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Solution of a q-deformed Linear System Containing Zero Conditions at Infinity

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Abstract

In this study, we investigate the eigenvalues and eigenvectors of a quadratic pencil of q-difference equations. The results obtained are then used to solve the corresponding system of differential equations with boundary and initial conditions and zero conditions at infinity.

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1 Introduction

We consider the system of linear second order differential equations

$$\begin{aligned} c_n \frac{d^2 u_n(t)}{dt^2} &= r_n \frac{du_n(t)}{dt} + a_{n-1}u_{n-1}(t) + b_n u_n(t) + q a_n u_{n+1}(t), \\ n &\in \{0, 1, \ldots, N-1\}, \quad t \geq 0 \end{aligned} \tag{1.1}$$

with the “boundary” conditions

$$u_{-1}(t) = 0, \quad u_N(t) + h u_{N-1}(t) = 0, \quad t \geq 0 \tag{1.2}$$

and the conditions (initial conditions at $t = 0$ and zero “end” conditions at $t = \infty$)

$$u_n(0) = f_n, \quad \lim_{t \to \infty} u_n(t) = 0, \quad n \in \{0, 1, \ldots, N-1\}, \tag{1.3}$$

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where $N \geq 2$ is a positive integer, $q > 0$ is a fixed real number, $\{u_n(t)\}_{n=-1}^{N}$ is a desired solution, f_n $(n = 0, 1, \ldots, N-1)$ are given complex numbers, the coefficients c_n, r_n, a_n, b_n of equation (1.1) and the number h in the boundary conditions (1.2) are real, and

$$a_n \neq 0, \; c_n > 0, \tag{1.4}$$

$$\begin{gathered} b_0 - q\,|a_0| \geq 0, \quad b_{N-1} - hqa_{N-1} - |a_{N-2}| \geq 0, \\ b_n - |a_{n-1}| - q\,|a_n| \geq 0, \quad n \in \{1, 2, \ldots, N-2\} \end{gathered} \tag{1.5}$$

with strict inequality in at least one relation of (1.5). If $\{u_n(t)\}_{n=-1}^{N}$ is a solution of problem (1.1)–(1.3), then taking boundary conditions (1.2) into account, we have

$$\begin{aligned} c_0 \frac{d^2 u_0(t)}{dt^2} &= r_0 \frac{du_0(t)}{dt} + b_0 u_0(t) + qa_0 u_1(t), \\ c_n \frac{d^2 u_n(t)}{dt^2} &= r_n \frac{du_n(t)}{dt} + a_{n-1} u_{n-1}(t) + b_n u_n(t) + qa_n u_{n+1}(t), \\ n &= 1, 2, \ldots, N-2, \\ c_{N-1} \frac{d^2 u_{N-1}(t)}{dt^2} &= r_{N-1} \frac{du_{N-1}(t)}{dt} + a_{N-2} u_{N-2}(t) + (b_{N-1} - hqa_{N-1}) u_{N-1}(t). \end{aligned} \tag{1.6}$$

Consequently, finding a solution $\{u_n(t)\}_{n=-1}^{N}$ of problem (1.1)–(1.3) is equivalent to the problem of finding a solution $\{u_n(t)\}_{n=0}^{N-1}$ of system (1.6) that satisfies the conditions (1.3). Setting

$$u(t) = \begin{bmatrix} u_0(t) \\ u_1(t) \\ \vdots \\ u_{N-1}(t) \end{bmatrix}, \quad f = \begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_{N-1} \end{bmatrix},$$

$$C = \begin{bmatrix} c_0 & 0 & 0 & \cdots & 0 \\ 0 & c_1 & 0 & \cdots & 0 \\ 0 & 0 & c_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & c_{N-1} \end{bmatrix}, \quad R = \begin{bmatrix} r_0 & 0 & 0 & \cdots & 0 \\ 0 & r_1 & 0 & \cdots & 0 \\ 0 & 0 & r_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & r_{N-1} \end{bmatrix},$$

$$J = \begin{bmatrix} b_0 & qa_0 & 0 & \cdots & 0 & 0 & 0 \\ a_0 & b_1 & qa_1 & \cdots & 0 & 0 & 0 \\ 0 & a_1 & b_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_{N-3} & qa_{N-3} & 0 \\ 0 & 0 & 0 & \cdots & a_{N-3} & b_{N-2} & qa_{N-2} \\ 0 & 0 & 0 & \cdots & 0 & a_{N-2} & b_{N-1} - hqa_{N-1} \end{bmatrix}, \tag{1.7}$$

we can write problem (1.6), (1.3) in the form

$$C \frac{d^2 u(t)}{dt^2} = R \frac{du(t)}{dt} + Ju(t), \quad 0 \leq t < \infty, \tag{1.8}$$

$$u(0) = f, \quad \lim_{t\to\infty} u(t) = 0. \tag{1.9}$$

Thus problem (1.1)–(1.3) is equivalent to problem (1.8), (1.9), i.e., if $\{u_n(t)\}_{n=-1}^{N}$ is a solution of problem (1.1)–(1.3), then the vector-function $u(t) = \{u_n(t)\}_{n=0}^{N-1}$ forms a solution of problem (1.8), (1.9), and conversely, if $u(t) = \{u_n(t)\}_{n=0}^{N-1}$ is a solution of (1.8), (1.9) then $\{u_n(t)\}_{n=-1}^{N}$, where $u_{-1}(t) = 0$, and $u_N(t) = -hu_{N-1}(t)$, forms a solution of problem (1.1)–(1.3).

