2.4 are met, so the boundary value problem (1.4) and (1.2) has at least three positive solutions u_1, u_2, u_3 such that

$$\max_{t\in[0,l]_{\mathbb{T}}} u_1(t) < a,\ b < \min_{t\in[l,T]_{\mathbb{T}}} u_2(t) \text{ and } a < \max_{t\in[0,l]_{\mathbb{T}}} u_3(t) \text{ with } \min_{t\in[l,T]_{\mathbb{T}}} u_3(t) < b.$$

The proof is complete. □

Now, we consider the boundary value problem (1.4) and (1.3) and fix ξ such that

$$0 < \xi < \eta < T.$$

Define a cone $P_1 \subset E$ by

$$P_1 = \left\{u \in E \,\middle|\, u^{\Delta}(0) = 0,\ u \text{ is concave nonnegative on } [0,\sigma(T)]_{\mathbb{T}}\right\}.$$

Similar to Lemma 3.1, we have the following.

Lemma 3.3. *If $u \in P_1$, then*

(*i*) $u(t) \geq \dfrac{T-t}{T}\|u\|$ *for* $t \in [0,T]_{\mathbb{T}}$;

(*ii*) $(T-\eta)u(\xi) \leq (T-\xi)u(\eta)$.

Now define the nonnegative continuous concave functionals α_1 and φ_1 and nonnegative continuous convex functionals γ_1, β_1 and θ_1 on P_1 respectively by

$$\begin{aligned}\gamma_1(u) := \theta_1(u) := \max_{t\in[\eta,T]_{\mathbb{T}}} u(t) = u(\eta),\\ \alpha_1(u) := \min_{t\in[0,\xi]_{\mathbb{T}}} u(t) = u(\xi),\\ \beta_1(u) := \max_{t\in[\xi,T]_{\mathbb{T}}} u(t) = u(\xi),\end{aligned}$$

and

$$\varphi_1(u) := \min_{t\in[0,\eta]_{\mathbb{T}}} u(t) = u(\eta).$$

It is clear that $\alpha_1(u) = \beta_1(u)$ for all $u \in P_1$. Let

$$\begin{aligned} M_1 &= AG\left(\int_0^T a(r)\nabla r\right) + (T-\eta)G\left(\int_0^T a(r)\nabla r\right), \\ m_1 &= (B+T-\xi)G\left(\int_0^\xi a(r)\nabla r\right), \\ \lambda_\xi &= (A+T-\xi)G\left(\int_0^T a(r)\nabla r\right). \end{aligned}$$

We have the following result.

Theorem 3.4. *Assume that* (H_1) *and* (H_2) *hold and that there exist nonnegative real numbers* a, b, c *with* $0 < a < \dfrac{(T-\xi)b}{T} < \dfrac{(T-\xi)(T-\eta)c}{T^2}$, $M_1 b < m_1 c$, $f(t,u(t))$ *such that the following conditions are satisfied:*

(G_1) $f(t,x) < g\left(\dfrac{c}{M_1}\right)$ *for all* $(t,x) \in [0,T]_{\mathbb{T}} \times \left[0, \dfrac{Tc}{T-\eta}\right]$;

(G_2) $f(t,x) > g\left(\dfrac{b}{m_1}\right)$ *for all* $(t,x) \in [\xi,T]_{\mathbb{T}} \times \left[b, \dfrac{T^2 b}{(T-\eta)\eta}\right]$;

(G_3) $f(t,x) < g\left(\dfrac{a}{\lambda_\xi}\right)$ *for all* $(t,x) \in [0,T]_{\mathbb{T}} \times \left[0, \dfrac{Ta}{T-\xi}\right]$.