Our principal aim in this paper is to prove that problem (1.1)–(1.3) (or, equivalently, problem (1.8), (1.9)) has a unique solution and to investigate the structure of the solution, that is, to give an effective formula for it. To do so, we seek a nontrivial solution of equation (1.8), which has the form

$$u(t) = e^{\lambda t} y, \tag{1.10}$$

where λ is a complex constant and y is a constant vector (an element of the space $\mathbb{C}^N$) which depends upon λ and which we desire to be nontrivial, that is, not equal to 0, the null vector. Substituting (1.10) into (1.8), we obtain

$$(\lambda^2 C - \lambda R - J)y = 0. \tag{1.11}$$

Definition 1.1. A complex number λ_0 is said to be an eigenvalue of equation (1.11) (or of the quadratic pencil $\lambda^2 C - \lambda R - J$) if there exists a nonzero vector $y \in \mathbb{C}^N$ satisfying equation (1.11) for $\lambda = \lambda_0$. The vector y is called an eigenvector of equation (1.11), corresponding to the eigenvalue λ_0.

Thus the vector-function (1.10) is a nontrivial solution of equation (1.8) if and only if λ is an eigenvalue and y is a corresponding eigenvector of equation (1.11). Note that (1.11) is equivalent to the discrete boundary value problem

$$(\lambda^2 c_n - \lambda r_n - b_n) y_n - a_{n-1} y_{n-1} - q a_n y_{n+1} = 0, \quad n = 0, 1, \ldots, N-1, \tag{1.12}$$

$$y_{-1} = 0, \quad y_N + h y_{N-1} = 0, \tag{1.13}$$

that is, if $\{y_n\}_{n=-1}^{N}$ is a solution of problem (1.12), (1.13), then the vector $y = \{y_n\}_{n=0}^{N-1}$ forms a solution of equation (1.11), and conversely, if $y = \{y_n\}_{n=0}^{N-1}$ is a solution of (1.11), then $\{y_n\}_{n=-1}^{N}$, where $y_{-1} = 0$ and $y_N = -hy_{N-1}$, forms a solution of problem (1.12), (1.13).

Let us denote by $\lambda_1, \ldots, \lambda_m$ all eigenvalues of equation (1.11), and by $y^{(1)}, \ldots, y^{(m)}$ the corresponding eigenvectors. Then by linearity of equation (1.8) the vector-function

$$u(t) = \sum_{j=1}^{m} \alpha_j e^{\lambda_j t} y^{(j)} \tag{1.14}$$

will be a solution of equation (1.8), where $\alpha_1, \dots, \alpha_m$ are arbitrary constants independent of t. Now we must try to choose the constants α_j so that (1.14) will also satisfy the conditions in (1.9):

$$\sum_{j=1}^{m} \alpha_j y^{(j)} = f, \quad \lim_{t\to\infty} \sum_{j=1}^{m} \alpha_j e^{\lambda_j t} y^{(j)} = 0. \tag{1.15}$$

In this paper we show that under the conditions (1.4), (1.5), such a choice of constants α_j is possible if we pick out suitable eigenvalues λ_j. We also find formulas for the constants α_j. To this end we have to examine the eigenvalue problem (1.11) in detail.

The paper is organized as follows. In Section 2 we present some needed facts about second order self-adjoint q-difference equations because (1.12) is such an equation. In Section 3 reality of the eigenvalues and "orthogonality" of the eigenvectors are established. Section 4 contains further properties of the eigenvalues and eigenvectors. In Section 5 we find the form of a general solution of equation (1.8). Section 6 determines the number of the negative eigenvalues. Section 7 is devoted to the proof of the basisness of "half" of the eigenvectors. Finally, in Section 8, using the results obtained for the eigenvalue problem, we prove existence and uniqueness of solution to the problem (1.1)–(1.3) and present an effective formula for the solution.

Note that a comprehensive treatment of general matrix polynomials to which our quadratic pencil $\lambda^2 C - \lambda R - J$ belongs is given in [4]. However, due to the special structure (1.7) of the matrices C, R and J, and the conditions (1.4), (1.5), we have succeeded in obtaining, in this paper, more specific results. Similar problems involving usual difference equations ($q = 1$) were investigated earlier in [5, 6].

2 Auxiliary Facts on Linear q-difference Equations

For a treatment of q-calculus, we refer the reader to [7]. The theorems given below in this section and related to second order self-adjoint q-difference equations are similar to those for usual second order difference equations [9] and are not difficult to verify.

Definition 2.1. Let q be a fixed real number such that $q \neq 0$ and $q \neq 1$. Let us set

$$q^{\mathbb{Z}} = \{q^n : n \in \mathbb{Z}\} = \left\{\dots, q^{-2}, q^{-1}, q^0, q^1, q^2, \dots\right\}.$$

Let $y(x)$ be a complex-valued function defined for $x \in q^{\mathbb{Z}}$. The "q-difference" operator D_q is defined by

$$D_q y(x) = \frac{y(qx) - y(x)}{(q-1)x}, \quad x \in q^{\mathbb{Z}}. \tag{2.1}$$

The expression in (2.1) is called the q-derivative of the function y at x.

Higher order q-derivatives are defined by repeated application of the operator D_q. For example, the second order q-derivative is

$$D_q^2 y(x) = D_q(D_q y(x)) = \frac{y(q^2x) - (q+1)y(qx) + qy(x)}{q(q-1)^2x^2}.$$

Fundamental properties of D_q are given in the following theorem.

Theorem 2.2. *Assume $f, g : q^{\mathbb{Z}} \to \mathbb{C}$ are functions. Then*

(*i*) $D_q(y(x) + z(x)) = D_q y(x) + D_q z(x)$,

(*ii*) $D_q(cy(x)) = cD_q y(x)$ *if c is a constant,*

(*iii*) $D_q(y(x)z(x)) = (D_q y(x))z(x) + y(qx)D_q z(x) = y(x)D_q z(x) + (D_q y(x))z(qx)$,

(*iv*) $D_q\left(\dfrac{y(x)}{z(x)}\right) = \dfrac{(D_q y(x))z(x) - y(x)D_q z(x)}{z(x)z(qx)}$ *if $z(x)z(qx) \neq 0$.*

Example 2.3. We have the following:

(i) $D_q c = 0$ if c is a constant,

(ii) $D_q x = 1$,

(iii) $D_q x^2 = (q+1)x$,

(iv) $D_q x^3 = (q^2+q+1)x^2$,

(v) $D_q \ln x = \dfrac{\ln q}{(q-1)x}$.

Theorem 2.4. *If $D_q y(x)$ is identically zero on $q^{\mathbb{Z}}$, then $y(x)$ is constant on $q^{\mathbb{Z}}$.*

Let $p(x)$ and $r(x)$ be given functions defined on $q^{\mathbb{Z}}$ with $p(x) \neq 0$ for all $x \in q^{\mathbb{Z}}$. The second order self-adjoint linear homogeneous q-difference equation is defined to be

$$D_q\left[p\left(\frac{x}{q}\right) D_q y\left(\frac{x}{q}\right)\right] + r(x)y(x) = 0, \quad x \in q^{\mathbb{Z}}, \tag{2.2}$$

where $y(x)$ is a desired solution. We can also write equation (2.2) in the form

$$a\left(\frac{x}{q}\right) y\left(\frac{x}{q}\right) + b(x)y(x) + qa(x)y(qx) = 0, \quad x \in q^{\mathbb{Z}}, \tag{2.3}$$

where

$$a(x) = \frac{p(x)}{(q-1)^2q^2x^2}, \quad b(x) = r(x) - qa(x) - a\left(\frac{x}{q}\right).$$

Note that any equation written in the form of equation (2.3), where $a(x) \neq 0$ on $q^{\mathbb{Z}}$, can be written in the self-adjoint form of equation (2.2) by taking

$$p(x) = (q-1)^2 q^2 x^2 a(x), \quad r(x) = b(x) + qa(x) + a\left(\frac{x}{q}\right).$$

Setting $x = q^n$ ($n \in \mathbb{Z}$) in equation (2.3), we get

$$a(q^{n-1})y(q^{n-1}) + b(q^n)y(q^n) + qa(q^n)y(q^{n+1}) = 0, \quad n \in \mathbb{Z}.$$

Finally, denoting

$$a_n = a(q^n), \quad b_n = b(q^n), \quad y_n = y(q^n) \quad \text{for} \quad n \in \mathbb{Z},$$

we can write the last equation in the form

$$a_{n-1}y_{n-1} + b_n y_n + qa_n y_{n+1} = 0, \quad n \in \mathbb{Z}. \tag{2.4}$$

Further we will deal with equations of the form (2.4) assuming that q is any fixed positive real number. Note that equation (1.12) has the form of equation (2.4). Take a fixed integer $n_0 \in \mathbb{Z}$ and consider the initial conditions

$$y_{n_0} = c_0, \quad y_{n_0+1} = c_1, \tag{2.5}$$

where c_0 and c_1 are given numbers.

Theorem 2.5 (Existence and Uniqueness Theorem). *The initial value problem (IVP)* (2.4), (2.5) *has exactly one solution* $y = (y_n)$.

Corollary 2.6. *Let* (y_n) *be a solution of equation* (2.4). *If* y_n *is zero for two successive integer values of* n, *then* $y_n = 0$ *for all* $n \in \mathbb{Z}$.

Definition 2.7. Let $y = (y_n)$ and $z = (z_n)$ be solutions of equation (2.4). The Wronskian of these solutions is defined to be

$$W_n(y,z) = \begin{vmatrix} y_n & z_n \\ y_{n+1} & z_{n+1} \end{vmatrix} = y_n z_{n+1} - y_{n+1} z_n, \quad n \in \mathbb{Z}.$$

Theorem 2.8. *If* $y = (y_n)$ *and* $z = (z_n)$ *are solutions of equation* (2.4), *then*

$$W_n(y,z) = \frac{c}{q^n a_n}, \quad n \in \mathbb{Z},$$

where c *is a constant.*

Corollary 2.9. *If* $y = (y_n)$ *and* $z = (z_n)$ *are solutions of equation* (2.4), *then either* $W_n(y,z) = 0$ *for all* $n \in \mathbb{Z}$ *or* $W_n(y,z) \neq 0$ *for all* $n \in \mathbb{Z}$.

Theorem 2.10. *Any two solutions of equation* (2.4) *are linearly independent if and only if their Wronskian is different from zero.*

Theorem 2.11. *Equation* (2.4) *has two linearly independent solutions and every solution of equation* (2.4) *is a linear combination of these solutions.*

3 Properties of the Eigenvalues and Eigenvectors

In this section we establish the reality of the eigenvalues and get several "orthogonality" relations for the eigenvectors of problem (1.11) assuming that there exist the eigenvalues and eigenvectors. Existence of the eigenvalues and eigenvectors and their further properties will be established in the next section. Note that there are only a few works concerning spectral analysis of q-difference equations, see [1–3].

Consider the eigenvalue problem (1.11), where the matrices C, R, and J have the form (1.7), and we assume throughout that the conditions (1.4), (1.5) are satisfied. We will investigate the equation (1.11) in the linear space

$$\mathbb{C}^N = \left\{ y = (y_n)_{n=0}^{N-1} : y_n \in \mathbb{C}, \quad n = 0, 1, \ldots, N-1 \right\}$$

with the inner product

$$(y, z)_q = \sum_{n=0}^{N-1} q^n y_n \overline{z}_n, \tag{3.1}$$

where $\mathbb{C}$ denotes the set of complex numbers and the bar over a number denotes complex conjugation.

The following two lemmas are not difficult to prove.