Then the boundary value problem (1.4) *and* (1.3) *has at least three positive solutions* u_1, u_2, u_3 *such that*

$$\max_{t\in[\xi,T]_{\mathbb{T}}} u_1(t) < a,\ b < \min_{t\in[0,\xi]_{\mathbb{T}}} u_2(t) \text{ and } a < \max_{t\in[\xi,T]_{\mathbb{T}}} u_3(t) \text{ with } \min_{t\in[0,\xi]_{\mathbb{T}}} u_3(t) < b.$$

Proof. Define a completely continuous integral operator $\Phi : P \to E$ by

$$(\Phi u)(t) = B_1\left(G\left(\int_0^T a(r)f(r,u(r))\nabla r\right)\right) + \int_t^T G\left(\int_0^s a(r)f(r,u(r))\nabla r\right)\Delta s,$$

$t \in [0,T]_{\mathbb{T}}$. The rest of the proof is similar to that of Theorem 3.2, so we omit it here. □

4 An Example

In this section, we present a simple example to explain our result. Let

$$\mathbb{T} = \bigcup_{i=1}^{\infty}\left\{1 - \left(\frac{1}{2}\right)^i\right\} \cup \{1\} \text{ and } T{=}1.$$

Consider the following boundary value problem with $p = 3$.

$$\left(g\left(u^{\Delta}(t)\right)\right)^{\nabla} + a(t)f(t,u(t)) = 0,\ t \in [0,1]_{\mathbb{T}} \tag{4.1}$$

satisfying the boundary conditions

$$u(0) - 2(u^{\Delta}(0)) = 0,\ u^{\Delta}(1) = 0, \tag{4.2}$$

where

$$a(t) = t + \rho(t) \text{ for } t \in [0,1]_{\mathbb{T}},$$

and

$$f(t,u(t)) =: f(u) = \begin{cases} 1\times 10^{-6}, & u \in [0, 0.048],\\ 0.1963x - 9.214\times 10^{-3}, & u \in [0.048, 0.051],\\ 5.9\times 10^{-4}, & u \in [0.051, 0.816],\\ 8.3333\times 10^{-4}x - 8.997\times 10^{-5}, & u \in [0.816, 0.84],\\ 6.1\times 10^{-4}, & u \in [0.84, 1],\\ s(u), & u \in [1,\infty),\end{cases}$$

$s(u)$ satisfies $s(1) = 6.1\times 10^{-4}$ and $s(u) : \mathbb{R} \to \mathbb{R}^+$ is continuous. If we choose $A = B = 2$, $l = \dfrac{1}{8}$, $\eta = \dfrac{1}{4}$, then a direct calculation shows that

$$\begin{aligned} M &= 2G\left(\int_0^1 a(r)\nabla r\right) + \eta G\left(\int_0^1 a(r)\nabla r\right)\\ &= \left(2+\frac{1}{4}\right)G\left(\int_0^1 (r+\rho(r))\,\nabla r\right) = \frac{9}{4} = 2.25.\end{aligned}$$

Using a similar way, we have

$$m \approx 2.1083, \ \lambda_l = 2.125.$$

Furthermore, if we take

$$a = 0.006, \quad b = 0.051, \quad c = 0.21,$$

then $Mb = 0.11475 < 0.44274 \approx mc$ and $0 < a < lb < l\eta c$.

$$f(t,u) = 1 \times 10^{-6} < 7.9723 \times 10^{-6} = g\left(\frac{a}{\lambda_l}\right) \text{ for } (t,u) \in [0,1]_{\mathbb{T}} \times [0, 0.048],$$

$$f(t,u) \geq 5.9 \times 10^{-4} > 5.8516 \times 10^{-4} = g\left(\frac{b}{m}\right) \text{ for } (t,u) \in \left[\frac{1}{8}, 1\right]_{\mathbb{T}} \times [0.051, 0.816],$$

and

$$f(t,u) \leq 6.1 \times 10^{-4} < 8.7111 \times 10^{-3} = g\left(\frac{c}{M}\right) \text{ for } (t,u) \in [0,1]_{\mathbb{T}} \times \in [0, 0.84].$$

By Theorem 3.2, we see that the boundary value problem (4.1) and (4.2) has at least *three* positive solutions u_1, u_2 and u_3 such that

$$\max_{t \in \left[0, \frac{1}{8}\right]_{\mathbb{T}}} u_1(t) < 0.006 < \max_{t \in \left[0, \frac{1}{8}\right]_{\mathbb{T}}} u_3(t) \text{ and } \min_{t \in \left[\frac{1}{8}, 1\right]_{\mathbb{T}}} u_3(t) < 0.051 < \min_{t \in \left[\frac{1}{8}, 1\right]_{\mathbb{T}}} u_2(t).$$

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