Lemma 3.1. *The matrices C, R, and J are self-adjoint with respect to the inner product* (3.1)*, that is, each of them satisfies the relation*

$$(Ty, z)_q = (y, Tz)_q, \quad \forall y, z \in \mathbb{C}^N.$$

Lemma 3.2. *The matrices C and J are positive, that is,*

$$(Cy, y)_q > 0, \quad (Jy, y)_q > 0, \quad \forall y \in \mathbb{C}^N, \quad y \neq 0.$$

Note that the positiveness of J follows from the condition (1.5) by virtue of the following equality: For any real vector $y = \{y_n\}_{n=0}^{N-1} \in \mathbb{R}^N$,

$$\begin{aligned} (Jy, y)_q &= (b_0 - q\,|a_0|)y_0^2 + q^{N-1}(b_{N-1} - hqa_{N-1} - |a_{N-2}|)y_{N-1}^2 \\ &\quad + \sum_{n=1}^{N-2} q^n (b_n - |a_{n-1}| - q\,|a_n|)y_n^2 + \sum_{n=1}^{N-1} q^n\,|a_{n-1}|\,(y_{n-1} \pm y_n)^2, \end{aligned}$$

where the $\pm$ sign in $(y_{n-1} \pm y_n)^2$ is taken to be that of a_{n-1}.

Theorem 3.3. *Each eigenvalue λ of equation* (1.11) *is real, nonzero, and has the same sign as*

$$2\lambda(Cy, y)_q - (Ry, y)_q \neq 0, \tag{3.2}$$

where y is an eigenvector corresponding to λ.

Proof. Let $\lambda \in \mathbb{C}$ be an eigenvalue of equation (1.11) and $y = \{y_n\}_{n=0}^{N-1} \neq 0$ be a corresponding eigenvector. By forming the inner product of both sides of equation (1.11) with the vector y, we get

$$\lambda^2(Cy, y)_q - \lambda(Ry, y)_q - (Jy, y)_q = 0. \tag{3.3}$$

Since, by Lemma 3.2, $(Jy, y)_q > 0$, we get from (3.3) that $\lambda \neq 0$. Also, since $(Cy, y)_q > 0$, we have

$$\lambda = \frac{(Ry, y)_q \pm \sqrt{(Ry, y)_q^2 + 4(Cy, y)_q(Jy, y)_q}}{2(Cy, y)_q}. \tag{3.4}$$

Since $(Ry, y)_q$ is real by Lemma 3.1 and

$$(Ry, y)_q^2 + 4(Cy, y)_q(Jy, y)_q > 0,$$

we get from (3.4) that λ is real. Further, the product of λ with $2\lambda(Cy, y)_q - (Ry, y)_q$ is, by (3.3),

$$\lambda\left[2\lambda(Cy, y)_q - (Ry, y)_q\right] = \lambda^2(Cy, y)_q + (Jy, y)_q > 0,$$

so that (3.2) holds and the sign of λ is the same as the sign of the expression in (3.2). The theorem is proved. □

Theorem 3.4. *The eigenvectors y and z of equation* (1.11) *corresponding to the distinct eigenvalues λ and μ, respectively, satisfy the "orthogonality" relations*

$$(\lambda + \mu)(Cy, z)_q - (Ry, z)_q = 0, \tag{3.5}$$

$$\lambda\mu(Cy, z)_q + (Jy, z)_q = 0, \tag{3.6}$$

$$\lambda\mu(Ry, z)_q + (\lambda + \mu)(Jy, z)_q = 0. \tag{3.7}$$

Proof. Multiplying in the sense of the inner product the first of the equalities

$$\lambda^2 Cy - \lambda Ry - Jy = 0, \quad \mu^2 Cz - \mu Rz - Jz = 0$$

from the right by z and the second one from the left by y, and using the reality of the eigenvalues and Lemma 3.1, we get

$$\lambda^2(Cy, z)_q - \lambda(Ry, z)_q - (Jy, z)_q = 0,$$

$$\mu^2(Cy, z)_q - \mu(Ry, z)_q - (Jy, z)_q = 0.$$

Eliminating from these two equations in turn (Jy, z), (Ry, z), and (Cy, z), we obtain respectively the "orthogonality" relations (3.5), (3.6), and (3.7). □

4 Existence of Eigenvalues

To investigate the existence and further properties of the eigenvalues and eigenvectors of equation (1.11), we note that equation (1.11) is equivalent to the problem of finding a vector $\{y_n\}_{n=-1}^N$ that satisfies the boundary value problem (1.12), (1.13). We define the solution $y = \{\varphi_n(\lambda)\}_{n=-1}^N$ of equation (1.12) satisfying the initial conditions

$$\varphi_{-1}(\lambda) = 0, \quad \varphi_0(\lambda) = 1. \tag{4.1}$$

Using (4.1), we can recursively find $\varphi_n(\lambda)$, $n = 1, 2, \dots, N$, from the equation (1.12), and $\varphi_n(\lambda)$ is a polynomial in λ of degree $2n$. In fact, we can find that

$$\varphi_1(\lambda) = \frac{1}{qa_0}(\lambda^2 c_0 - \lambda r_0 - b_0),$$

$$\varphi_2(\lambda) = \frac{1}{q^2 a_0 a_1}(\lambda^2 c_0 - \lambda r_0 - b_0)(\lambda^2 c_1 - \lambda r_1 - b_1) - \frac{a_0}{qa_1},$$

$$\varphi_n(\lambda) = \frac{c_0 c_1 \cdots c_{n-1}}{q^n a_0 a_1 \cdots a_{n-1}}\lambda^{2n} + \dots \quad .$$

It is easy to see that every solution $\{y_n(\lambda)\}_{n=-1}^N$ of equation (1.12) satisfying the initial condition $y_{-1} = 0$ is equal to $\{\varphi_n(\lambda)\}_{n=-1}^N$ up to a constant factor:

$$y_n(\lambda) = \alpha\varphi_n(\lambda), \quad n = -1, 0, 1, \dots, N, \tag{4.2}$$

with $\alpha = y_0(\lambda)$. Indeed, both sides of (4.2) are solutions of equation (1.12) and they coincide for $n = -1$ and $n = 0$. Hence (4.2) holds by the uniqueness of solution. We get

$$y_N(\lambda) + hy_{N-1}(\lambda) = \alpha[\varphi_N(\lambda) + h\varphi_{N-1}(\lambda)].$$

Consequently setting

$$\chi(\lambda) = \varphi_N(\lambda) + h\varphi_{N-1}(\lambda), \tag{4.3}$$

we have the following lemma.

Lemma 4.1. *The eigenvalues of equation* (1.11) *are roots of the recursively constructed polynomial* $\chi(\lambda)$*. To each eigenvalue* λ_0 *corresponds, up to a constant factor, a single eigenvector which can be taken to be the vector* $\{\varphi_n(\lambda_0)\}_{n=0}^{N-1}$.

The function $\chi(\lambda)$ is called the *characteristic function* of problem (1.12), (1.13) (or of (1.11)). By Lemma 4.1 the eigenvalues of equation (1.11) coincide with the roots of the function $\chi(\lambda)$. On the other hand the eigenvalues of equation (1.11) coincide, obviously, with the roots of the polynomial $\det(\lambda^2 C - \lambda R - J)$. Since both $\chi(\lambda)$ and $\det(\lambda^2 C - \lambda R - J)$ are polynomials in λ of degree $2N$, it follows therefore that they differ by at most a constant factor from each other. This factor is easily found. To this end it suffices to compare the coefficients of λ^{2N} in these polynomials. This yields

$$\det(\lambda^2 C - \lambda R - J) = q^N a_0 a_1 \cdots a_{N-1}\chi(\lambda).$$

Lemma 4.2. *There exist $2N$ distinct eigenvalues.*

Proof. Since $\varphi_n(\lambda)$ for each n is a polynomial of degree $2n$, by (4.3) $\chi(\lambda)$ is a polynomial of degree $2N$. Therefore $\chi(\lambda)$ has $2N$ roots. Now we show that the roots of $\chi(\lambda)$ are simple. Hence the statement of the lemma will follow. Differentiating the equation

$$(\lambda^2 c_n - \lambda r_n - b_n)\varphi_n(\lambda) - a_{n-1}\varphi_{n-1}(\lambda) - qa_n\varphi_{n+1}(\lambda) = 0$$

with respect to λ, we get

$$(2\lambda c_n - r_n)\varphi_n(\lambda) + (\lambda^2 c_n - \lambda r_n - b_n)\dot{\varphi}_n(\lambda) - a_{n-1}\dot{\varphi}_{n-1}(\lambda) - qa_n\dot{\varphi}_{n+1}(\lambda) = 0,$$

where the dot over a function indicates the derivative with respect to λ. Multiplying the first equation by $\dot{\varphi}_n(\lambda)$ and the second one by $\varphi_n(\lambda)$, and subtracting the left and right members of the resulting equations, we get

$$\begin{gathered}(2\lambda c_n - r_n)\varphi_n^2(\lambda) + a_{n-1}[\varphi_{n-1}(\lambda)\dot{\varphi}_n(\lambda) - \dot{\varphi}_{n-1}(\lambda)\varphi_n(\lambda)] \\ -qa_n[\varphi_n(\lambda)\dot{\varphi}_{n+1}(\lambda) - \dot{\varphi}_n(\lambda)\varphi_{n+1}(\lambda)] = 0.\end{gathered}$$

Summing the last equation multiplied by q^n, for the values $n = 0, 1, \ldots, m$ ($m \leq N-1$) and using the initial conditions (4.1), we get

$$q^{m+1}a_m[\varphi_m(\lambda)\dot{\varphi}_{m+1}(\lambda) - \dot{\varphi}_m(\lambda)\varphi_{m+1}(\lambda)] = \sum_{n=0}^{m} q^n(2\lambda c_n - r_n)\varphi_n^2(\lambda). \tag{4.4}$$

Let us assume $\chi(\lambda_0) = 0$. In particular, setting in (4.4), $m = N-1$ and $\lambda = \lambda_0$, and using the equality $\varphi_N(\lambda_0) = -h\varphi_{N-1}(\lambda_0)$, which follows from the assumption $\chi(\lambda_0) = 0$, we have

$$q^N a_{N-1}\dot{\chi}(\lambda_0)\varphi_{N-1}(\lambda_0) = \sum_{n=0}^{N-1} q^n(2\lambda_0 c_n - r_n)\varphi_n^2(\lambda_0). \tag{4.5}$$

The right-hand side of (4.5) is not zero by virtue of Theorem 3.3. Consequently $\dot{\chi}(\lambda_0) \neq 0$, that is, the root λ_0 of the function $\chi(\lambda)$ is simple. □

We can summarize the results obtained above in the following theorem.

Theorem 4.3. *The equation* (1.11) *has precisely* $2N$ *real distinct eigenvalues* λ_j $(j = 1, \ldots, 2N)$. *These eigenvalues are different from zero. To each eigenvalue* λ_j *there corresponds, up to a constant factor, a single eigenvector which can be taken to be* $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$, *where* $\{\varphi_n(\lambda)\}_{n=-1}^{N}$ *is the solution of equation* (1.12) *satisfying initial conditions* (4.1).

5 General Solution of the Problem (1.1), (1.2)

Now we determine a general form of solutions of problem (1.1), (1.2).

Lemma 5.1. *If $\varphi^{(1)}, \ldots, \varphi^{(2N)}$ are eigenvectors of equation* (1.11)*, corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_{2N}$, respectively, then the vectors $\Phi_j = [\varphi^{(j)}, \lambda_j \varphi^{(j)}] \in \mathbb{C}^N \times \mathbb{C}^N$ ($j = 1, \ldots, 2N$) form a basis for $\mathbb{C}^N \times \mathbb{C}^N$.*

Proof. Consider the linear space $\mathbb{C}^N \times \mathbb{C}^N$ of vectors denoted by $[y, z]$, where $y, z \in \mathbb{C}^N$. Define on this space the bilinear form by the formula

$$\langle [y, z], [u, v] \rangle = (Cy, v)_q + (Cz, u)_q - (Ry, u)_q, \tag{5.1}$$

where $(\cdot, \cdot)_q$ in the right-hand side denotes the inner product in $\mathbb{C}^N$ defined by the formula (3.1). Note that the formula (5.1) does not define an inner product in the space $\mathbb{C}^N \times \mathbb{C}^N$, because for the nonzero vectors $[y, z]$ the numbers $\langle [y, z], [y, z] \rangle$ are not necessarily positive (they may also be zero or negative). In view of formula (3.5) of Theorem 3.4, the vectors

$$\Phi_j = [\varphi^{(j)}, \lambda_j \varphi^{(j)}], \quad j = 1, \ldots, 2N$$

are orthogonal with respect to the bilinear form $\langle \cdot, \cdot \rangle$:

$$\langle \Phi_j, \Phi_l \rangle = 0, \quad j \neq l. \tag{5.2}$$

Further, it is remarkable that, by virtue of Theorem 3.3, we have

$$\rho_j = \langle \Phi_j, \Phi_j \rangle = 2\lambda_j (C\varphi^{(j)}, \varphi^{(j)})_q - (R\varphi^{(j)}, \varphi^{(j)})_q \neq 0 \tag{5.3}$$

and the sign of ρ_j coincides with the sign of λ_j,

$$\text{sign}\rho_j = \text{sign}\lambda_j. \tag{5.4}$$

From (5.2) and (5.3) it follows that $\Phi_1, \ldots, \Phi_{2N}$ are linearly independent in the space $\mathbb{C}^N \times \mathbb{C}^N$. Since the number of them is equal to $2N$ and $\dim(\mathbb{C}^N \times \mathbb{C}^N) = 2N$, they form a basis for the space $\mathbb{C}^N \times \mathbb{C}^N$. The theorem is proved. □

By Lemma 5.1, for an arbitrary vector $[f, g]$ that belongs to $\mathbb{C}^N \times \mathbb{C}^N$, we have the unique expansion

$$[f, g] = \sum_{j=1}^{2N} \beta_j \Phi_j, \quad \text{i.e.,} \quad f = \sum_{j=1}^{2N} \beta_j \varphi^{(j)}, \quad g = \sum_{j=1}^{2N} \beta_j \lambda_j \varphi^{(j)}, \tag{5.5}$$

and

$$\beta_j = \frac{1}{\rho_j} \langle [f, g], \Phi_j \rangle = \frac{1}{\rho_j} \{\lambda_j (Cf, \varphi^{(j)})_q + (Cg, \varphi^{(j)})_q - (Rf, \varphi^{(j)})_q\}, \tag{5.6}$$

where ρ_j is defined by formula (5.3).

Remark 5.2. To prove Lemma 5.1 we could also use the orthogonality relation (3.6) or (3.7). In the case of (3.6) we must use on $\mathbb{C}^N \times \mathbb{C}^N$ the bilinear form

$$\langle [y,z],[u,v]\rangle = (Jy,u) + (Cz,v), \tag{5.7}$$

and in the case of (3.7)

$$\langle [y,z],[u,v]\rangle = (Jy,v) + (Jz,u) + (Rz,v).$$

The bilinear form (5.7), in contrast to the bilinear form (5.1), is an inner product in $\mathbb{C}^N \times \mathbb{C}^N$. But in connection with the formulas (5.3) and (5.6) for ρ_j and β_j, the bilinear form (5.1) has more advantages, since both the matrices C and R presented in it are diagonal.

Theorem 5.3. *The general solution* $\{u_n(t)\}_{n=-1}^{N}$ *of problem* (1.1), (1.2) *has the form*

$$u_n(t) = \sum_{j=1}^{2N} \gamma_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \ldots, N, \tag{5.8}$$

where $\gamma_1, \ldots, \gamma_{2N}$ *are arbitrary constants.*

Proof. From the definitions of λ_j and $\varphi_n(\lambda)$ it follows that (5.8) satisfies (1.1), (1.2) for arbitrary constants $\gamma_1, \ldots, \gamma_{2N}$. Conversely, assume that $\{u_n(t)\}_{n=-1}^{N}$ is an arbitrary solution of (1.1), (1.2). Define the vectors $f = \{f_n\}_{n=0}^{N-1}$ and $g = \{g_n\}_{n=0}^{N-1}$ by setting

$$u_n(0) = f_n, \quad \frac{du_n(0)}{dt} = g_n, \quad n \in \{0, 1, \ldots, N-1\}, \tag{5.9}$$

and then determine the constants $\beta_1, \ldots, \beta_{2N}$ using the expansion (5.5) of the vector $[f,g]$, and put

$$v_n(t) = \sum_{j=1}^{2N} \beta_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \ldots, N.$$

Then the vector-function $v(t) = \{v_n(t)\}_{n=0}^{N-1}$ satisfies the initial value problem

$$C\frac{d^2 v(t)}{dt^2} = R\frac{dv(t)}{dt} + Jv(t), \quad 0 \le t < \infty, \tag{5.10}$$

$$v(0) = f, \quad \frac{dv(0)}{dt} = g. \tag{5.11}$$

On the other hand, $u(t) = \{u_n(t)\}_{n=0}^{N-1}$ also satisfies (5.10), (5.11), and it is well known that a problem of the kind (5.10), (5.11) which can be written in the form of a first order linear system has a unique solution. Hence $u(t) = v(t)$, so that we have (5.8) with $\gamma_j = \beta_j$ ($j = 1, \ldots, 2N$). $\square$

6 Number of the Negative Eigenvalues

Above in Section 4 we showed that under the conditions (1.4), (1.5) equation (1.11) has exactly $2N$ eigenvalues λ_j $(j = 1, \ldots, 2N)$ which are real, distinct, and different from zero. We will assume that these eigenvalues are arranged in increasing order:

$$\lambda_1 < \lambda_2 < \ldots < \lambda_{2N}. \tag{6.1}$$

Note also that since $a_n \neq 0$ $(n = 0, 1, \ldots, N-1)$ and $q > 0$, it follows from the condition (1.5) that

$$b_0 > 0,\ b_1 > 0,\ \ldots\ ,\ b_{N-2} > 0,\ b_{N-1} - hqa_{N-1} > 0. \tag{6.2}$$

Theorem 6.1. *Half of the eigenvalues of equation* (1.11) *are negative and the other half positive, that is, under the assumption* (6.1) *we have*

$$\lambda_j < 0 \textit{ for } j = 1, \ldots, N \quad \textit{and} \quad \lambda_j > 0 \textit{ for } j = N+1, \ldots, 2N.$$

Proof. Consider the auxiliary eigenvalue problem

$$[\lambda^2 C - \lambda \varepsilon R - J(\varepsilon)]y = 0 \tag{6.3}$$

depending on a parameter $\varepsilon \in [0, 1]$, where the matrix $J(\varepsilon)$ is obtained from the matrix J by means of multiplication of all its nondiagonal elements by ε. It is obvious that the analog of the conditions (1.4) and (1.5) is fulfilled for all $\varepsilon \in (0, 1]$. The eigenvalues of equation (6.3) are nonzero for all $\varepsilon \in [0, 1]$ and coincide with the roots of the polynomial

$$\det[\lambda^2 C - \lambda \varepsilon R - J(\varepsilon)]. \tag{6.4}$$

For each $\varepsilon \in (0, 1]$, the roots of the polynomial (6.4) are distinct by virtue of Lemma 4.2, being applicable to the equation (6.3). Denote them by

$$\lambda_1(\varepsilon) < \lambda_2(\varepsilon) < \ldots < \lambda_{2N}(\varepsilon).$$

The equation (6.3) is equivalent to the pair of equations

$$\begin{aligned} z &= \lambda y, \\ \varepsilon C^{-1} R z + C^{-1} J(\varepsilon) y &= \lambda z. \end{aligned}$$

Therefore, the eigenvalues of equation (6.3) coincide with the eigenvalues of the matrix

$$A(\varepsilon) = \begin{bmatrix} 0 & I \\ C^{-1} J(\varepsilon) & \varepsilon C^{-1} R \end{bmatrix}$$

of dimension $2N \times 2N$. Since $\lambda_j(\varepsilon)$ $(j = 1, \ldots, 2N)$ are the eigenvalues of the matrix $A(\varepsilon)$, being continuous in $\varepsilon \in [0, 1]$, they are continuous functions of ε (see [8, Chapter

2, §5]). Note that at the point $\varepsilon = 0$ we do not state that $\lambda_1(\varepsilon), \lambda_2(\varepsilon), \ldots, \lambda_{2N}(\varepsilon)$ are distinct. Now we show that for all values of $\varepsilon \in (0,1]$ half of $\lambda_j(\varepsilon)$ ($j = 1, \ldots, 2N$) are negative and the other half positive:

$$\lambda_j(\varepsilon) < 0 \; (j = 1, \ldots, N), \quad \lambda_j(\varepsilon) > 0 \; (j = N+1, \ldots, 2N).$$

Hence, in particular, for $\varepsilon = 1$ the statement of the lemma will follow. Assume the contrary. Let for some value of $\varepsilon \in (0,1]$

$$\lambda_j(\varepsilon) < 0 \; (j = 1, \ldots, k), \quad \lambda_j(\varepsilon) > 0 \; (j = k+1, \ldots, 2N), \tag{6.5}$$

where $0 \le k \le 2N$ and $k \ne N$ (for $k = 0$ all the eigenvalues $\lambda_j(\varepsilon)$ are understood to be positive, and for $k = 2N$ to be negative). Since $\lambda_j(\varepsilon)$ ($j = 1, \ldots, 2N$) are different from zero and are distinct and continuous functions for all values of $\varepsilon \in (0,1]$, the inequalities (6.5) are valid for all values of $\varepsilon \in (0,1]$ with the same value of k. Passing in (6.5) to the limit as $\varepsilon \to 0$, we get

$$\lambda_j(0) \le 0 \; (j = 1, \ldots, k), \quad \lambda_j(0) \ge 0 \; (j = k+1, \ldots, 2N).$$

But this is a contradiction, since for $\varepsilon = 0$, the roots of the polynomial (6.4) are the numbers

$$\pm\sqrt{\frac{b_j}{c_j}} \; (j = 0, 1, \ldots, N-2), \quad \pm\sqrt{\frac{b_{N-1} - hqa_{N-1}}{c_{N-1}}},$$

half of which are negative and the other half positive by virtue of (6.2). Thus the theorem is proved. □

7 Basisness of "Half" of the Eigenvectors

Theorem 7.1. *The eigenvectors of equation* (1.11), *corresponding to the negative (or positive) eigenvalues form a basis for* $\mathbb{C}^N$.

Proof. We may assume, by Theorem 6.1, that

$$\lambda_1 < \ldots < \lambda_N < 0 < \lambda_{N+1} < \ldots < \lambda_{2N}.$$

To see that the eigenvectors $\varphi^{(1)}, \ldots, \varphi^{(N)}$ corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_N$, respectively, form a basis for $\mathbb{C}^N$, let $z = \{z_n\}_0^{N-1} \in \mathbb{C}^N$ and

$$(z, \varphi^{(j)})_q = 0, \quad j = 1, \ldots, N. \tag{7.1}$$

It suffices for us to establish that then $z = 0$, the null vector. Applying (5.5) and (5.6) to the vectors $f = 0$ and $g = C^{-1}z$, we have

$$0 = \sum_{j=1}^{2N} \beta_j \varphi^{(j)}, \quad C^{-1}z = \sum_{j=1}^{2N} \beta_j \lambda_j \varphi^{(j)}, \tag{7.2}$$

where

$$\beta_j = \frac{1}{\rho_j}(z, \varphi^{(j)})_q, \quad j = 1, \ldots, 2N. \tag{7.3}$$

From (7.3), in view of (7.1), we have $\beta_j = 0$, $j = 1, \ldots, N$, and therefore, (7.2) takes the form

$$0 = \sum_{j=N+1}^{2N} \beta_j \varphi^{(j)}, \quad C^{-1}z = \sum_{j=N+1}^{2N} \beta_j \lambda_j \varphi^{(j)}.$$

Multiplying the last equalities by z in the sense of the inner product in $\mathbb{C}^N$, we get

$$0 = \sum_{j=N+1}^{2N} \beta_j (\varphi^{(j)}, z)_q = \sum_{j=N+1}^{2N} \rho_j \left|\beta_j\right|^2, \tag{7.4}$$

$$(C^{-1}z, z)_q = \sum_{j=N+1}^{2N} \beta_j \lambda_j (\varphi^{(j)}, z)_q = \sum_{j=N+1}^{2N} \rho_j \lambda_j \left|\beta_j\right|^2. \tag{7.5}$$

By virtue of (5.4) we have $\rho_j > 0$, $j = N+1, \ldots, 2N$. Consequently from (7.4) it follows that $\beta_j = 0$, $j = N+1, \ldots, 2N$, and from (7.5) we then get $(C^{-1}z, z)_q = 0$. Hence $z = 0$, since

$$(C^{-1}z, z)_q = \sum_{n=0}^{N-1} \frac{q^n}{c_n} \left|z_n\right|^2$$

and $q > 0$, $c_n > 0$. The theorem is proved. □

8 Solution of the Problem (1.1)–(1.3)

We now give an application of the results obtained above.

Theorem 8.1. *The problem* (1.1)–(1.3) *has a unique solution* $\{u_n(t)\}_{n=-1}^{N}$ *that is representable in the form*

$$u_n(t) = \sum_{j=1}^{N} \alpha_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \ldots, N, \tag{8.1}$$

in which $\lambda_1, \ldots, \lambda_N$ *are the negative eigenvalues of equation* (1.11)*, and* $\{\varphi_n(\lambda)\}_{n=-1}^{N}$ *is the solution of equation* (1.12) *satisfying initial conditions* (4.1)*. Further, the coefficients* $\alpha_1, \ldots, \alpha_N$ *are defined with the help of the unique expansion*

$$f = \sum_{j=1}^{N} \alpha_j \varphi^{(j)}, \tag{8.2}$$

in which $\varphi^{(j)} = \{\varphi_n(\lambda_j)\}_{n=0}^{N-1}$.

Proof. By Theorem 7.1, the vectors $\varphi^{(1)}, \ldots, \varphi^{(N)}$ form a basis of $\mathbb{C}^N$. Therefore, for the vector $f = \{f_n\}_0^{N-1} \in \mathbb{C}^N$ presented in the first condition of (1.3), there exist the numbers $\alpha_1, \ldots, \alpha_N$ uniquely determined by the expansion (8.2). Next we define $u_n(t)$ by formula (8.1). Then $\{u_n(t)\}_{n=-1}^N$ will be a solution of problem (1.1), (1.2) satisfying the first condition in (1.3) in view of (8.2), and the second one in view of that $\lambda_1, \ldots, \lambda_N$ are negative. For proof of the uniqueness of solution, we note that by Theorem 5.3 the general solution of problem (1.1), (1.2) has the representation

$$u_n(t) = \sum_{j=1}^{2N} \gamma_j e^{\lambda_j t} \varphi_n(\lambda_j), \quad n = -1, 0, 1, \ldots, N, \tag{8.3}$$

where $\gamma_1, \ldots, \gamma_N$ are arbitrary constants. Let (8.3) satisfy the conditions (1.3). Then from the second condition in (1.3) it follows that there it must be $\gamma_{N+1} = \ldots = \gamma_{2N} = 0$, since $\lambda_j > 0$ for $j = N+1, \ldots, 2N$ and are distinct. Further, setting $t = 0$ in (8.3) and using the first condition in (1.3), and (8.2) we get $\gamma_j = \alpha_j$, $j = 1, \ldots, N$. The theorem is proved. □

Remark 8.2. The expansion (8.2) written in the form

$$f_n = \alpha_1 \varphi_n(\lambda_1) + \alpha_2 \varphi_n(\lambda_2) + \ldots + \alpha_N \varphi_n(\lambda_N), \quad n = 0, 1, \ldots, N-1,$$

gives us a linear nonhomogeneous system of equations in unknowns $\alpha_1, \alpha_2, \ldots, \alpha_N$.

Remark 8.3. As follows from (8.1), for the solution of problem (1.1)–(1.3) we have

$$u_n(t) = O(e^{-\delta t}), \quad n = -1, 0, 1, \ldots, N,$$

as $t \to \infty$, where $\delta = \min\{|\lambda_1|, \ldots, |\lambda_N|\}$.

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